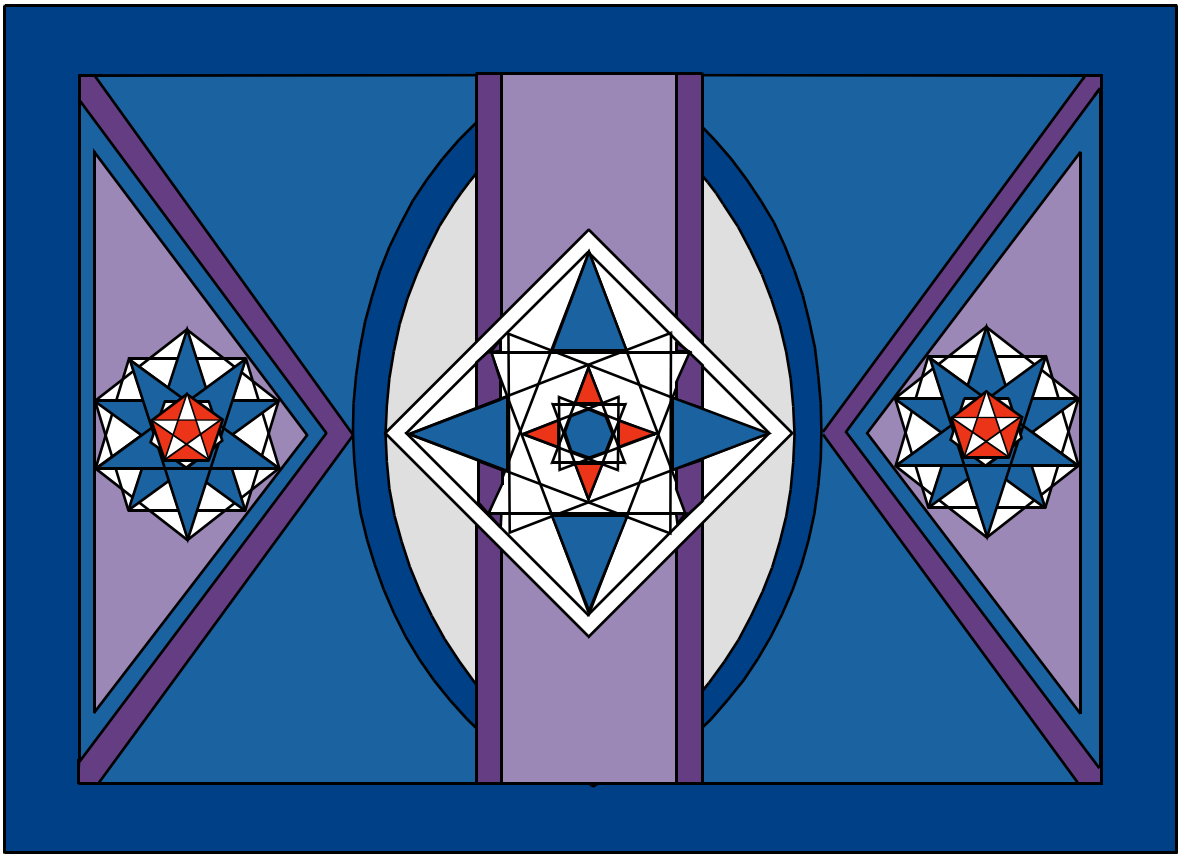

PHYSICS
with
CALCULUS

(Volume I)

CRAIG FLETCHER

P H Y S I C S
w i t h
C A L C U L U S

Volume I -- Classical Mechanics

Third Edition, Fourth Year

by
Craig Fletcher

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TABLE OF CONTENTS

NOTES TO THE STUDENT

Chapter 1
MATHEMATICS REVIEW

Scalars 1
Vectors 1
Graphical Vector Manipulation 3
Algebraic Vector Manipulation--Polar Coordinates 5
Algebraic Vector Manipulation--Cartesian Coordinates 7
Conversion: *Unit Vector* to *Polar Notation* 9
Conversion: *Polar* to *Unit Vector Notation* 11
Dot Product in Polar Notation 12
Dot Product in Unit Vector Notation 13
Dot Product in General 14
Cross Product in Polar Notation 15
Cross Product in Unit Vector Notation 17
Cross Product in General 18
QUESTIONS 23

Chapter 2
DERIVATIVES, DIFFERENTIALS, and DIFFERENTIAL EQUATIONS

Derivatives--Preliminaries 25
Derivatives--In the Beginning 27
Derivative Rules 30
The Chasm Between Mathematicians and Physicists 30
The Chain Rule 31
Partial Derivatives 33
The *Del Operator* 34
The *Gradient* 35
The *Divergence* 36
The *Curl* 38
QUESTIONS ..... --

Chapter 3
INTEGRATION

The Integral in Theory 43
The Area Under a *Velocity versus Time* Graph 43
Differential Time Interval 44
Differential Area Function 44
Example of Integral Summing 47
QUESTIONS ..... --

Chapter 4
ELEMENTARY DEFINITIONS and the KINEMATIC EQUATIONS

Speed 51
Velocity: Magnitude and Direction 52
Acceleration: Magnitude and Direction 56
Sign Significance for Velocity and Acceleration 59
Kinematic Equations 62
Hand-waving Derivations--Where *Kinematics* Come From 64
One-Dimensional Kinematics Problem--In General 69
One-Dimensional Kinematics Problem--Freefall 71
Kinematics in Two Dimensions--Projectile Motion 72
QUESTIONS 80

Chapter 5
NEWTON'S LAWS

Newton's Three Laws 85
Newton's Second Law and Types of Forces 87
Newton's *Second Law*--the APPROACH 93
Example Problems 94
Friction and Freefall 106
Center-Seeking (Centripetal) Forces 110
A Note About *Mass* 122
Fictitious Forces: Centrifugal and Others 124
Determining a Time Dependent Velocity *v(t)* Using N.S.L. 128
QUESTIONS 132

Chapter 6
ENERGY CONSIDERATION

Work 139
Work Due to Variable Forces 143
The *Work/Energy* Theorem 151
Conservative Forces 157
Preamble to the *Gravitational Potential Energy* Function 160
Comments and Problems--*Potential Energy Functions* in General 163
Deriving Potential Energy Function For Known Force Function 166
The Potential Energy Function For An Ideal Spring 171
Deriving Force Function For Known Potential Energy Function 174
Modified Conservation of Energy Theorem--The Bottom Line 175
Energy and Multiple-body systems 185
Power 187
QUESTIONS 189

Chapter 7
MOMENTUM

Momentum and Impulse 195
Center of Mass for Discrete Masses 197
Center of Mass for Continuous Masses 199
Systems of Bodies and Their Collective Motion 204
The Modified *Conservation of Momentum* Equation 205
Collision Examples 209
Confusion Be Gone--What Is Conserved When 220
A Bit of Conceptual Nastiness 222
Exotica--The *Center of Mass* Reference Frame 225
QUESTIONS 230

Chapter 8
ROTATIONAL MOTION--PART I

Preliminary Comments and Basic Definitions 235
Rotational Kinematic Equations 242
A Plug for Rotational Parameters 246
Rotational Inertia (*Moment of Inertia*) 248
Moment of Inertia Expressions for Various Objects 257
QUESTIONS 258

Chapter 9
ROTATIONAL MOTION--PART II

Torque 261
Rigid Body *Equilibrium* Problems 266
Rotational Analog to *Newton's Second Law* 269
Rotation And Translation Acting Together--Newton's Second Law ... 270
Weird But Effective Alternative 276
Energy Considerations and Rotational Motion 281
Energy Considerations--Examples 284
Comment on Test Questions: N.S.L. and ENERGY Considerations . 290
Conservation of Angular Momentum 291
Simultaneous Sliding and Rolling 297
Parting Shot and a Bit of Order 300
QUESTIONS 301

Chapter 10
GRAVITATION

Introduction--A Little History 307
Newton's Gravitational Law (in general) 309
Kepler's Laws 312
Gravitation Inside a Massive Object 317
Potential Energy and Gravitational Fields 321
Potential Energy in a System of Bodies 322
Orbital Motion and Energy Considerations 323
Energy Symmetry 328
QUESTIONS 330

Chapter 11
VIBRATORY MOTION

Vibratory Motion--Basic Concepts 333
The Mathematics of *Simple Harmonic Motion* 336
Angular Frequency (w) 340
Phase Shift (f) 342
Energy in a Vibrating System 349
A Summary Example 349
Another Kind of Oscillatory Motion--The Pendulum 350
QUESTIONS 352

Chapter 12
WAVE MOTION

Characteristics of Waves 355
Mathematics of *Traveling Waves* 360
Resonance in General 361
Resonance in String Systems 362
Resonance in a Steel Bar 369
Resonance of Sound in an Air Column 373
Odds and Ends: Beats and Doppler Effect 376
QUESTIONS 381

CHAPTER REVIEWS

Review Prologue 385
Chapter 1--MATH REVIEW 387
Chapter 2--DIFFERENTIALS AND DERIVATIVES ..... -----
Chapter 3--INTEGRALS ..... -----
Chapter 4--KINEMATICS 390
Chapter 5--NEWTON'S SECOND LAW 393
Chapter 6--ENERGY 397
Chapter 7--MOMENTUM 401
Chapter 8--ROTATIONAL MOTION I 405
Chapter 9--ROTATIONAL MOTION II 408
Chapter 10--GRAVITATION 411
Chapter 11--VIBRATORY MOTION 414
Chapter 12--WAVE MOTION 416

A.P. Preparation
MULTIPLE CHOICE TESTS

Preamble to Multiple Choice Tests 419
Multiple Choice Test I 421
Multiple Choice Test II 427
Multiple Choice Test III 435
Multiple Choice Test IV 443

SOLUTIONS and INDEX

Solutions Prologue 453
Chapter 1--MATH REVIEW 455
Chapter 2--DIFFERENTIALS AND DERIVATIVES ..... -----
Chapter 3--INTEGRALS ..... -----
Chapter 4--KINEMATICS 461
Chapter 5--NEWTON'S SECOND LAW 477
Chapter 6--ENERGY 499
Chapter 7--MOMENTUM 513
Chapter 8--ROTATIONAL MOTION I 527
Chapter 9--ROTATIONAL MOTION II 535
Chapter 10--GRAVITATION 557
Chapter 11--VIBRATORY MOTION 567
Chapter 12--WAVE MOTION 577
Multiple Choice Prologue 587
Multiple Choice Test I 589
Multiple Choice Test II 599
Multiple Choice Test III 617
Multiple Choice Test IV 633
Index 649

Notes to the Student

ABOUT THIS TEXT BOOK:

1.) This text is a *reference source* designed solely for your use while taking advanced physics here at Poly. It is very much like an expanded set of class notes. That is, there is nothing within the text that will not be discussed in class, and there is very little that will be discussed in class that is not covered by the text. Put another way, if you are an *ear-learner* and absorb material by listening to lectures, you may successfully complete this class without ever opening the book (though I wouldn't suggest trying it). If you are an *eye-learner* and learn through reading, this book will do it all for you. In either case, the text is for your use and convenience. Utilize it in any way you wish.

2.) On the assumption that complex ideas are more easily absorbed when presented in chunks, this text has been written in an *outline form*.

3.) There are six to fifteen questions at the end of each chapter, many of which have multiple parts to them. If you can think each problem through to completion *without the use of your book*, you will understand both the *theoretical concepts* and the *mathematical techniques* involved in the chapter. This is what you should be shooting for.

In short, the problems are designed to make you *think about the ideas being presented*. You will not turn them in for a grade. In fact, the *complete answers* to *all the problems* are found at the end of the text.

Note: Though your use of the problems is strictly up to you, I give you fair warning. When you do the problems, I WOULD *STRONGLY SUGGEST* THAT YOU DO NOT LOOK AT A GIVEN PROBLEM, THINK ABOUT IT FOR FIFTEEN SECONDS, THEN GIVE UP AND LOOK AT MY SOLUTION IN THE BACK OF THE BOOK. Doing so simply rides on what *I* know, not on what *you* know, and that defeats the purpose of the problems.

Your best chance for survival on a given test depends upon your ability to look at a problem you have never seen before and have the wherewithal to

untangle it with nothing more than your brains and your wits at hand. If you can't do the chapter-end problems cold, you don't understand the material well enough to pass the next test.

My suggestion, then, is that you treat the chapter-end problems as though they were *test problems*. That is, do them without looking back in your book, asking friends, or above all else, looking at the solutions in the back of the book. You only have so many of these problems available to you. Don't squander them. Getting an answer isn't worth the paper it's written on if you didn't think through the *how to* of the problem totally on your own. Understanding the *how to* is what will help you when test-time comes.

ABOUT THE CLASS:

1.) Grades:

a.) You may expect from 4 to 6 tests per quarter (a test every one-to-two weeks). Tests will make up 75% of your grade.

b.) Labs will make up 25% of your grade. Labs will be held on Thursday or Friday, depending upon your free A/L periods. More will be said about labs during the first lab session.

c.) Formal homework assignments will not be given. You are big people--a year from now you will be roaming the college of your choice completely footloose and fancy-free. It is time for you to begin to develop study habits that will allow you to deal with all that freedom without going glug-glug in the process.

For those of you who feel naked without some kind of assignment guidance, here are my suggestions:

i.) YOU WANT TO LEARN THIS STUFF AS FAST AS YOU POSSIBLY CAN. The tests are going to require you to *think on your feet*. That means you need to give the ideas time to crash around in your head before you have to use them on a test. It also means that if you wait until a day or two before the test to try the chapter-end problems, etc., stuff is going to be swirling around you like a tornado when it comes time to take the test ((this completely ignores the fact that approaching the chapter-end problems in this way will give you no time to come in and ask questions should you find yourself

confused). That is not going to help you deal with the test questions, *none of which you will have ever seen before*.

ii.) On the first night after you have taken a test, *skim* the next chapter of the book (even if you are an ear-learner). This should take no more than fifteen to twenty minutes (by skim I mean look at the headings and pictures, read a few captions, etc.).

iii.) On the second night after having taken a test, *skim* the associated chapter in the *Conceptual Manual*.

iv.) On the third night, though you don't want to be anal about this, read as much of the chapter as you can (if you can do the whole thing in a night or two, great!). Things always make more sense the second time you come in contact with them, so even if the reading makes little sense the first time around, the topics will make more sense than they otherwise might when I present them in class.

v.) At some point in that first three or four days, try the first end-of-chapter question (they are in relatively sequential order). If you cannot figure the question out, LEAVE IT. Allow your mind to chew on it for a while. DO NOT GO TO THE SOLUTIONS, at least not yet. There will still be plenty of time to do that later if need be.

iv.) NOT TRUE! Street wisdom and your gut probably suggest that you don't want to peek too soon, that you don't want to learn something so far ahead of the test that you forget it in the meantime. For this class, THIS IS NOT A GOOD WAY TO THINK.

You will, on occasion, be running into new material the day before the test. If you master the topics as they come, all you'll have to worry about just before the test will be the new topic and a quick review of the material you've already learned. Leave everything until just before the test and you will end up with a mountain of stuff to master all at once. The natural consequence of that is that you will enter the test with everything swirling around you as mentioned above.

In other words, LEARN THE MATERIAL, ESPECIALLY THE MATERIAL AT THE BEGINNING OF A CHAPTER, *AS YOU RUN INTO IT*. You may forget a little between the time you learn it and

the time the test comes, but because you learned it once, just a little bit of review should bring it all back.

Put yet another way, don't put things off *especially* if there is a topic that looks absolutely awful at hand. If you don't master it on the spot, it will look just as awful the night before the test as it did the first time you saw it.

v.) Some time within three days of the test, come in and ask me questions about the material you don't understand.

2.) Assignment deadlines:

a.) Deadline extensions will be generally frowned upon unless unusual, extenuating circumstances dictate leniency in the policy.

b.) Having made the above statement, let me add a human touch to the pronouncement. I realize you will have a lot of things to do over the course of the year--much more than you probably realize considering the added burden of college applications, etc. I have been dealing with seniors for twenty years and know there will be times when things get a bit tight.

Requiring you to get things in on time is not my way of making your life miserable. I am trying to get you to think more about the way you study. You will know *exactly* when each lab is due (we will decide on a fixed day--it will be approximately one week after the completion of a given lab) and tests will be announced with at least *one weekend* between the test and the announcement (usually about a week in advance). If you are being conscientious, you should be able to adjust your study schedule to accommodate those deadlines.

IF YOU HAVE AN UNFORESEEN PROBLEM, DON'T BE BASHFUL ABOUT COMING TO TALK TO ME (I don't bite, at least not hard). I am *not* averse to giving deserving students a break if there is a good reason to do so.

c.) *Legitimate* absence (i.e. for sickness) is automatically grounds for an extension, but God help you if I find your "sickness" was intimately related to your need to write a paper for another class.

3.) A.P. review:

a.) Students characteristically have two major problems in preparing for both the A.P. test and a final exam. They are:

i.) At the end of the year, how does one bring to mind all of the itsy bitsy pieces of information that are relevant and important if one is to understand the physics that has been covered, but that also hasn't been thought about since the start of the second semester or, worse yet, since the beginning of the year?

ii.) Schools whose physics programs are only one year long (versus being two years long) have little time for in-depth classroom discussions of the conceptual consequences of the theories being presented. As such, given the fact that the A.P. folks are now stressing conceptually oriented questions in their multiple choice section (they don't even *allow* the use of calculators in that section any longer), how does one prepare for such questions?

b.) Solution: For review, you will find four *multiple choice tests* with *Solutions* at the end of each volume (there are a total of 125 questions in the mechanics volume and 160 questions in the E&M volume). The problems are mixed between calculator-driven questions designed to help you review processes, and conceptual questions designed to nudge you into thinking along conceptual lines. If you follow the provided *strategy for use* when the time comes, this should be a fairly effective way to remind yourself of all of the good things you've forgotten over the year while additionally allowing you to see how conceptual questions are presented. The Solutions will highlight how you *should* approach each.

4.) Cheating:

a.) The following will be discussed during our first few class periods:
i.) Cheating on labs;
ii.) Cheating on tests;
iii.) Cheating on homework;
iv.) Cheating on make-up tests.

5.) Final exam: There will be a final exam at the end of the first semester AND at the end of the second semester (just before you leave for senior projects). Both will be cumulative.

6.) Dropping the class: You may drop the class *without* teacher approval within the first three weeks of the first quarter. After that, you're all mine.

Parting Shot:

Physics can be a bit of a challenge, but it can also be a lot of fun if you get the knack of it. My job is to make it enjoyable while teaching you the fundamentals of the subject. If I do my job well and you are at all open-minded, the two of us should be able to make the class palatable, enlightening, maybe even a bit amusing while delving into the rigor of it all. This book is the first step. Use it well.

SOME PHYSICAL CONSTANTS

QUANTITY	SYMBOL	VALUE
speed of light in a vacuum	c	3.00×10^{8} m/s
Coulomb's constant	$\frac{1}{4\pi\varepsilon_o}$	9×10^{9} nt·m^2/C^2
permittivity of free space	ε_o	8.85×10^{-12} F/m
permeability of vacuum	μ_o	$4\pi \times 10^{-7}$ H/m
elementary charge unit	e	1.6×10^{-19} C
mass of electron	m_e	9.1×10^{-31} kg
mass of proton	m_p	1.673×10^{-27} kg
mass of neutron	m_n	1.675×10^{-27} kg
Universal Gravitational Constant	G	6.67×10^{-11} m^3/kg·s^2
Planck's constant	h	6.63×10^{-34} J·s

Student's Name:_____

Advisor:_____

Chapter 1

MATHEMATICS REVIEW

A.) Scalars:

1.) A *scalar* is a variable that has *magnitude* (i.e., size) but does not have *direction* associated with it.

a.) Example 1: Temperature measurements are not recorded as "25^o C *upward*." Why? Because temperature is a scalar, it has a magnitude but it does *not* have direction.

b.) Other examples of scalars: length, time, mass, and speed.

2.) *Scalars* add and subtract in the same way dollars and cents add and subtract.

a.) Example 2: If we know the temperatures $T_1 = 115^o$ and $T_2 = 98^o$, then $T_2 - T_1 = -17^o$.

B.) Vectors:

1.) A *vector* is a variable that has both *magnitude and direction* associated with it.

a.) Example: Force is a vector. Push a box up an incline and you will find that the *direction* of the force is as important as the *magnitude* of force. Specifically, the force required to move the box up the incline in Case 1 (see Figure 1.1) will be considerably greater than the force required in Case 2.

Both *magnitude* and *direction* are important with vectors.

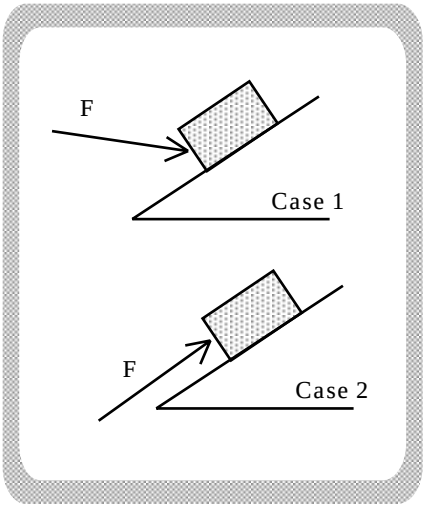


FIGURE 1.1

2.) Because *direction* makes a difference, vectors do *not* add or subtract like scalars.

a.) Example: A three newton force added to a four newton force will not necessarily sum to an equivalent force of seven newtons. Figure 1.2 depicts a situation in which the vector sum of those two forces generates an equivalent force of five newtons at an angle of 37^{o} with the horizontal.

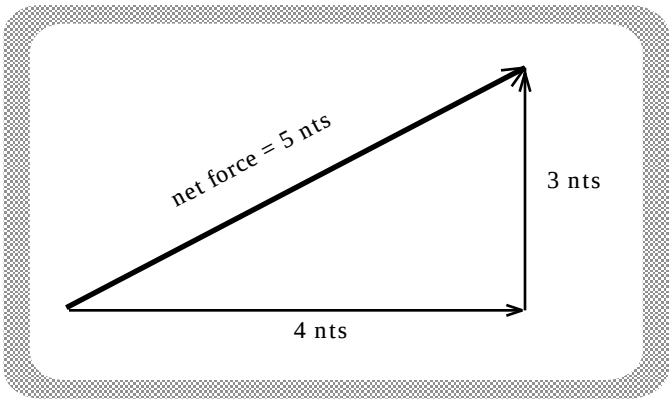


FIGURE 1.2

3.) A vector can be algebraically represented either by a *letter* with an arrow over it ($\vec{A}$) or by a letter in bold-face type (**A**). The former is most often used in classroom lectures and when doing problems longhand. The latter is used in texts as it is easier to represent on a computer. This book will use bold-face letters to represent vectors.

4.) There will be times when a vector's *magnitude* is all that is important. Notationally, the magnitude of the vector is represented as $|\mathbf{A}|$. On some occasions, this relatively formal approach is inconvenient. An alternate approach uses an unadorned letter like "A" (without the quotes) or a subscripted letter such as "A_x."

5.) The *graphical* representation of a vector is drawn as an arrow. The arrow's orientation depicts the vector's *direction* and the arrow's length is scaled to reflect the vector's *magnitude.*

a.) Example: Figure 1.3 depicts a force vector ***F*** whose magnitude is 36 newtons and whose orientation is at 45^{o} with the horizontal. The scaling factor in the sketch is *one-half inch per 12 newtons.*

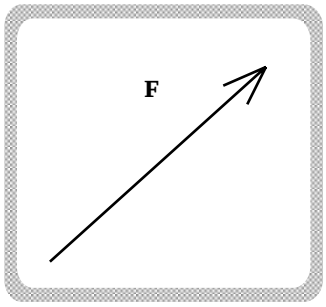


FIGURE 1.3

6.) Manipulating vectors, even for relatively mundane operations like addition or subtraction, requires special approaches. The two most commonly used are *graphical* manipulation and *algebraic* manipulation.

C.) Graphical Vector Manipulation:

1.) Consider the two velocity vectors ***A*** and ***B*** shown in Figure 1.4. To graphically add these, proceed as follows:

a.) Reproduce either vector (we'll use ***B***), drawing it to scale and keeping its orientation exactly as presented in the original sketch (Figure 1.5).

The scale used will be 50 *meters per second* per inch.

b.) Reproduce the second vector (vector ***A***) so that its *tail* is positioned at the *head* of the first vector (Figure 1.6). Again, make the drawing to scale and keep the vector's orientation intact.

c.) Draw the *resultant* vector ***C*=*A*+*B*** (Figure 1.7). This new vector will begin where the sketch began at the tail of ***B*** and end where the sketch ended at the head of ***A***.

d.) To determine the *magnitude* of the resultant vector ***C***, use a ruler or centimeter stick to measure ***C's*** length, then multiply by the scaling factor. In this case we get approximately 87 newtons.

e.) To determine ***C's*** direction *relative to the horizontal*, a protractor yields approximately 20 degrees.

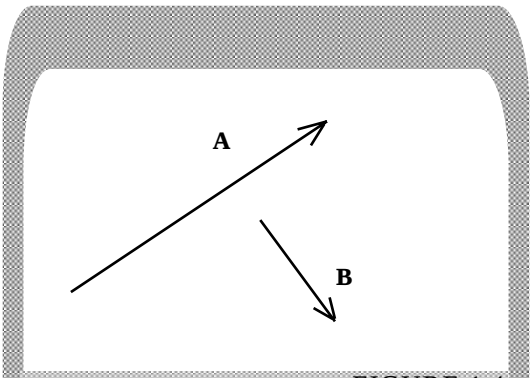


FIGURE 1.4

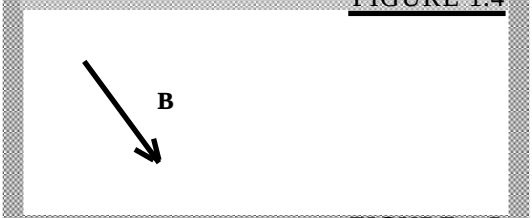


FIGURE 1.5

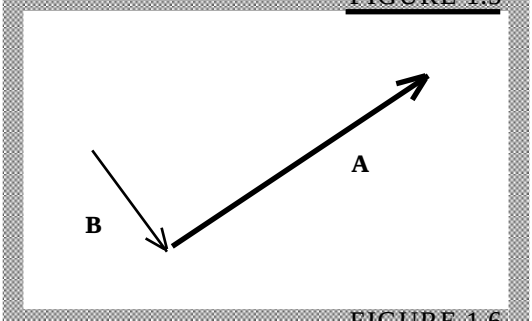


FIGURE 1.6

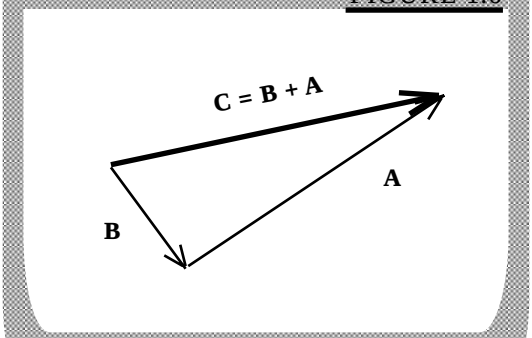


FIGURE 1.7

Note: Adding more than two vectors follows a similar pattern with the tail of each successive vector being placed at the head of the previous one until all the vectors are coupled (see Figure 1.8). Everything must be drawn to scale with relative directions kept intact. The *resultant* will be a vector that starts where you started and ends where you ended.

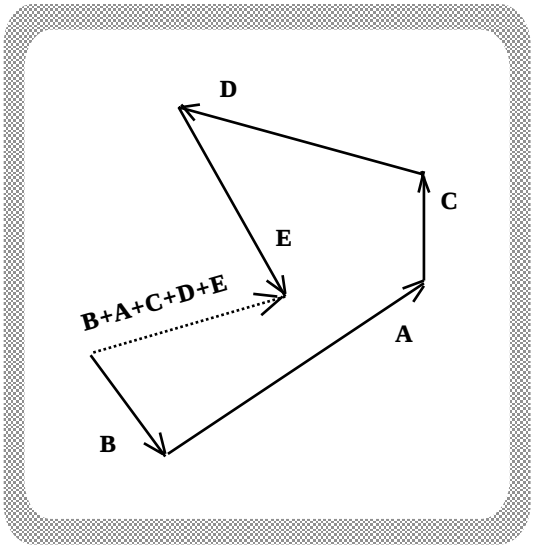


FIGURE 1.8

2.) Vectors can be multiplied by scalars as shown in Figures 1.9a through 1.9e below. Figure 1.9a depicts vector ***A***, then presents vectors *(1/2)**A**, 2**A**, -**A**,* and *-2**A*** respectively.

Note: A vector multiplied by a positive *scalar* either increases or decreases the *magnitude* of the original vector but does not change its orientation. Multiplying by *-1* effectively flips the vector so that its orientation is opposite that of the original vector, and multiplying by a *negative scalar* changes both the *magnitude* and the *direction*.

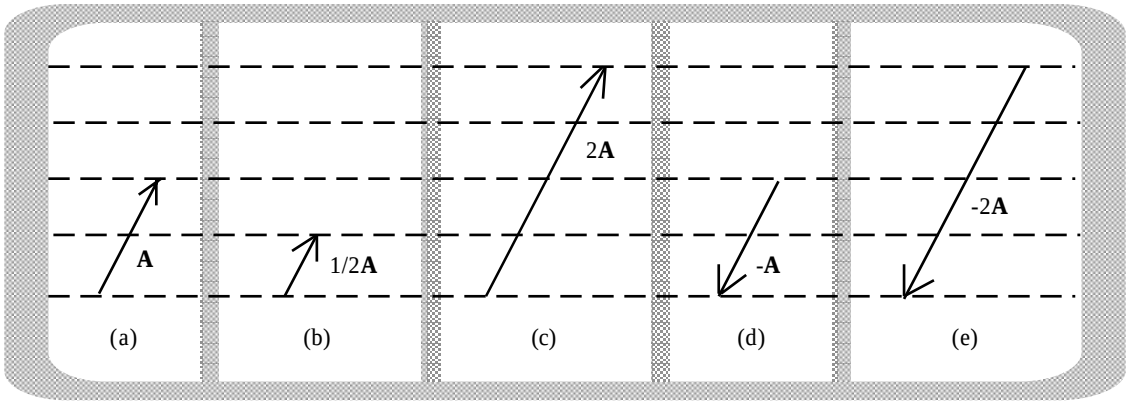


FIGURE 1.9

3.) Although it has its place, graphical vector manipulation is generally awkward. From a mathematical standpoint, it is usually more convenient to deal with vectors algebraically in the context of coordinate axes. We will make use of two such grid systems: one that uses Cartesian coordinates and one that uses polar coordinates.

D.) Algebraic Vector Manipulation-- Polar Notation:

1.) Polar notation defines a vector by designating the vector's *magnitude* $|\mathbf{A}|$ and *angle* θ relative to the + *x axis* (see Figure 1.10). Using that notation, the vector is written $\boldsymbol{A} = |A| \angle \theta$.

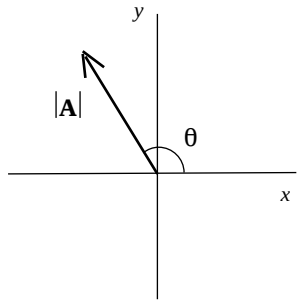


FIGURE 1.10

Note: Some math texts use an ordered paired vector notation like ($|\mathbf{A}|$, θ) to present polar information. As physicists do not usually use this notation, we will not use it in this book.

a.) Example 1: A force vector ***F*** with a *magnitude* of 12 newtons oriented at 210^o with the *+x axis* would be characterized as $\boldsymbol{F} = 12 \angle 210^o$ (Figure 1.11).

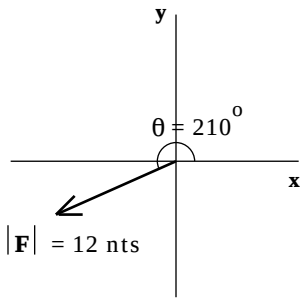


FIGURE 1.11

b.) Example 2: A force vector ***F*** with a magnitude of 12 newtons oriented along the *-x axis* would be characterized as $\boldsymbol{F} = 12 \angle 180^o$ (Figure 1.12).

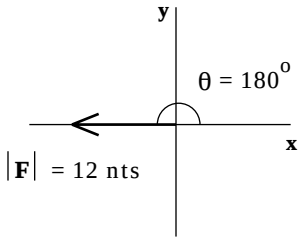


FIGURE 1.12

2.) Students should be able to characterize a graphically presented vector in *polar notation*.

a.) Example: From the graph in Figure 1.13, characterize vectors ***C*** and ***D*** in *polar notation* (Answer: $\boldsymbol{C} = 2 \angle 30^o$, $\boldsymbol{D} = 4 \angle -50^o$).

Note: Negative angles are measured from the *+x axis* clockwise.

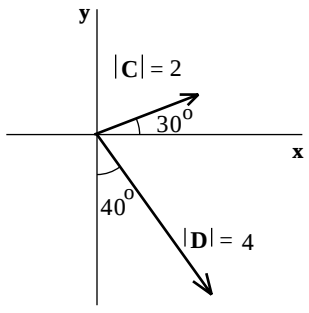


FIGURE 1.13

3.) Students should be able to graph a vector characterized in *polar notation*.

a.) Example: Graph $\boldsymbol{A} = 4 \angle 150^o$ and $\boldsymbol{B} = 6 \angle -30^o$ (the solution is shown in Figure 1.14 on next page).

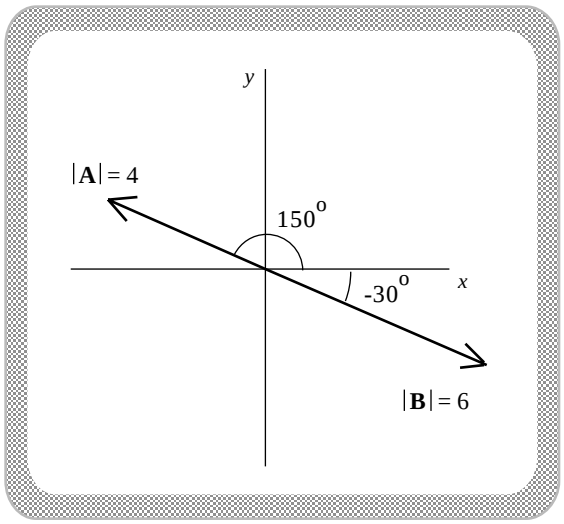


FIGURE 1.14

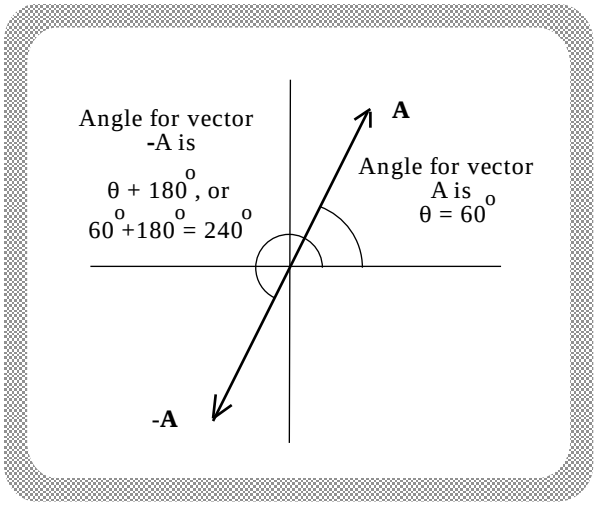


FIGURE 1.15

4.) Multiplying a vector by a *positive scalar* characterized in *polar notation* changes the *magnitude* of the vector but not the *direction* (i.e., the angle).

a.) Example: If $\mathbf{A} = 12 \angle 210^o$, then $3\mathbf{A} = 36 \angle 210^o$.

5.) Multiplying a vector by (-1) does not change the vector's *magnitude*, but it does reverse its *direction* (see Figure 1.15). As direction in *polar notation* is denoted by the vector's angle, a reversal of direction effectively adds 180^o to the angle.

a.) Example 1: If $\mathbf{D} = 4 \angle 110^o$, then $-\mathbf{D} = 4 \angle 290^o$.

Note: $-\mathbf{D}$ is *not* $-4 \angle 110^o$.

b.) Example 2: If $\mathbf{D} = 4 \angle 110^o$, then $-(1/2)\mathbf{D} = 2 \angle 290^o$.

Note: Angles should never be greater than 360^o. If $\boldsymbol{A} = 12 \angle 220^o$, $-\boldsymbol{A} = 12 \angle (220^o+180^o) = 12 \angle 400^o = 12 \angle 40^o$.

E.) Algebraic Manipulation in Cartesian Coordinates:

1.) An effective if not obscure system called *unit vector notation* is used to denote vectors presented within a Cartesian (x-y-z) coordinate system. Within that system, the vector shown in Figure 1.16 would be characterized as $\boldsymbol{A} = 4\boldsymbol{i} + 3\boldsymbol{j}$.

As this notation will be new to many students, the following will hopefully explain the rationale behind it:

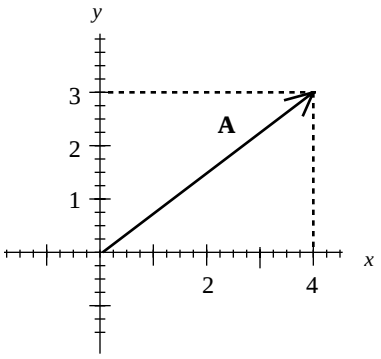


FIGURE 1.16

a.) Consider Figure 1.17. Graphical addition suggests that the vector ***A*** is equal to the sum of "a vector in the *+x direction* whose magnitude is 4" and "a vector in the *+y direction* whose magnitude is 3."

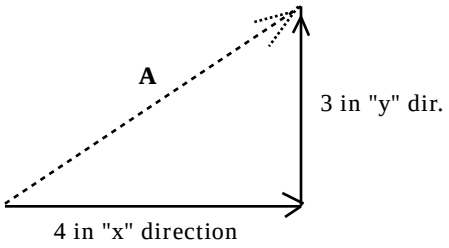


FIGURE 1.17

b.) We now define a special vector $\hat{\mathbf{i}}$ whose magnitude is always *one* and whose direction is always in the *+x direction* (Figure 1.18). Such a vector is called a *unit vector* in the *+x direction*.

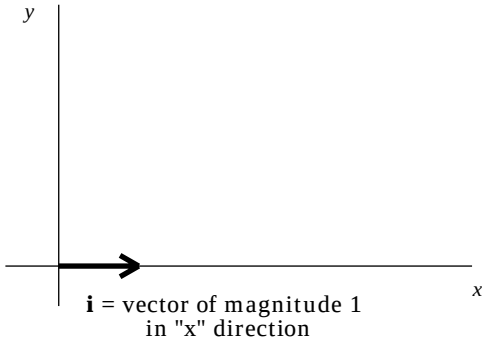


FIGURE 1.18

Note: Physics texts that use arrowheads over vector quantities replace those arrowheads with a *hat* when referring to *unit vectors*. To be technically complete, therefore, a *hat* has been included over the "**i**" term shown above in *Part b*. Although you must use a *hat* whenever writing unit vectors out longhand (you will notice that I always use a *hat* in class), this book will assume that all boldface **i**, **j**, and **k** vectors are UNIT VECTORS and will *not* subsequently include the hat in the text.

Put another way, when you see a boldface **j** in this text, know that it is being used to denote a *unit vector* in the *y direction* even though it has no hat above it.

c.) With ***i*** as defined above, notice that ***4i*** is a vector "whose magnitude is 4 and whose direction is along the *+x axis*" (Figure 1.19). This is half of the vector sum needed to carry out the vector addition required for the production of vector ***A*** as denoted in *Part a* above.

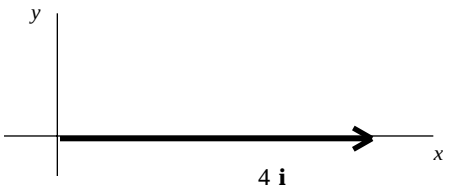


FIGURE 1.19

d.) Now define a *unit vector* directed along the *+y axis*; call it **j** (Figure 1.20).

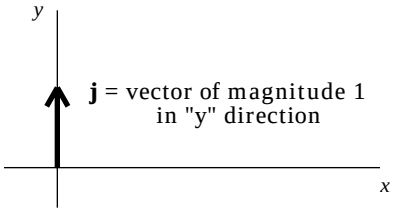


FIGURE 1.20

e.) Notice that ***3j*** is a vector "whose magnitude is 3 and whose direction is along the *+y axis*" (Figure 1.21). This is the other half of the vector sum needed to carry out the addition required for the production of vector ***A*** as denoted in *Part a*.

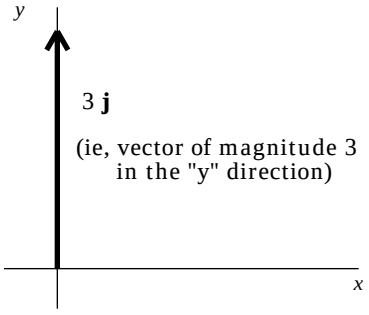


FIGURE 1.21

f.) Having defined the idea of a unit vector, the unit vector characterization of vector ***A*** is ***A = 4i + 3j*** (Figure 1.22).

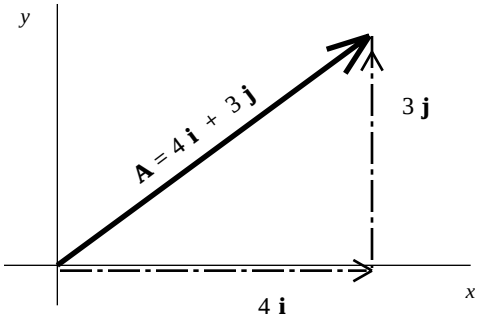


FIGURE 1.22

Note 1: For three dimensional situations, the *+z direction unit vector* is defined as **k**.

Note 2: Consider the vector ***A = -4k***. Technically, the *direction* should be associated with the *unit vector* ***k***. That is, the formally correct way of characterizing the vector would be as ***A = 4 (-k)***. Unfortunately, embedded signs can be overlooked, so it is

acceptable to put the negative sign in front of the expression as first depicted.

Note 3: In the vector $\boldsymbol{A} = -4\boldsymbol{i} + 3\boldsymbol{j}$, the "-4" part of the expression is called "the *x component* of $\boldsymbol{A}$," or A_x, and the "+3" term is called "the *y component* of $\boldsymbol{A}$," or A_y. *Components* are not really *magnitudes*--they can be negative.

2.) Students should be able to characterize a graphically presented vector, writing it out in *unit vector notation*.

a.) Example: From the graph in Figure 1.23, characterize vectors $\boldsymbol{J}$ and $\boldsymbol{K}$ in *unit vector notation* (Answer: $\boldsymbol{J} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{K} = 2\boldsymbol{i} - 5\boldsymbol{j}$).

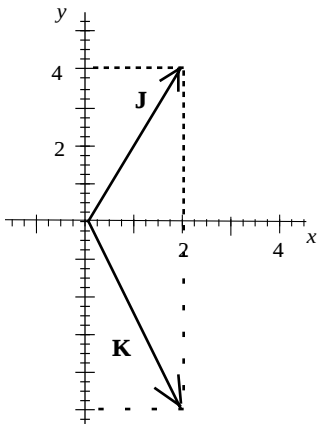


FIGURE 1.23

3.) Students should be able to graph a vector characterized in *unit vector notation*.

a.) Example: Graph vectors $\boldsymbol{G} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{H} = -3\boldsymbol{i} - 2\boldsymbol{j}$ (solution shown in Figure 1.24).

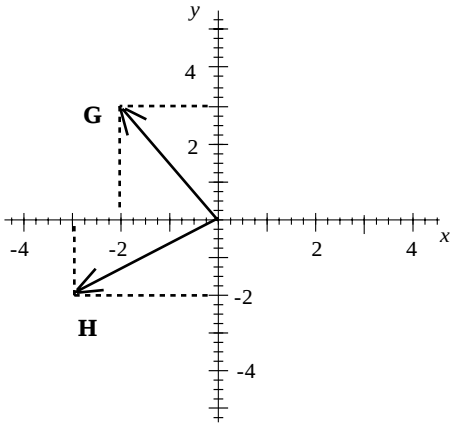


FIGURE 1.24

F.) Conversion: *Unit Vector* to *Polar Notation*:

1.) Consider the known vector $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j}$ shown in Figure 1.25. To characterize it in the *polar notation*, $\boldsymbol{A} = |\boldsymbol{A}|\angle\theta$.

a.) To determine $|\mathbf{A}|$: the right triangle shown in Figure 1.25 coupled with the Pythagorean relationship yields a vector magnitude:

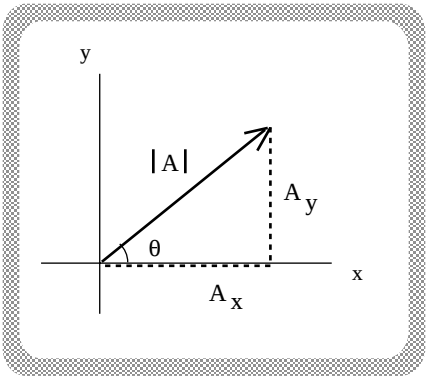


FIGURE 1.25

$$|\mathbf{A}| = [A_x^2 + A_y^2]^{1/2}.$$

b.) To determine θ: The tangent of θ is defined as the ratio of the side opposite θ (i.e., A_y) divided by the side adjacent (A_x), or *tan* $\theta = A_y/A_x$. This implies that:

$$\theta = \tan^{-1}(A_y/A_x).$$

Note: This is expressed as θ, *the angle whose tangent is* (A_y/A_x).

c.) Example: The conversion of the velocity vector ***A*** *= (-4**i** +3**j**)* m/s to *polar notation* (see Figure 1.26):

$$\mathbf{A} = |\mathbf{A}| \angle \theta$$
$$= [A_x^2 + A_y^2]^{1/2} \angle [\tan^{-1}(A_y/A_x)]$$
$$= [(-4)^2 + (3)^2]^{1/2} \angle [\tan^{-1}(3/(-4))]$$
$$= 5 \text{ m/s} \angle -36.9^{o}.$$

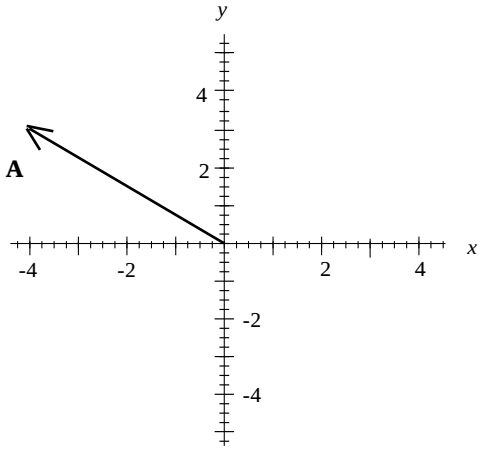


FIGURE 1.26

Note: There is SOMETHING WRONG here. ***A*** is a *second quadrant vector*, while the angle given by your calculator implies a fourth quadrant angle.

The problem lies in your calculator's inability to tell the difference between $tan^{-1}[3/(-4)]$--a second quadrant vector--and $tan^{-1}[(-3)/4]$--a fourth quadrant vector. As a consequence, all calculators assume they are dealing with *fourth quadrant values* whenever they are fed tangent arguments that are negative. In such cases, the correct *second quadrant angle* can be generated

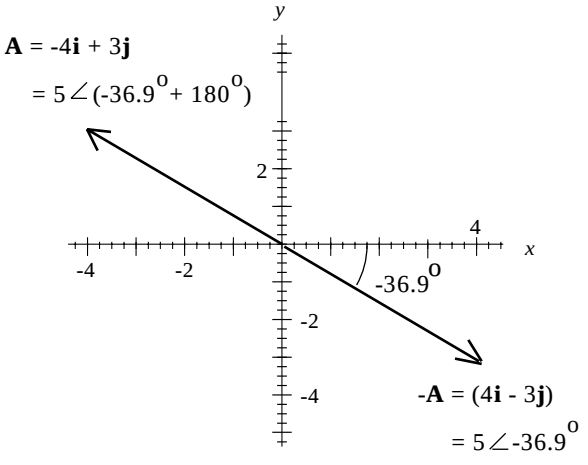


FIGURE 1.27

by adding 180^o to the calculator's value (see Figure 1.27 on previous page).

This means $\boldsymbol{A} = 5\ m/s\ \angle\ (-36.9^o+180^o) = 5\ m/s\ \angle\ 143.1^o$.

Note: A similar problem exists between first and third quadrant vectors. That is, *third quadrant* vectors have a tangent ratio of $(-A_y)/(-A_x)$, which is positive just as are *first quadrant* angles. All calculators assume that positive tangents are *first quadrant vectors*. Again, 180^o must be added to the calculator-generated angle if that is not the case.

G.) Conversion: *Polar* to *Unit Vector Notation*:

1.) There are two ways to do this. One is by memorizing a formula that always works; the other is by using your head (ooh, scary). As you'll undoubtedly forget the memorized equation sooner or later, understanding the *seat of your pants* approach is important. I'll outline both.

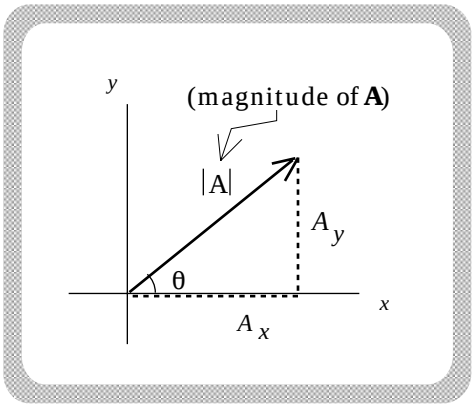


FIGURE 1.28

2.) Consider the known vector $\boldsymbol{A} = |\boldsymbol{A}|\angle\theta$ shown in Figure 1.28. To characterize it formally (i.e. in memorizable form) in *unit vector notation*:

a.) To determine A_x: *Cosine* $\angle\theta_1$ is defined as the ratio of the *side adjacent to* θ (i.e., A_x) divided by the *hypotenuse* ($|\boldsymbol{A}|$), or $\cos\ \theta = (A_x)/(|\boldsymbol{A}|)$. This implies that $A_x = |\mathbf{A}| \cos\ \theta$.

b.) Similar reasoning produces A_y as $A_y = |\mathbf{A}| \sin\ \theta$.

c.) These expressions will work for *any* angle (hence, if memorized, will never fail you . . . unless your memory fails you). As an example, the conversion of the velocity vector $\boldsymbol{A} = 7\ m/s\ \angle 130^o$ into *unit vector notation* (Figure 1.29) becomes:

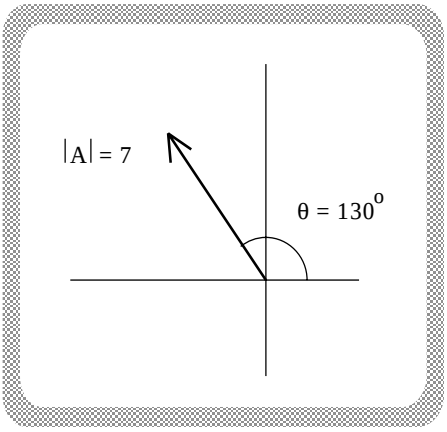


FIGURE 1.29

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$$
$$= (|\mathbf{A}| \cos\theta)\,\mathbf{i} + (|\mathbf{A}| \sin\theta)\,\mathbf{j}$$
$$= [(7 \text{ m/s}) \cos 130^o]\,\mathbf{i} + [(7 \text{ m/s}) \sin 130^o]\,\mathbf{j}$$
$$= (-4.5\,\mathbf{i} + 5.4\,\mathbf{j}) \text{ m/s}.$$

Note: The *only* nice thing about the memorized approach is that it always gives each component's correct sign.

3.) Assuming your memory is as miserable as mine, the more intelligent way to do the conversion outlined above is by *the seat of your pants* approach. That is:

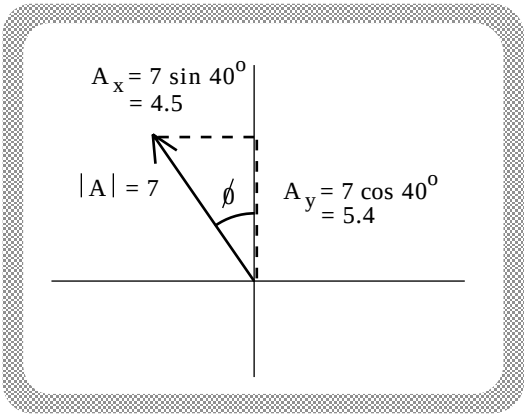


FIGURE 1.30

a.) Create a convenient *right triangle* like the one shown in Figure 1.30.

b.) Use the appropriate trig functions to determine the vector's components.

c.) Once you have the magnitude of the components, add signs and you're done (i.e., write the *x component* as negative). Doing so yields $\boldsymbol{A} = (-4.5\,\boldsymbol{i} + 5.4\,\boldsymbol{j})\ m/s$.

H.) *Dot Product* In Polar Notation:

1.) In polar notation, consider the two vectors: $\boldsymbol{A} = |\boldsymbol{A}| \angle\theta_1$ and $\boldsymbol{B} = |\boldsymbol{B}| \angle\theta_2$ (see Figure 1.31). The *dot product* between $\boldsymbol{A}$ and $\boldsymbol{B}$ produces a scalar quantity. The *magnitude* of the *scalar product* is defined as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\phi,$$

where ϕ is the *net angle between the line of the two vectors*.

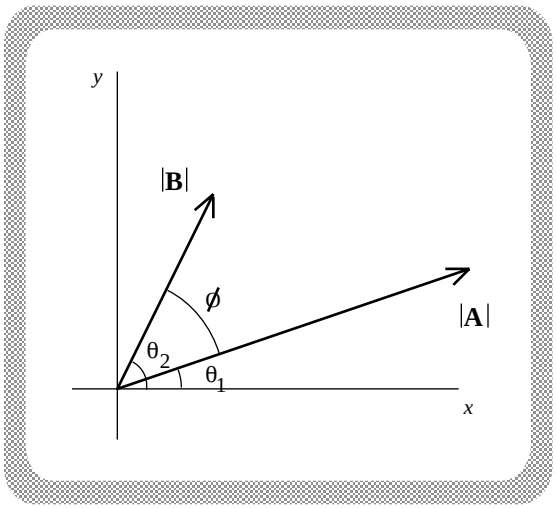


FIGURE 1.31

2.) Example: Let $\boldsymbol{A} = 5\ nt \angle 30^o$ and $\boldsymbol{B} = 12\ m \angle 180^o$ (see Figure 1.32 on next page). What is $\boldsymbol{A} \cdot \boldsymbol{B}$?

a.) Following the definition of the *dot product*:

$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta,$
$= (5 \text{ nt})(12 \text{ m}) \cos(150^o)$
$= -52 \text{ nt}\cdot\text{m}.$

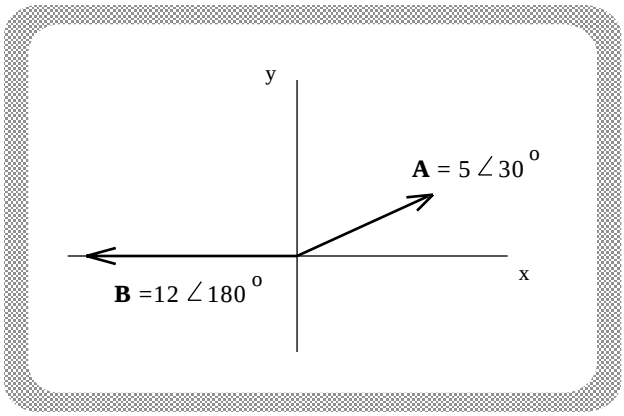


FIGURE 1.32

I.) *Dot Product* In Unit Vector Notation:

1.) In unit vector notation, consider the vectors: $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ and $\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$. The *dot product* between $\boldsymbol{A}$ and $\boldsymbol{B}$ produces a <u>scalar</u> quantity that is mathematically equal to:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

Note that this is easily derived:

a.) Beginning with $\boldsymbol{A} \cdot \boldsymbol{B} = (A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}) \cdot (B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k})$.

b.) Calling upon the distributive nature of *dot products*, we get a whole string of mini *dot products*:

$$\mathbf{A} \cdot \mathbf{B} = [(A_x\mathbf{i}) \cdot (B_x\mathbf{i})] + [(A_x\mathbf{i}) \cdot (B_y\mathbf{j})] + \ldots$$

c.) As the angle between two vectors in the $\boldsymbol{i}$ direction is zero degrees, the first mini *dot product* shown above is equal to $(A_x)(B_x)(\cos 0^o) = A_xB_x$. The implication is that *like-termed* products will *not* be zero (assuming neither A_x nor B_x are zero).

d.) As the angle between the $\boldsymbol{i}$ direction and the $\boldsymbol{j}$ direction is 90^o, the second mini *dot product* is equal to $(A_x)(B_y)(\cos 90^o) = 0$. The implication is that all *cross-termed* products *will* be zero.

e.) <u>Bottom line</u>: In *unit vector notation*, $\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$.

2.) Example: Let $\boldsymbol{A} = (3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k})$ *newtons* and $\boldsymbol{B} = (2\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k})$ *meters*. What is $\boldsymbol{A} \cdot \boldsymbol{B}$? Following the derived expression for the *dot product* expressed in unit vector notation:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= (3 \text{ nt})(2 \text{ m}) + (-4 \text{ nt})(7 \text{ m}) + (-5 \text{ nt})(3 \text{ m}) \\ &= -37 \text{ nt}\cdot\text{m}. \end{aligned}$$

J.) ***Dot Product*** **In General:**

1.) To understand the physical significance of the *dot product*, consider Figure 1.33. In it, the vector $\boldsymbol{B}$ has been split into two components--one <u>parallel</u> to the *line of* $\boldsymbol{A}$ and one <u>perpendicular</u> to the *line of* $\boldsymbol{A}$. Notice that the component parallel to the line of $\boldsymbol{A}$ has a magnitude of $|\boldsymbol{A}||\boldsymbol{B}| \cos \phi$.

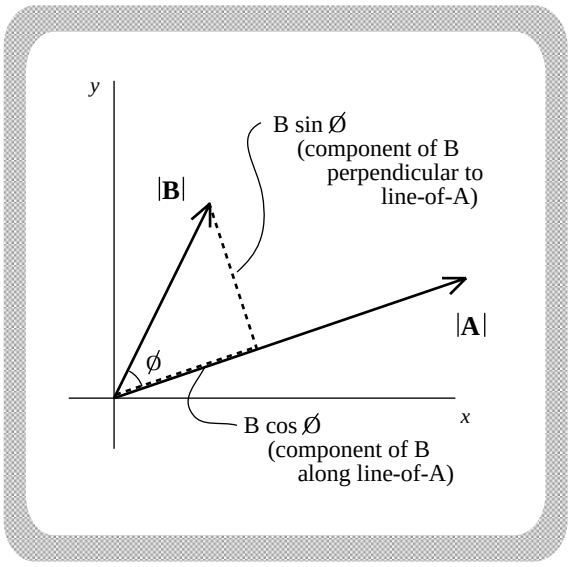


FIGURE 1.33

a.) Conclusion: When the *dot product* is taken between two vectors, it generates a quantity equal to the product of:

i.) The *magnitude* of one vector ($|\boldsymbol{A}|$ in this case) and,

ii.) The *magnitude* of the second vector's component that runs *parallel* to the first vector (i.e., $|\boldsymbol{B}| \cos \phi$).

Note 1: It doesn't matter whether you take $(|\boldsymbol{A}|)(|\boldsymbol{B}| \cos \phi)$ or $(|\boldsymbol{A}| \cos \phi)(|\boldsymbol{B}|)$. Both will work.

Note 2: For the sake of visualization, if $\boldsymbol{A} = 5 \text{ nt} \angle 30^o$ and $\boldsymbol{B} = 12 \text{ m} \angle 180^o$, the component of **A** along the *line of* $\boldsymbol{B}$ is shown in Figure 1.34 and the

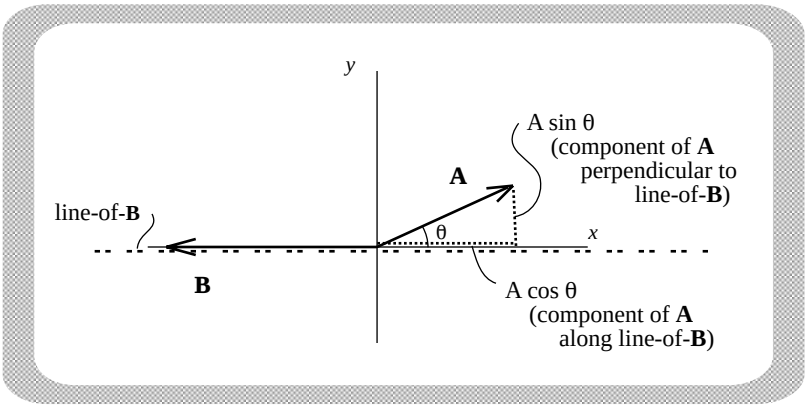


FIGURE 1.34

component of ***B*** along the *line of* ***A*** is shown in Figure 1.35.

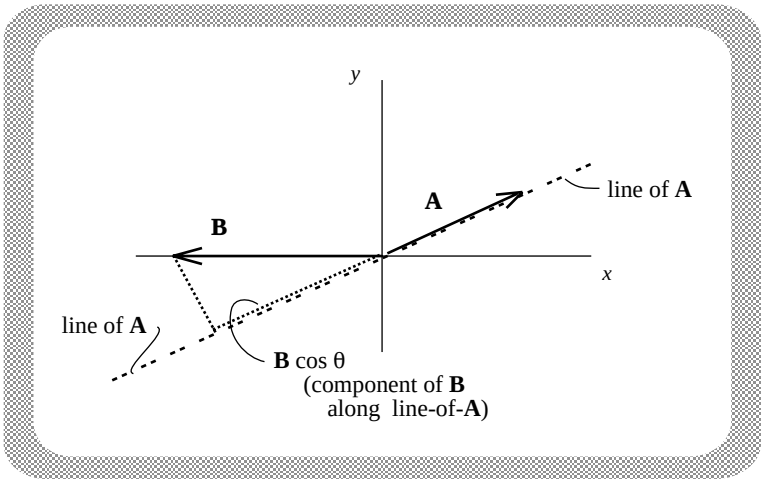


FIGURE 1.35

2.) Example of a situation in which the *dot product* comes in handy: Consider Fig-ure 1.36.

a.) A force ***F*** is applied to a moving object as it traverses over a frictionless surface for a distance ***d***. Notice that the force will increase the object's speed. Notice also that the force component that makes the object increase its speed is *the component of* ***F*** *along the line of* ***d***, or $|\mathbf{F}|\cos\theta$.

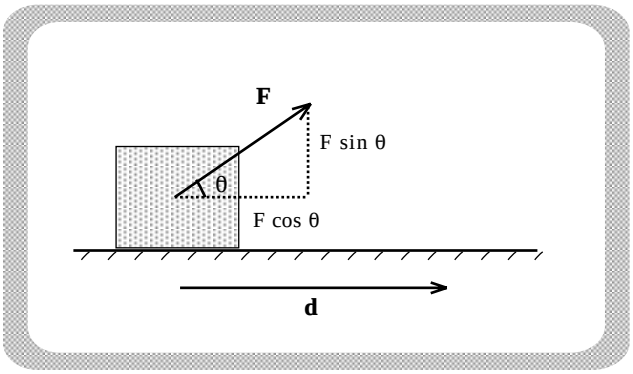


FIGURE 1.36

b.) As the amount of speed the object picks up is dependent only upon the distance over which the force acts (i.e., the *magnitude* of the displace-ment vector ***d***) and the component of ***F*** along the line of ***d*** (i.e., $|\boldsymbol{F}|\ cos\ \theta$), the product of those two quantities is deemed important enough to be given a special name--*WORK*.

c.) In short, the *work* done by the force ***F*** acting on an object whose dis-placement is defined by a distance ***d*** is mathematically defined as $W_F = \boldsymbol{F} \cdot \boldsymbol{d}$.

K.) *Cross Product* In Polar Notation:

1.) Consider two vectors $\boldsymbol{A} = |\boldsymbol{A}| \angle\theta_1$ and $\boldsymbol{B} = |\boldsymbol{B}| \angle\theta_2$. The *cross product* between ***A*** and ***B*** produces a <u>vector</u> whose *magnitude* is mathematically defined as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\ \phi,$$

where ϕ is the angle between the two vectors.

2.) Example: Let $\boldsymbol{A} = 5$ *newtons* $\angle\ 30^o$ and $\boldsymbol{B} = 12$ *meters* $\angle\ 180^o$ (look back at Figure 1.32). What is $|\boldsymbol{B}x\boldsymbol{A}|$?

a.) Following the definition of the *magnitude* of the *cross product*:

$$\begin{aligned} |\mathbf{B}\text{x}\mathbf{A}| &= |\mathbf{B}|\ |\mathbf{A}|\ \sin\phi, \\ &= (12\text{ m})(5\text{ nt})\ \sin(150^o) \\ &= 30\text{ nt}\cdot\text{m}. \end{aligned}$$

b.) When dealing with vectors in *polar notation*, the *direction* of a cross product vector can be found using the right-hand rule. This rule is outlined below:

i.) The line of both vectors should be extended until they intersect (they already intersect in the problem we are examining);

ii.) With the wrist of the right hand placed at the intersection of the two vectors, the straightened fingers of the open right hand should be positioned so that they are parallel to the direction of the first vector (vector $\boldsymbol{B}$ in this case);

iii.) The fingers of the right hand should then be curled (waved) in the direction of the second vector (vector $\boldsymbol{A}$ in this case). *Note that you may have to flip your hand over to do this.*

iv.) If the thumb of the right hand is held out at a right angle to the fingers, the *direction* of the thumb will point in the direction of the cross product vector.

Note: Both vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are in the plane of the paper, whereas the direction of the right thumb alluded to above will be *out of* or *into* that plane. As peculiar as this may seem now, the direction of a *cross product* is always perpendicular to the plane defined by the vectors being crossed.

c.) In the problem above, your thumb should end up pointing downward into the page. Assuming the $+\boldsymbol{i}$ *direction* is to the right and the $+\boldsymbol{j}$ *direction* is upward toward the top of the page, the cross product's direction will be in the *negative z* direction characterized by a $-\boldsymbol{k}$ in unit vector notation.

Note: Assuming we are not dealing with an angle of 180^o, there are two angles between any given vectors--one less than 180^o and one more than 180^o. The "wave" should always be through the angle *less than* 180^o.

d.) Putting everything together, we get $\boldsymbol{B}$x$\boldsymbol{A}$ = *30 nt·m (-**k**)*. Or, if you do not like having the negative sign embedded in the middle of an expression, the cross product could be written $\boldsymbol{B}$x$\boldsymbol{A}$ = *-30 nt·m (**k**)*.

Note 1: There will be times when the two vectors being crossed will not have a common starting point. When that occurs, it is very important that you extend the line of the two vectors until they intersect before trying to execute the right-hand rule.

Note 2: An alternate approach to determining *cross product* directions: Position your open right hand so that your thumb is in the direction of the first vector (***B*** in this case) and your straightened fingers are in the direction of the second vector (you may have to flip your hand over to accomplish this). The direction the palm of your right hand faces (i.e., along a vector coming out of the palm) will be the direction of the cross product. Try it for the above situation (Figure 1.37) and you will find that your palm faces downward.

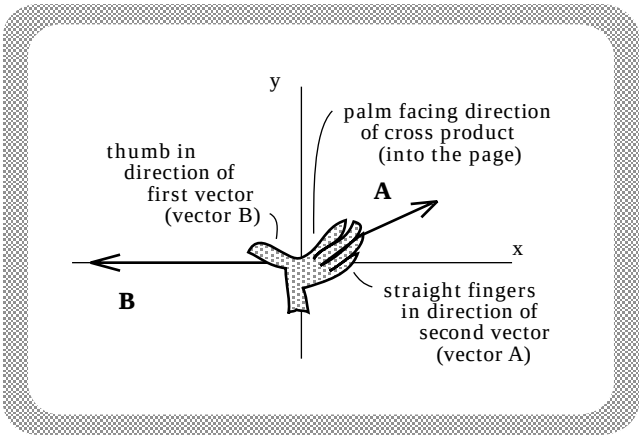


FIGURE 1.37

Note 3: If you are wondering why anyone would want an operation that takes two *x-y plane* vectors and produces a third vector whose direction is *perpendicular* to the *x-y plane*, an example is coming in the section CROSS PRODUCT IN GENERAL.

L.) *Cross Product* In Unit Vector Notation:

1.) In unit vector notation, consider the vectors $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ and $\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$. The *cross product* between ***B*** and ***A*** produces a vector the *magnitude* and *direction* of which can be determined by evaluating the following matrix:

$$\mathbf{B} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{i}\,[(B_y)(A_z) - (B_z)(A_y)] + \mathbf{j}\,[(B_z)(A_x) - (B_x)(A_z)] + \mathbf{k}\,[(B_x)(A_y) - (B_y)(A_x)].$$

Note: Because there are unit vectors directly placed within the matrix, its evaluation will automatically give you both the magnitude *and direction* of the cross product. You do not have to mess with the right-hand rule when evaluating cross products in u.v.n.

2.) To begin with, memorizing the end-result of the *cross product* matrix is a bit of a waste of time. It is much better to simply learn how to evaluate such a matrix (you will need to know how to do this later when we get to *circuit analysis*). The obvious question is, *how do you do that?*

3.) Consider the following example. Let $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{B} = -4\boldsymbol{i} - 5\boldsymbol{j} - 6\boldsymbol{k}$. What is $\boldsymbol{B} \times \boldsymbol{A}$?

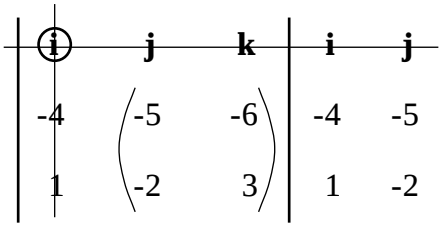
a.) The matrix requires that you array the unit vectors across the top row as shown. The first vector ($\boldsymbol{B}$ in this case) in the cross product has its *components* (signs included) placed in the second row of the matrix, and the second vector has its components placed in the third row. So:

$$\mathbf{B} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -5 & -6 \\ 1 & -2 & 3 \end{vmatrix}$$

b.) There are two ways to proceed from here. I prefer the approach that is, in my opinion, the simplest. To follow that technique, the first two columns must be reproduced to the right of the matrix as shown below (the reason for doing this will become evident shortly).

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ -4 & -5 & -6 & -4 & -5 \\ 1 & -2 & 3 & 1 & -2 \end{vmatrix}$$

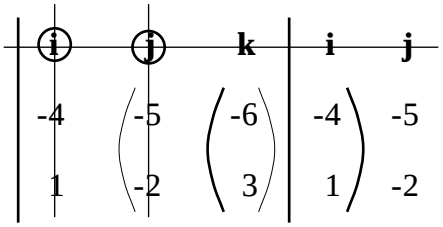
c.) Once set up, circle the first ***i*** unit vector, then cross out the row and column in which it resides.



d.) The ***x*** component of the cross product will be ***i*** (i.e., the unit vector that is circled) times the evaluation of the 2x2 matrix shown in the sketch. Note that the upper left-hand piece of that 2x2 matrix is *one column over, one row down* from the circled ***i***).

e.) To evaluate a 2x2 matrix, take the upper left-hand piece times the bottom right-hand piece MINUS the upper right-hand piece times the lower left-hand piece. In our example, that will be (-5)(3) - (-6)(-2) = -27. Once you multiply that by ***i*** to get *-27**i***, you have the *x component* of the cross product.

f.) The exact same thing is done with the ***j*** column and row (this is usually overlaid on top of the previous set-up--it isn't likely that you are going to want to recreate the entire matrix to do the next step). That is shown below.



g.) For this part, multiply the circled ***j*** times the evaluation of the 2x2 matrix shown in the sketch above (the upper left-hand piece is *-6*). The evaluation for this part will be (-6)(1) - (-4)(3) = +6. Multiplying by the ***j*** yields *+6**j***, and you have the *y component* of the cross product.

h.) The *z component* is taken care of similarly (try it) yielding a final cross product, unit vectors and all, of:

$$\mathbf{B} \times \mathbf{A} = \mathbf{i}\,[(-5)(3) - (-6)(-2)] + \mathbf{j}\,[(-6)(1) - (-4)(3)] + \mathbf{k}\,[(-4)(-2) - (-5)(1)]$$
$$= -27\mathbf{i} + 6\mathbf{j} + 13\mathbf{k}.$$

M.) ***Cross Product*** **In General:**

1.) To understand the physical significance of the cross product, re-consider Figure 1.33.

a.) Vector $\boldsymbol{B}$ has been split into two components--one *parallel* to the line of $\boldsymbol{A}$ and one *perpendicular* to the line of $\boldsymbol{A}$. Notice that the component perpendicular to the line of $\boldsymbol{A}$ has a magnitude of $|\boldsymbol{B}| \sin \phi$.

b.) Evidently, cross products generate a vector whose *magnitude* is equal to the product of:

i.) The *magnitude* of one of the vectors ($|\boldsymbol{A}|$ in Fig 1.33), and

ii.) The *magnitude* of the second-vector's-component that runs *perpendicular* to the first vector ($|\boldsymbol{B}| \sin \phi$).

c.) An example of such a situation follows in *Part 3* of this section.

Note: Again, it makes no difference whether the magnitude is obtained by determining $(|\boldsymbol{A}|)(|\boldsymbol{B}| \sin \phi)$ or $(|\boldsymbol{A}| \sin \phi)(|\boldsymbol{B}|)$. A sketch of the information required to determine the former is shown in Figure 1.33, whereas the information required to determine the latter is shown in Figure 1.38.

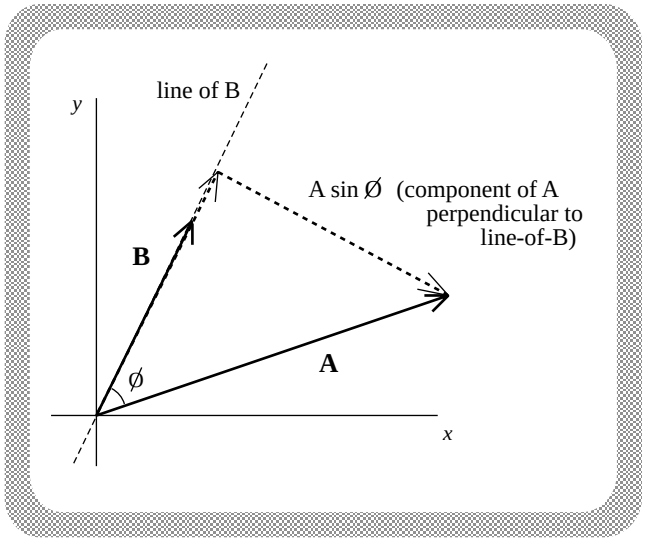


FIGURE 1.38

2.) The significance of a cross product's *direction* depends upon the situation in which the *cross product* is used. For instance, a charged particle moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ will feel a magnetic force $\boldsymbol{F}$ that is proportional to $\boldsymbol{v}\boldsymbol{x}\boldsymbol{B}$. In this case, the *direction* of the *cross product* is the *direction* of the magnetic force as it is applied to the particle (the apparently odd fact that the *direction* of a magnetic force on a moving charged particle is always

perpendicular to the plane defined by the *velocity* and *magnetic field* vectors was deduced experimentally).

Note: Concerning this "perpendicular-to-the-plane-of-the-two-vectors" characteristic of *cross-product directions*, notice that if the vectors being crossed are in the *x-y plane*--a situation that will *always* be the case when working in *polar notation*--the *cross product* direction will be either in the $+\boldsymbol{k}$ or $-\boldsymbol{k}$ *direction*.

3.) Example of a situation in which the *cross product* comes in handy: Consider Figure 1.39.

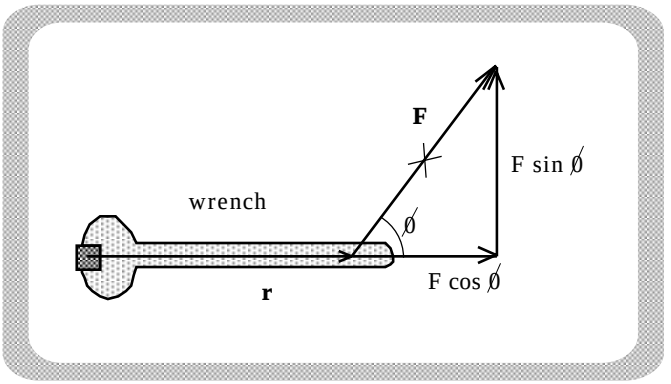


FIGURE 1.39

a.) A force $\boldsymbol{F}$ is applied to a wrench at a distance $\boldsymbol{r}$ units from the axis of rotation. Note that:

i.) The greater $|\boldsymbol{r}|$, the less difficult it will be to rotate the bolt;

ii.) The greater $|\boldsymbol{F}|$, the less difficult it will be to rotate the bolt; and

iii.) The force component that will make the bolt rotate will be the component *perpendicular* to the line of $\boldsymbol{r}$ (i.e., $|\boldsymbol{F}| \sin \phi$).

b.) As ease of rotation is related to $|\boldsymbol{r}|$ and $(|\boldsymbol{F}| \sin \phi)$, the product of those two variables is deemed important enough to be given a special name--*torque* ($\boldsymbol{\Gamma}$). In short, the magnitude of the torque applied by $\boldsymbol{F}$ about the axis of rotation will be $|\boldsymbol{\Gamma}| = |\boldsymbol{r} x \boldsymbol{F}|$.

c.) Assuming $\boldsymbol{r}$ and $\boldsymbol{F}$ are in the x-y plane, the direction of the *cross product* will either be in the $+\boldsymbol{k}$ or $-\boldsymbol{k}$ *direction* (using the right-hand rule outlined above, it turns out to be the $+\boldsymbol{k}$ *direction*).

Note 1: For straight-line motion, the use of the $\boldsymbol{i}, \boldsymbol{j},$ and $\boldsymbol{k}$ unit vectors to denote, say, a velocity's direction, is easy to decipher. In such a case, they quite literally tell you the *direction* in which the body is moving. Unfortunately, when dealing with planar rotational motion and the *cross products* that define them, the idea of *direction* is a little more complicated.

Note 2: If *Part d* (below) does not make perfect sense now, don't worry about it. All you really need to be able to do at this point is to calculate a *cross product* (both magnitude and direction). The significance of the calculation will become much more evident when we begin using the math in specific situations.

d.) Direction of a *cross product*:

i.) To begin with, notice that the motion of a spinning object (we'll assume it is spinning in the *x-y plane*) does not really *go* anywhere--it just sits there spinning. That means that the *direction* of, say, the *rotational velocity* of the object can have nothing to do with *linear displacement*.

ii.) In fact, direction as related to spinning objects addresses one question only: Is the rotation *clockwise* or *counterclockwise*?

iii.) As there are no *unit vectors* defined in the clockwise direction, a little fancy footwork is required to convey the *sense of rotation*, using the unit vector notation we already have at our disposal.

iv.) The vectors used to identify the torque that started the object spinning in the first place (i.e., $\boldsymbol{F}$ and $\boldsymbol{r}$) are in the *x-y plane*, but notice that the object rotates about a line parallel to the z-axis.

v.) Physicists and mathematicians traditionally define the sense-of-rotation by denoting the axis about which the object rotates.

vi.) If, therefore, a rotation in the *x-y plane* is clockwise (this kind of rotation would screw a bolt INTO the *x-y plane*), the axis of rotation is along the *z axis* (inward) and the unit vector assigned to denote that fact would be $-\boldsymbol{k}$.

By the same token, if a rotation in the *x-y plane* is counterclockwise (i.e., screwing the bolt OUT), the axis of rotation is along the *z axis* (outward) and the unit vector assigned to denote that fact would be $+\boldsymbol{k}$.

vii.) Using that designation, torques that produce *counterclockwise* motion unscrew bolts *out* of the page and are designated as having a $+\boldsymbol{k}$ *direction*. Torques that produce clockwise motion screw a bolt *into* the page and are defined as having a $-\boldsymbol{k}$ *direction*.

viii.) In light of all this, it shouldn't be surprising to find that a torque that makes a bolt rotate, say, clockwise, will have a *cross product* whose direction is along the $-\boldsymbol{k}$ *axis*. Put another way, when a torque calculation readings $\boldsymbol{r}\times\boldsymbol{F} = (3\ nt\cdot m)(-\boldsymbol{k})$, the $-\boldsymbol{k}$ part of the cross product tells you that the torque in question has a *clockwise* sense.

QUESTIONS

Note: If it is now at least three days into the course, I would suggest you begin with Question #1.3. Do Questions #1.1 and #1.2 only if you have time--they are *not* terribly important concepts.

1.1) With some urgency, a newlywed couple on their honeymoon hires a not-so-bright boatman to row them across a 100 meter wide river to their honeymoon hotel. In calm water, the boatman (we'll call him Jack *the idiot*, for reasons that will become obvious shortly) can row his dinghy 5 miles per hour. The river moves at 2 miles per hour.

When sitting directly across from the hotel, the boatman points his bow at the hotel and proceeds to row like mad without a second glance. In blissful ignorance, he, the newlyweds, and the boat do not move directly across the river but instead move across and down the river--Jack has not compensated for the current. (Landing nowhere close to the hotel, the couple is less than delighted with Jack's performance, hence tagging him "the idiot.")

a.) Using *graphical manipulation*, determine the boat's actual velocity (as a *vector*) relative to dry land.

b.) (This one is a stinker--don't spend a lot of time on it). Changing the problem slightly, let's assume the boat starts 30 meters down river from the original starting position (that is, 30 meters below a line drawn directly across the river to the hotel--see Figure I). If Jack is clever, he can point the boat upstream to compensate for the moving water and make the boat's net movement travel directly toward the hotel. How would he have to orient the boat to do so and, as a consequence, how fast would the boat move relative to dry land? Again, use graphical means to determine this!

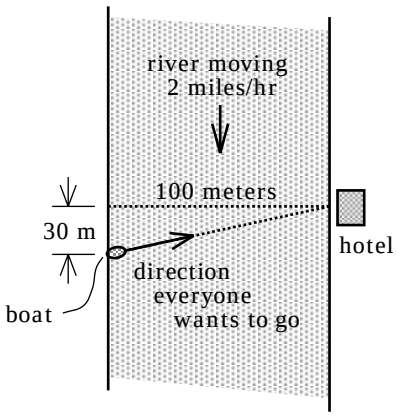


FIGURE I

1.2) A plane flies 80 miles north, then makes a north-westerly turn (versus south-westerly turn) at an unknown angle and flies an additional 60 miles. Upon landing, the pilot realizes she is exactly 30^o west of north, *relative to her starting point*.

a.) Use graphical manipulation to determine the "unknown angle" of her westerly turn.

b.) What was her *net* distance traveled (i.e., her net displacement)?

Note: Graphical manipulation is only rarely used in physics, but it is something that is usually included in physics classes for the sake of completeness.

1.3) Graph the vectors $\boldsymbol{T} = 8\boldsymbol{i} - 12\boldsymbol{j}$ and $\boldsymbol{P} = 7\angle(-60^o)$.

1.4) From the graph in Figure II, characterize vectors ***A*** and ***C*** in *unit vector notation* and vectors ***B*** and ***D*** in *polar notation*.

1.5) On the graph shown in Figure II, vectors ***B*** and ***D*** are almost the same length. Why is the magnitude of ***B*** only 1.3 when ***D's*** magnitude is around 2.6?

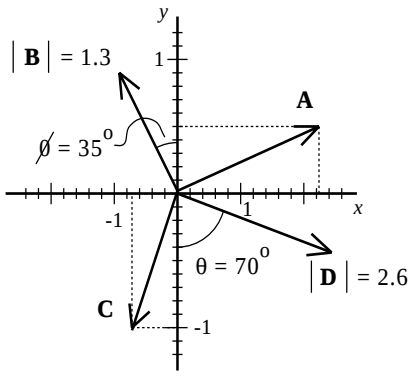


FIGURE II

1.6) Assume:

$\mathbf{A} = -8\mathbf{i} + 12\mathbf{j}$
$\mathbf{B} = -4\mathbf{i} - 3\mathbf{j}$
$\mathbf{C} = 5\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}$
$\mathbf{D} = 7 \angle (-60^o)$
$\mathbf{E} = 12 \angle 225^o$
$\mathbf{F} = 2 \angle 105^o$.

In the problems below, use the above vectors *as they are denoted above* (that is, if asked to do $\boldsymbol{A}\times\boldsymbol{B}$, do so in unit vector notation if ***A*** and ***B*** are *given* in u.v.n.). With that in mind, determine:

a.) $-(1/3)\mathbf{A}$;
b.) $-6\mathbf{E}$;
c.) $\mathbf{A} + \mathbf{B} - \mathbf{C}$;
d.) **E** converted to unit vector notation;
e.) **F** converted to u.v.n.
f.) **A** converted to polar
g.) **B** converted to polar
h.) $\mathbf{A}\cdot\mathbf{C}$;
i.) $\mathbf{D}\cdot\mathbf{E}$;
j.) $\mathbf{A}\times\mathbf{B}$;
k.) $\mathbf{C}\times\mathbf{B}$;
l.) $\mathbf{D}\times\mathbf{E}$.

1.7) What does $\boldsymbol{A}\cdot\boldsymbol{B}$ really tell you?

1.8) What does $\boldsymbol{A}\times\boldsymbol{B}$ really tell you?

Chapter 2

DERIVATIVES, DIFFERENTIALS, AND DIFFERENTIAL EQUATIONS

Note: THIS CHAPTER IS FOR REFERENCE ONLY. If you are feeling insecure about your understanding of Calculus, read Chapters 2 and 3 closely. Otherwise, skim them. In either case, don't panic if the material looks obscure. We'll ease into it gently (besides, the first test will primarily be over Chapter 1).

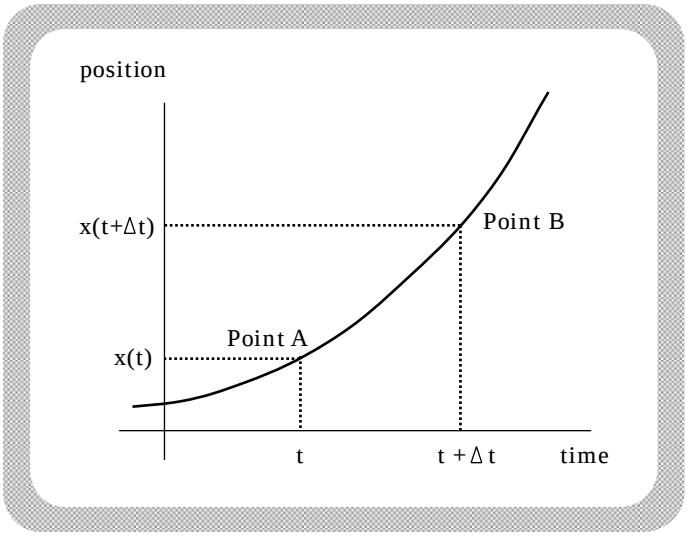


FIGURE 2.1

A.) Derivatives--Preliminaries:

1.) An eccentric's pet ant is constrained to move in one dimension. Figure 2.1 presents a graph of its motion in time.

a.) At time *t*, the ant is located at *Point A*. While there, its position coordinate is *x(t)*. At time $t + \Delta t$, the ant is located at *Point B* with a position coordinate $x(t + \Delta t)$.

b.) In Figure 2.2, a *secant* is drawn between *Points A* and *B*.

i.) Note that the *slope of the secant* is equal to the secant's "rise over run," or:

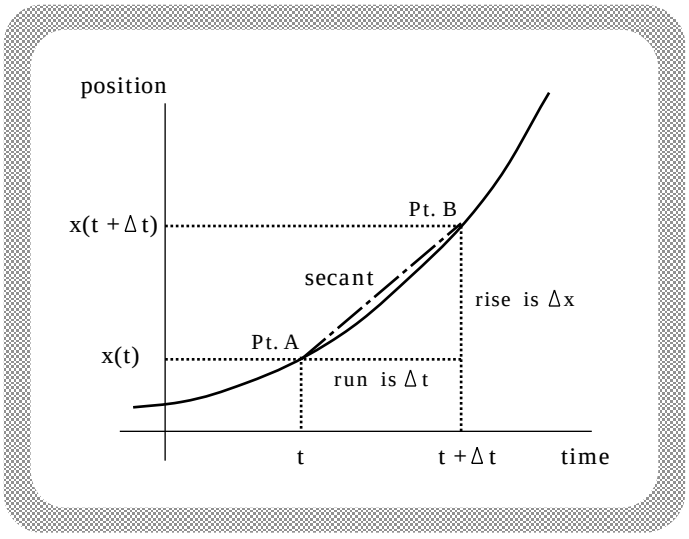


FIGURE 2.2

$$\text{slope} = \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t}$$
$$= \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

2.) If we hold *Point A* fixed while allowing Δt to become very small, *Point B* approaches *Point A* and the *secant* approaches the *tangent to the curve* at *Point A*. Writing this out mathematically, we get:

$$\text{slope of tangent} = \text{slope of } \mathbf{secant}$$
$$= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

Note: Although the denominator goes to zero, the *ratio* of the numerator and denominator converges to a finite number.

a.) This operation is called a *derivative*. In general, a *derivative* yields a new function that defines the *rate of change of the original function with respect to one of its variables*. In the above case, it defines the *rate of change of "x" with time*.

b.) Books from different disciplines denote *derivative* operations in different ways.

i.) Some math books, for instance, use the notation:

$$x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

ii.) Most physics books, on the other hand, use the notation:

$$\frac{d(x)}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

Note: In the latter notation, *dt* is NOT *d* multiplied by a time *t* any more than Δt is Δ multiplied by a time *t*.

c.) Mathematicians treat *dx*/*dt* as a <u>single symbol</u> denoting a particular mathematical operation (a derivative). As you will see shortly, physicists are not so formal in their treatment of the notation.

3.) Bottom line: In general, the *time derivative* of a time-varying function *f(t)* gives you a second function *df(t)/dt* that defines the *slope of the tangent to f(t)* at any point on *f(t)'s* curve. Put another way, it gives us a general function that defines *the rate at which f(t) changes with time*. This function can then be evaluated at any given time-of-interest *t*.

B.) Derivatives–In the Beginning . . . :

1.) In most beginning Calculus classes, the *definition of the derivative*:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t},$$

is used to generate derivative-type expressions for relatively common functions. Although the approach is limited, it is effective in presenting the theoretical underpinnings of the derivative. As such, we will do a few examples to show how it works.

a.) Consider the function *f(t) = 3t +2* (see Figure 2.3). What is the *time rate of change of the function*? That is, what is the *new function* that defines how *f(t)* changes as *t* changes?

i.) This is an admittedly easy problem. Seeing that the function is linear and noting that its slope (i.e., its rate of change) is constant and equal to 3, we know the answer to the derivative calculation before we start. Nevertheless:

ii.) Following through with the *definition of the derivative*:

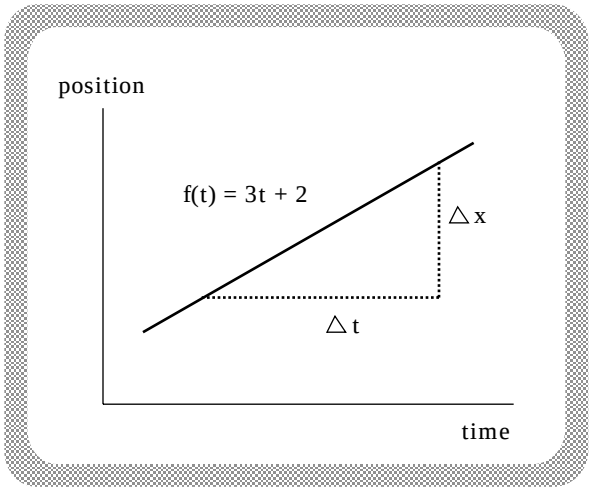


FIGURE 2.3

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[3(t+\Delta t)+2] - [3t+2]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3t + 3\Delta t + 2 - 3t - 2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3\Delta t}{\Delta t}$$

$$= 3.$$

iii.) As the Δt*'s* in the fourth line canceled, we didn't need to invoke the limit (i.e., have Δt go to zero). Still, the process leads us to a function (a constant in this case) that defines the *rate* at which $x = 3t + 2$ changes with time.

b.) Consider the function $x(t) = kt^3$ (see Figure 2.4 below), where k is a proportionality constant equal to $1\ m/s^3$ (without k, x would have the units of *seconds cubed*). What is the *rate of change* of the function? That is, what is the *new function* that defines how x changes as t changes?

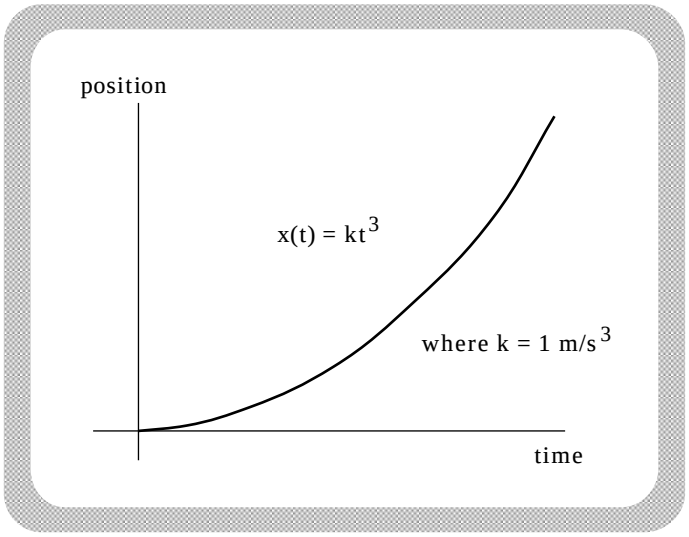


FIGURE 2.4

i.) This is not a trivial situation. Setting $k = 1$ (we don't need to carry it through the calculation as it was included only to make the units acceptable), and following through with the *definition of the derivative*, we get:

$$\frac{dx(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[(t+\Delta t)^3\right] - \left[t^3\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{t^3 + 3t^2(\Delta t) + 3t(\Delta t)^2 + (\Delta t)^3 - t^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3t^2(\Delta t) + 3t(\Delta t)^2 + (\Delta t)^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[3t^2 + 3t(\Delta t) + (\Delta t)^2\right]$$

$$= 3t^2 \qquad \text{(or } 3kt^2 \text{ if we replace the "k").}$$

ii.) What happened to all the Δt's? Some were canceled; some went away when we invoked the limit.

iii.) What does this mean? At *t = 2 seconds*, for instance, the body is at $x = kt^3 = (1\ m/s^3)(2\ s)^3 = 8\ meters$. In addition, according to our derivation, the *slope of the tangent to the curve* at that time (i.e., the body's *rate of change of position with time*, a.k.a. its *velocity*) is $dx/dt = 3kt^2 = 3(1\ m/s^3)(2\ s)^2 = 12\ m/s$.

c.) If we had done this process for k_2t^2, we would have found the derivative equal to $2k_2t$ (try it!). If we had done the process for k_4t^4, the derivative would have been $4k_4t^3$.

Generalizing, we can write:

If $x = k_1t$, then $dx/dt = 1k_1$.

If $x = k_2t^2$, then $dx/dt = 2k_2t$.

If $x = k_3t^3$, then $dx/dt = 3k_3t^2$.

If $x = k_4t^4$, then $dx/dt = 4k_4t^3$.

etc.

From this we can deduce that if:

$$x = k_n t^n, \qquad \text{then} \qquad dx/dt = nk_n t^{n-1}.$$

d.) This is exactly what Calculus students do for the first month or so of the *Derivatives* section of their course. They use the *definition of the derivative* to determine specific derivative functions, then they generalize if possible.

e.) Other derivative rules that might come in handy:

i.) $d(Ae^{-kt})/dt = -Ake^{-kt}$, where A and k are constants;

ii.) $d[\sin(kt)]/dt = k\cos(kt)$;

iii.) $d[\cos(kt)]/dt = -k\sin(kt)$;

iv.) $d[\ln(kt)]/dt = k(1/t)$, where "ln" designates a *natural log*.

v.) Noting the $(k/t) = kt^{-1}$, we get $d(kt^{-1})/dt = (-1)kt^{-2} = -k/(t^2)$;

vi.) $d[f(x)g(x)]/dx = [df(x)/dx][g(x)] + [f(x)][dg(x)/dx]$ (this is called *the product rule*). Example: $d[(4x)(x^2)]/dx = (4x)(2x) + 4(x^2) = 12x^2$ (note that $(4x)(x^2) = 4x^3$, and that the derivative of $4x^3$ is, indeed, $12x^2$ . . . it works).

C.) The Chasm Between Mathematicians and Physicists:

1.) As has already been mentioned, mathematicians have very definite ideas about how calculus-oriented terms are to be defined and manipulated. On the other hand, physicists are (within limits) willing to play fast and loose with the notation. Specifically:

a.) Mathematicians demand that *dt* be read as "an *infinitesimally small* time interval" (i.e., no real duration at all). Physicists think of *dt* as *a very, very small* time interval. The appeal of the latter description? It implies there exists a tiny period of time during which something *can* happen.

b.) A similar situation holds for the symbol *dx*. From a mathematician's perspective, *dx* is "an *infinitesimally small* displacement with no *physical* reality to it at all." Physicists treat *dx* as a *very, very small* displacement.

c.) Although *dx*/*dt* denotes a specific operation, physicists are more than happy to treat *dx* and *dt* as separate entities, even to the extent of manipulating them algebraically. For instance, $dx/dt = c$ implies that the derivative of a function *x* equals a constant *c*. Mathematicians use formal algorithms and transforms to show that if this is true, it is also true that $dx = c\,dt$ (this reads *a differential change of position "dx" equals a constant "c" times the differential time interval "dt" over which the displacement takes place*). Physicists know how to use the algorithms, etc., but aren't willing to bother. Instead, they short cut the process by simply multiplying both sides of $dx/dt = c$ by *dt* yielding $dx = c\,dt$. Mathematicians hate this kind of nonchalance, but it works if you're only interested in the bottom line.

2.) To what does this all come down? As aesthetically irritating as it might be to mathematicians, the Calculus is both easier to visualize and easier to use in the context of *real world problems* when *dx's* and *dt's* are afforded a physical significance.

3.) An example: Understanding *the Chain Rule:*

a.) Consider the function $f(x) = x^2$.

i.) Note that it is easy to see how the function changes with *changes of position* as its derivative with respect to *x* is:

$$\frac{df(x)}{dx} = 2x.$$

ii.) IMPORTANT: There is nothing wrong with manipulating this into the form $df(x) = (2x)dx$, then replacing the *2x* by $df(x)/dx$. This probably seems a bit circular, but in doing so we get an interesting differential equation. Specifically:

$$df(x) = (2x)\,dx$$
$$= \left[\frac{df(x)}{dx}\right]dx.$$

This equation states that the *differential change of the function* (i.e., *df*) as one moves some *differential distance dx* along the *x axis* equals the *rate at which the function changes with "x"* (i.e., $df/dx = 2x$) times the SIZE of the change (i.e., the *displacement dx*).

b.) Assume now that x changes with time (i.e., $x = x(t)$). Specifically, for example, assume that $x(t) = kt^3$. How does the function $f(x)$ change with time (i.e., what is $df(x)/dt$)?

c.) The approach that allows us to determine df/dt is called the Chain Rule. Stated mathematically, it is:

$$\frac{df(x)}{dt} = \left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\right].$$

The question? Where did this come from and what does it mean?

d.) In *Part 3a-ii*, we decided that the *change of the function df* as we move some *differential distance dx* along the *x axis* equals the *rate at which the function changes as we proceed along the x axis* (i.e., df/dx) times the SIZE of the *differential displacement* (dx). Mathematically, this was written:

$$df(x) = \left[\frac{df(x)}{dx}\right]dx \qquad \text{(Equation A)}.$$

e.) We can follow a similar path in expressing the differential displacement dx in terms of dt. That is, the *change* of the position function dx as we move through some *differential time interval dt* equals the *rate at which x changes in time* (i.e., dx/dt) times the SIZE of the *differential time interval dt* during which the change occurs. This is written:

$$dx = \left[\frac{dx(t)}{dt}\right]dt \qquad \text{(Equation B)}.$$

f.) As Equation B is an expression for the *differential change* of x (i.e., dx) in terms of time, we can use it in Equation A to write:

$$\begin{aligned} df(x) &= \left[\frac{df(x)}{dx}\right] \quad dx \\ &= \left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\,dt\right]. \end{aligned}$$

g.) Dividing both sides by *dt* (oops, there goes Fermat, rolling over in his grave again), we get the Chain Rule, or:

$$\frac{df(x)}{dt} = \left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\right].$$

h.) Example: Consider $f(x) = x^2$ (derivative $df(x)/dx = 2x$). If $x(t) = kt^3$ (derivative $dx(t)/dt = 3kt^2$), then df/dt is:

$$\begin{aligned}\frac{df(x)}{dt} &= \left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\right] \\ &= \left[2x\right]\left[3kt^2\right] \\ &= \left[2(kt^3)\right]\left[3kt^2\right] \\ &= 6k^2t^5.\end{aligned}$$

i.) Bottom Line: It is not correct to treat the symbolic representation of a derivative as though it were made up of algebraic symbols. Nevertheless, doing so has allowed us to derive a Chain Rule that looks and acts just like the one derived by mathematicians.

D.) Partial Derivatives and the Del Operator:

Note: The *del operator* is a mathematical entity that most students first encounter at the university level. It is, nevertheless, a tool we will use later in this course. Of the material presented below, the *only* operations you will be expected to know will be the basic execution of a *partial derivative* and the basic execution of a *gradient*. Both will be discussed in class; neither will show up until much later in the course. The rest of the material has been included for the sake of completeness and for your own personal amusement.

1.) Consider a function *f(x,y)* similar to the one graphed in Figure 2.5 to the right. Note that a *plane* parallel to the *y-z* plane has been placed at x_1, and that it cuts the

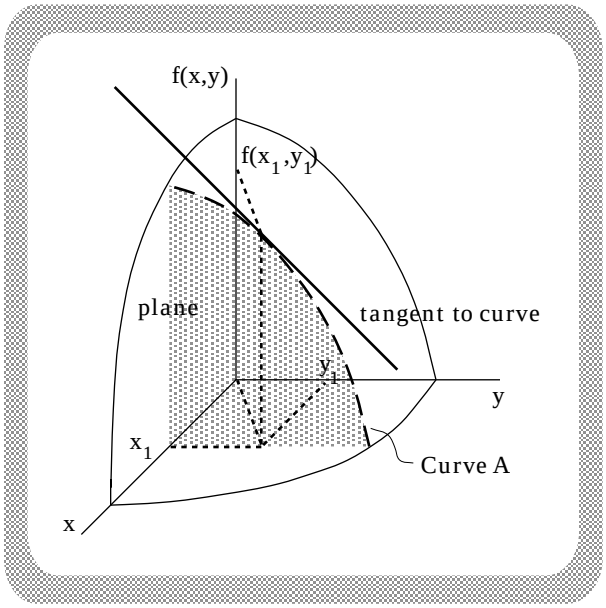


FIGURE 2.5

surface of *f(x,y)* along a curve as shown (the sketch labels this curve "Curve A").

a.) The question? How do we derive a function that tells us the *rate of change of f(x,y)* while holding the *x parameter* constant? Put another way, how do we mathematically define the *slope of the tangent to Curve A* in the sketch?

b.) In Figure 2.5, a tangent has been drawn on the curve at an arbitrary point $f(x_1, y_1)$. The slope of that tangent is:

$$\lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}.$$

c.) This somewhat unusual operation (i.e., determining the slope of *f(x,y)* holding the *x* variable constant) is called a *partial derivative.* The symbol and formal definition for a partial derivative is shown below:

$$\frac{\partial f(x, y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}.$$

d.) The *partial derivative* quoted above reads: the change of *f(x,y)* with respect to *y* holding *x* constant. Put another way, it is the derivative of *f(x,y)* with respect to *y* treating all other variables as constants.

e.) Example: $\partial(x^2y^4) / \partial y = (x^2)(4y^3)$.

2.) Partial derivatives are interesting in and of themselves, but they become really useful when used in the context of what is called the *del operator*.

a.) The *del operator* is defined as:

$$\nabla = \left[\frac{\partial()}{\partial x}\mathbf{i} + \frac{\partial()}{\partial y}\mathbf{j} + \frac{\partial()}{\partial z}\mathbf{k} \right].$$

Note 1: Bold-face letters denote vectors. The bold-face letters **i**, **j**, and **k** denote *unit vectors* in the *x*, *y*, and *z* directions respectively. As such, the *del operator* is a *vector operator*.

Note 2: An operator has no significance by itself. That is, it must operate on a function, be it scalar or vector, to derive any meaning.

Note 3: The "()'s" found in the *del operator* expression normally house the function being operated upon. Examples are coming.

3.) The *del operator* and the GRADIENT of a scalar:

a.) A *scalar field* is a function that defines a magnitude at every point in a given volume. *Temperature* is a *scalar field* as there exists a value for it everywhere.

b.) Consider the temperature *scalar field* $T = 3ky^2 + 20$ (this could model a room in which the temperature increases as one moves upward away from the floor). Assume that $k = 1\ ^o/m^2$ (it is included for the sake of units).

c.) Note that if we travel in the $+x$ direction, the temperature stays the same. Only when we move upward or downward does it change. In fact, the MAXIMUM POSITIVE CHANGE of T is along the "+**j**" direction.

d.) Using the *del operator* in conjunction with T, we get:

$$\nabla T = \left[\frac{\partial(3ky^2 + 20)}{\partial x} i + \frac{\partial(3ky^2 + 20)}{\partial y} j + \frac{\partial(3ky^2 + 20)}{\partial z} k \right]$$
$$= (6ky)j.$$

e.) Two things to observe:

i.) The DIRECTION of $\boldsymbol{\nabla} T$ is $+\boldsymbol{j}$. This is the same direction as the MAXIMUM POSITIVE CHANGE OF T at any point.

ii.) Though it may not be immediately evident, the RATE OF CHANGE OF THE FUNCTION IN THE DIRECTION OF THE MAXIMUM POSITIVE CHANGE at any point is equal to the MAGNITUDE of $\boldsymbol{\nabla} T$ evaluated at that point.

f.) Bottom Line: When a *del operator* acts on a scalar function, it yields a vector function. The *direction* of the vector is THE DIRECTION IN WHICH THE SCALAR FUNCTION CHANGES THE FASTEST (in a positive sense), and the *magnitude* of the vector equals the RATE AT WHICH THE SCALAR FUNCTION CHANGES IN THAT FASTEST-CHANGING DIRECTION.

g.) This operation is called *the GRADIENT*.

NOTE: THE REST OF THIS CHAPTER IS FOR YOUR OWN EDIFICATION. READ IT BUT DO NOT STRESS OVER IT!

4.) The *del operator* and the DIVERGENCE of a vector:

a.) A *vector field* is a *vector function* that defines a magnitude and a direction at every point in a given volume. As an example, the earth's *gravitational field* is a vector field in the sense that at every point around the earth: **i.)** there is a number that defines the *magnitude* of the acceleration an object will experience if released at that point, and **ii.)** there is a *direction* defined at every point for that acceleration.

b.) Consider a cylindrical cavity that is closed at both ends and that holds air at high pressure (see Figure 2.6a).

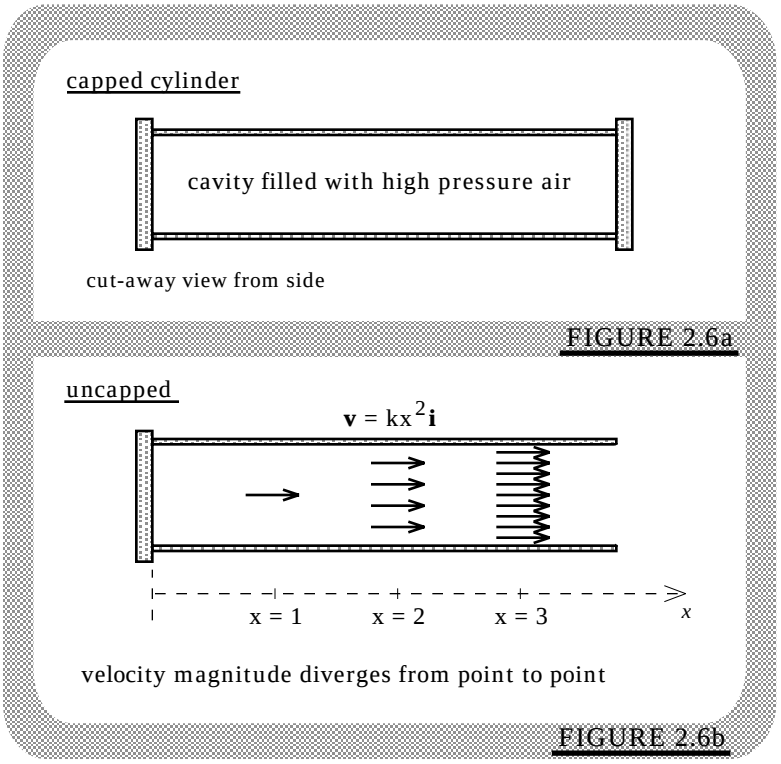


FIGURE 2.6a

FIGURE 2.6b

Note: This example was inspired by a problem in Hugh Skilling's book *Fundamentals of Electric Waves.*

i.) When the cavity is opened, the compressed air will expand out and rush from the cavity in the process.

ii.) Just after the cavity is opened, the air's velocity near the *closed end* will be relatively small while the air's velocity near the *open end* will be large. For the sake of argument, assume the spatially varying velocity is defined as $\boldsymbol{v} = kx^2\boldsymbol{i}$, where k is a constant with the appropriate units and x is defined from the closed end (see Figure 2.6b).

iii.) The air's velocity is clearly diverging from point to point.

iv.) If we dot the *del operator* into the *velocity vector*, we get:

$$\nabla \bullet \mathrm{v} = \left[\frac{\partial()}{\partial \mathrm{x}} \mathrm{i} + \frac{\partial()}{\partial \mathrm{y}} \mathrm{j} + \frac{\partial()}{\partial \mathrm{z}} \mathrm{k} \right] \bullet (\mathrm{kx}^2 \mathrm{i} + 0\mathrm{j} + 0\mathrm{k})$$
$$= \left[\frac{\partial(\mathrm{kx}^2)}{\partial \mathrm{x}} \right]$$
$$= 2\mathrm{kx}.$$

v.) This scalar quantity tells us the rate at which the velocity changes as we move along the line of the velocity vector. Put another way, it gives us the divergence of the velocity--*the rate of change of velocity with position*. This is *the DIVERGENCE* of the velocity vector.

vi.) In general, the DIVERGENCE yields *the rate of change of a vector function (in the direction of the vector) with position.*

Note: Although the above example is that of a diverging velocity, the divergence is most often associated with the *differential form of Gauss's Law* as it pertains to *electric fields* generated by *symmetric, extended charge* configurations. Most Calculus based physics texts (Halliday and Resnick's *Fundamentals of Physics*, for instance) do not present this form of Gauss's Law (they only give Gauss's Law in *INTEGRAL form*), but the *DIFFERENTIAL form* presented below is useful as well as being the *form of choice* in most advanced *electricity and magnetism* books. It has been included here because it is a practical example of the use of the divergence operation, and because it allows the student a peek at a relatively sophisticated mathematical tool physicists have at their disposal.

c.) In the world of static electricity, the evaluation of the *divergence* of an electric field ***E*** at a point is found to be proportional to the *charge density r* at the point. Mathematically, this is written:

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_o},$$

where the symbol ε_o is a constant. Called the *differential form* of *Gauss's Law*, this is one of Maxwell's equations.

Note: Again, when we get to Gauss's Law later in the year, we will deal with it in its *integral form*. Even so, the *differential* version definitely has its use.

5.) The *del operator* and the CURL of a vector:

Note: This operation is useful when dealing with the *differential form of Maxwell's equations* as they pertain to *magnetic fields*. As before, YOU WILL NOT BE TESTED ON THIS CONCEPT.

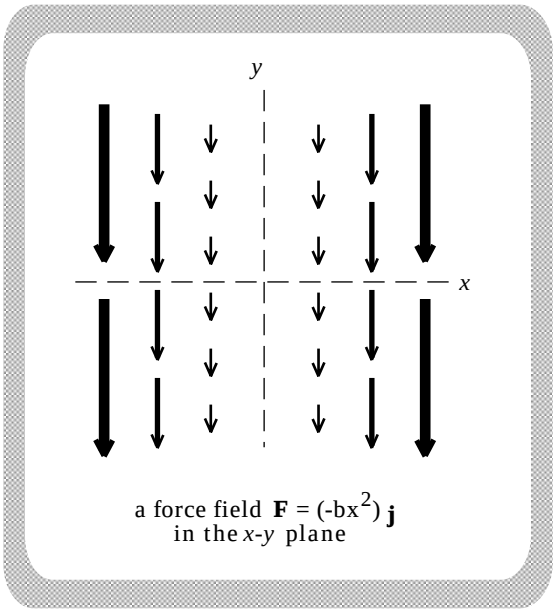


FIGURE 2.7

a.) Consider the force field $\boldsymbol{F} = (-bx^2)\boldsymbol{j}$ shown in Figure 2.7 to the right.

b.) Assume $b = 1$ *newton/meter* (i.e., it is there solely for units conformity). Both the *del operator* and $\boldsymbol{F}$ are vectors, so let's do a *cross product* between the two and see what we get:

$$\nabla \mathrm{x} F = \begin{vmatrix} i & j & k \\ \frac{\partial()}{\partial x} & \frac{\partial()}{\partial y} & \frac{\partial()}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial()}{\partial x} & \frac{\partial()}{\partial y} & \frac{\partial()}{\partial z} \\ 0 & -bx^2 & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial(0)}{\partial y} - \frac{\partial(-bx^2)}{\partial z}\right] + j\left[\frac{\partial(0)}{\partial z} - \frac{\partial(0)}{\partial x}\right] + k\left[\frac{\partial(-bx^2)}{\partial x} - \frac{\partial(0)}{\partial y}\right]$$

$$= (-2bx)k.$$

c.) Physical interpretation: Assume our *field* models the force of water flowing in a stream. That is, assume there is something in the middle of a stream which makes the flow negligible down the middle while allowing the water to flow faster as one proceeds out away from the central axis (we are obviously ignoring the fact that the stream's flow will slow down at the stream's shoreline). With this in mind, consider:

i.) Three paddle-wheels are placed in the field (see Figure 2.8)--one at *Point A*, one at *Point B*, and one at *Point C*. Each is positioned so that its *axis* is *perpendicular* to the stream's surface (i.e., in the *z direction*).

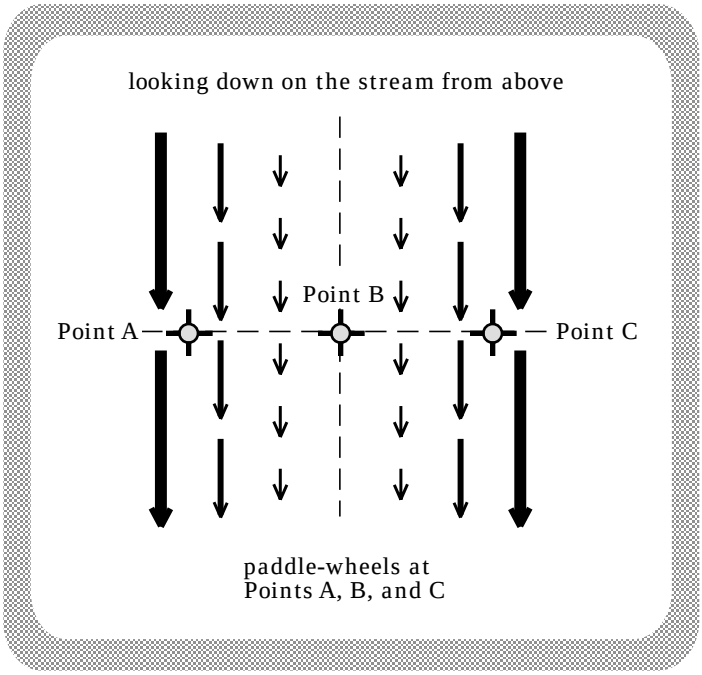


FIGURE 2.8

ii.) Due to the symmetry of the situation, there is no net force-of-rotation (i.e., no torque) on the paddle-wheel at *Point B*. That is, there is as much force pushing on the paddle to the left of the origin as there is on the paddle to the right of the origin.

Note also that at this point, the *CURL* of the force field $\boldsymbol{\nabla \cdot F} = (-2bx)\boldsymbol{k}$ is zero as $x = 0$.

iii.) Due to the asymmetry of the situation, there is a net torque acting on the paddle-wheel at *Point A* that makes it turn counterclockwise.

Note: The sign convention presented in the last chapter for the *direction* of a torque is as follows: A torque's UNIT VECTOR defines *the direction of the axis about which the rotation occurs:* the SIGN of the torque's *unit vector* defines the sense of the rotation (i.e., a *negative* sign implies clockwise rotation whereas a *positive* sign implies counterclockwise rotation).

iv.) Using our sign convention for torque, the stream-produced torque at *Point A* will, by inspection, be in the $+\boldsymbol{k}$ *direction* (the paddle-wheel will rotate counterclockwise).

v.) The *curl of* $\boldsymbol{F}$ (i.e., $\boldsymbol{\nabla \cdot F} = \boldsymbol{(-2bx)k}$) has a direction of $+\boldsymbol{k}$ for all negative values of x (including that at *Point A*). In other words, the *curl* of $\boldsymbol{F}$ at *Point A* has the same sign and axis direction as the torque on the paddle-wheel at *Point A*.

vi.) A similar analysis for *positive x* (i.e., for the paddle-wheel at *Point C*) yields a *curl* direction equal to $-\boldsymbol{k}$. This is the same direction as that of the paddle-wheel torque in *that* situation.

vii.) Lastly, although it is probably not as obvious, the MAGNITUDE of the *curl* yields the *rate* at which a paddle wheel will rotate at a given point in the field.

d.) In summary:

i.) The MAGNITUDE of the *curl* of a vector yields the *rate at which the field circulates* at a given point in the field.

ii.) The DIRECTION of the *curl* of a vector at a particular point tells you the *axis* about which the field *circulates* at that point. It also denotes the sense of the rotation (i.e., clockwise or counterclockwise).

Note: The word *circulate* is being used loosely here. Its significance depends upon the system in question. In our example, it was associated with the direction of the torque provided by the stream on a paddle-wheel. We could as easily have been dealing with electrical phenomena. For example, a magnetic field $\boldsymbol{B}$ that *changes with time* induces an electric field $\boldsymbol{E}$ that circles. The rate at which the magnetic field changes $\partial \boldsymbol{B} / \partial t$ is related to the *curl of* $\boldsymbol{E}$ such that:

$$\nabla \mathrm{x} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}}.$$

The moral of the story? How one interprets the word "circulation" is driven by the context of the problem.

QUESTIONS

There are none! This chapter was not intended to be an exercise in skills-learning (hence, no need for questions to test your understanding). It was designed as a discussion aimed at giving you a feel for how Calculus works and, more to the point, how physicists treat Calculus in the analysis of *real-world* problems. If the material has made sense, great. If not, don't worry about it. You will have plenty of time to become acquainted with Calculus, physics style, as we go.

Chapter 3

INTEGRATION

Note: As was the case in the last chapter, this material is for your edification only! You will not be directly tested on its contents.

A.) The Integral:

1.) Preliminaries:

a.) *Velocity* is defined as a body's *time-rate-of-change of position*. Mathematically, it is written as:

$$v(t) = \frac{d[x(t)]}{dt}.$$

b.) Consider the VELOCITY VERSUS TIME graph shown in Figure 3.1.

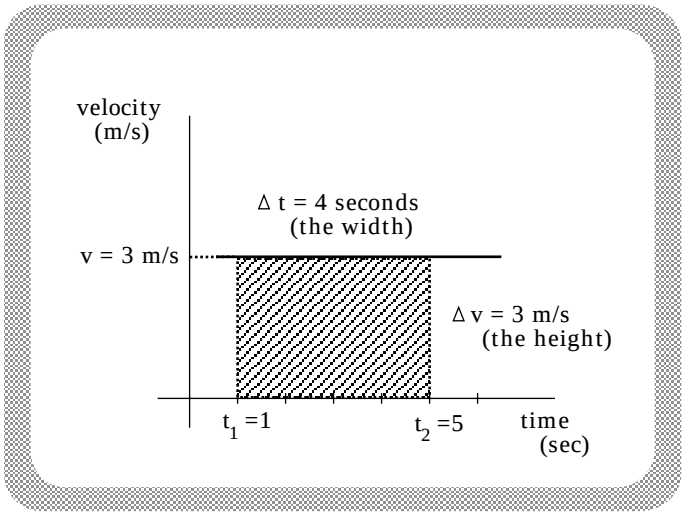


FIGURE 3.1

i.) The distance Δx traveled during the time interval between t_1 and t_2 is $v\,\Delta t = (3\ m/s)(4\ s) = 12\ m$ (this shouldn't be surprising: it is the old "*distance* equals *rate* times *time*" formula you learned when you were a wee one).

ii.) Notice: the distance traveled (i.e., the change of position Δx) is equal to the AREA under the *VELOCITY vs. TIME* curve.

How so? The height of the rectangle is $v = 3\ m/s$ while the width is $\Delta t = 4\ seconds$. As the area of a rectangle is its *height times width*, the area is $v\,\Delta t$ (or Δx).

2.) Assume a body's *position function* does not change in a linear way (i.e., a function whose derivative is not a constant). How do we determine the

body's *net displacement* during a large time interval (i.e., this will be the area under the *VELOCITY vs. TIME* curve during that interval)?

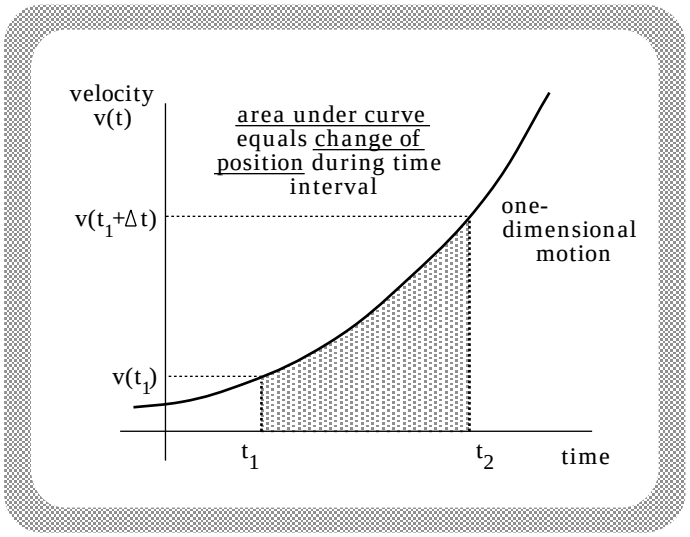


FIGURE 3.2

a.) Assume one-dimensional motion and a *VELOCITY vs. TIME* graph as shown in Figure 3.2.

b.) The distance traveled during the time interval between t_1 and t_2 equals the shaded area under the curve. As the function varies continuously, determining this area is not as easy as was the area-calculation in the previous problem. To accommodate:

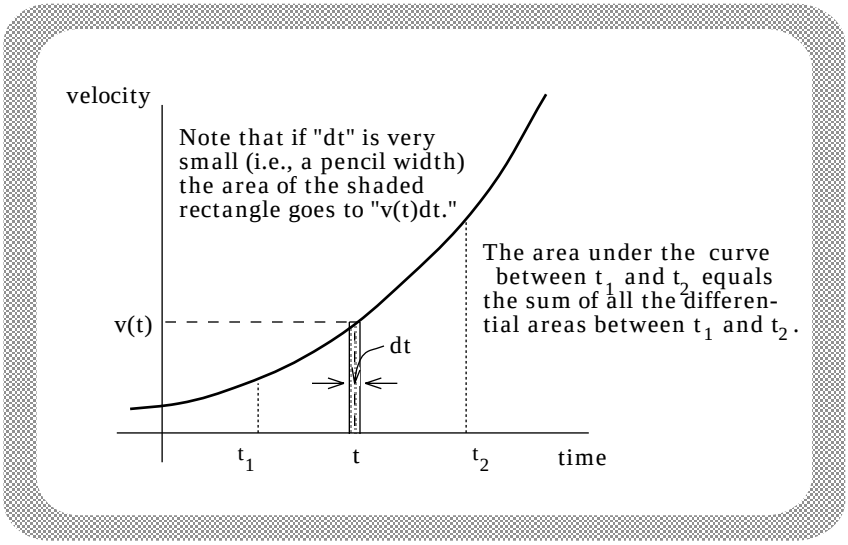


FIGURE 3.3

c.) Consider an arbitrary time *t*:

i.) Place a differential time interval *dt* about time *t* (see Figure 3.3).

Note: This interval should be no thicker than a pencil lead (if that). It has been expanded in the sketch for ease of viewing.

ii.) Call the differential area under the curve bounded by *dt* the *differential displacement dx*. We want to calculate that area (hence, determine that displacement).

iii.) The *differential area* under the curve over the interval *dt* is approximately that of a rectangle (this approximation is made

clear in Figure 3.4 to the right). Its area (i.e., *dx*) equals *v(t)dt*. By definition, *v(t) = dx/dt*. As such:

$$dx = \left[v(t)\right]\, dt$$
$$= \left[\frac{dx(t)}{dt}\right] dt.$$

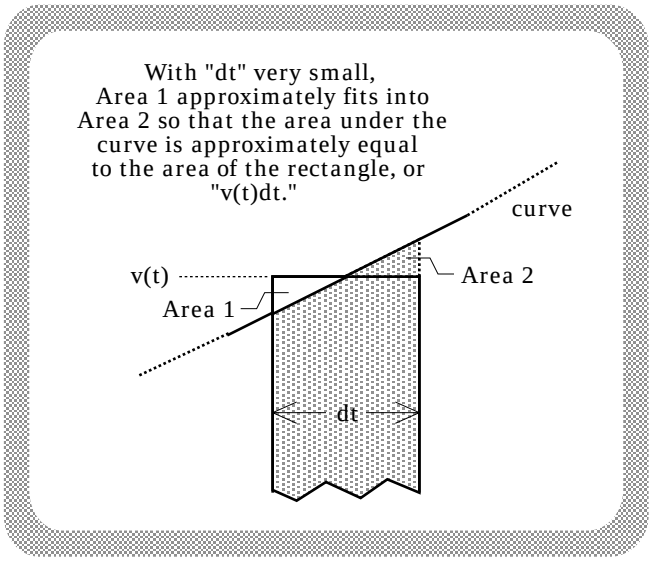


FIGURE 3.4

iv.) In English, this equation reads: the *differential displacement dx* of the body equals *the rate at which its position changes with time* (i.e., *dx(t)/dt*) times *the time interval "dt" over which the change occurs.*

Note: Because we chose *t* to be any arbitrary time, our expression is good for *any* time.

d.) If we executed this area-finding process for all *dt's* between times t_1 and t_2, then added them all up, we end up with the *total displacement of the body* (i.e., its net *change of position*) between times t_1 and t_2.

3.) The temptation is to use a conventional summation sign "∑" to write out this information in mathematical form. The problem? A "∑" sign is used to denote the summing of *discrete quantities*, not to show the summing of a *continuously varying function.*

4.) When a continuous function is summed, a different sign is used. Called an *integral,* the symbol looks like "∫". With it, we can write:

a.) The sum of the *differential displacements* (i.e., the sum of all the individual *dx's*) between t_1 and t_2 equals the net displacement $x(t_2) - x(t_1)$, or:

$$\int dx = \left[x(t_2) - x(t_1)\right].$$

b.) The sum of the *differential displacements dx* between t_1 and t_2 equals the *rate of change of position with time dx/dt* times the *differential time interval dt*, all summed over the time interval between t_1 and t_2. That is:

$$\int dx = \int_{t_1}^{t_2} \left[\frac{dx(t)}{dt} \right] dt .$$

Note: Notice how the limits are placed in the integral.

c.) Putting *Parts a* and *b* together, we can write:

$$\int_{t_1}^{t_2} \left[\frac{dx(t)}{dt} \right] dt = \left[x(t_2) - x(t_1) \right]$$
$$= \Delta \left[x(t) \right].$$

d.) The function *dx(t)/dt* and the function *x(t)* are obviously related to one another (one is the derivative of the other). What is interesting is that *both functions* have shown up in our final equation. Specifically:

i.) We are GIVEN the *time-rate-of-change of the position function* (i.e., *dx/dt*)--that is the quantity inside the integral on the left side of the equal sign.

ii.) We are LOOKING FOR the *position function x(t)* the change Δx of which is denoted on the right side of the equal sign.

5.) Bottom Line: When you find yourselves with an integral like

$$\int_{t=2}^{7} (2t) dt = \ ?$$

you are really looking for a new function (call it *x(t)*) whose *rate of change* (i.e., whose *derivative*) is the function inside the integral. This new function *x(t)* is called *the anti-derivative* of the function inside the integral.

a.) In the context of our example, it means we are looking for a function the derivative of which is *2t* (that function happens to be $x(t) = t^2$).

b.) Once we know that anti-derivative (i.e., *x(t)*), we can evaluate it at the limits to determine how its value *changes* over that time interval (i.e., we can determine $\Delta x(t)$).

c.) To evaluate a function, the upper limit (*t = 7 seconds*) is put into the function (i.e., $x(t) = t^2 = (7)^2 = 49$) to determine the function's value at time *t = 7 seconds*. From that the evaluation of the function at *t = 2 seconds* is subtracted. In that way, the *change of the function over the time interval*--in this case, the net distance traveled between *t = 2 seconds* and *t = 7 seconds*--is determined.

d.) Writing this out formally, we get:

$$\int_{t=2}^{7} (2t)dt = (t^2)\Big|_{t=2}^{7}$$
$$= (7^2) - (2^2)$$
$$= 45.$$

Note: The limits about which the integral's solution is to be evaluated (in this case, *t = 2* and *t = 7 seconds*) are placed as shown above. Occasionally, a bracket notation "$\left[t^2\right]_{t=2}^{7}$" may be used instead.

B.) Conclusions and an Example:

1.) *Integration* allows us to determine the AREA UNDER A CURVE over some specified interval. The solution to an integral is a function *the derivative of which we know*. That is, if we ignore the *dt* (or *dx* or whatever the differential variable happens to be), everything else under the *integral sign* is the derivative of the function we are trying to determine.

2.) Because mathematicians don't attribute physical significance to *differential variables* like *dt* or *dx*, those variables have no meaning in an integral aside from specifying the variable over which the integral summation is to be carried out. By attributing physical significance to such variables, it is possible to think through problems that might otherwise be obscure.

3.) A simple example of this kind of thinking:

a.) Assume we know the circumference of a circle is $2\pi r$, where *r* is the circle's radius. How can we derive an expression for the *area* of a circle whose radius is *R*?

b.) Begin by taking a differential HOOP of radius *r* and differential thickness *dr*. A sketch of the hoop is shown in Figure 3.5 to the right.

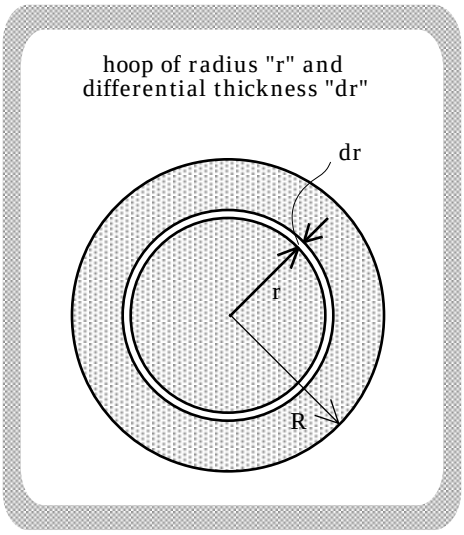


FIGURE 3.5

Note 1: To make it viewable, the differential thickness *dr* has been rendered considerably larger than it actually is.

Note 2: Why take *dr* instead of *dx* or whatever? The variable *r* is usually used when referring to the *radial direction* in POLAR and/or SPHERICAL POLAR notation. Following the convention, the *radial distance* from the circle's center to the arbitrarily chosen hoop is called *r*, and the *differential thickness* of the hoop follows as *dr*.

c.) If we can determine the *differential area* of one arbitrarily chosen hoop, we could do the calculation for all possible differential hoops. Having done so, we could then sum up all those little differential areas to find the *total area* of the circle.

Put mathematically, if we can derive an expression for the *differential* (read this, "small in comparison to the whole") *area dA* of our *hoop*, integrate that expression to sum over all possible hoop areas, then evaluate that integral's solution between $r = 0$ to $r = R$, we will end up with an expression for the total area of the circle.

d.) Executing that operation:

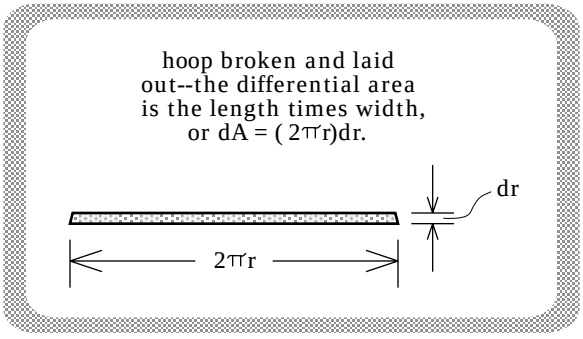


FIGURE 3.6

i.) If *dr* is tiny, the *differential area* of the hoop is equal to the *circumference of the hoop (2πr)* times the hoop's thickness *dr* (see Figure 3.6 to the right).

That is, $dA = (2\pi r)dr$.

ii.) Summing yields:

$$\begin{aligned} A = \int dA &= \int_{r=0}^{R} (2\pi r)\,dr \\ &= 2\pi \int_{r=0}^{R} (r)\,dr \\ &= 2\pi \left[\frac{r^2}{2} \right]_{r=0}^{R} \\ &= 2\pi \left[\left(\frac{R^2}{2} \right) - (0) \right] \\ &= \pi R^2. \end{aligned}$$

What fun!

QUESTIONS

As was the case with Chapter 2, there are no questions for this section. If you understand this chapter's material, fine. If not, don't panic. It will be a while before we do anything too exotic with integrals.

Note 1: The most important point in this chapter is the idea that problems can be set up by attributing physical significance to differential quantities like *dx* and *dt*. Approaching problems in this way allows one to create *differential equations* that physically reflect the situations one is trying to model.

Note 2: Approaching Calculus with the idea that the *dt's* and *dx's* have significance is particularly useful when one is trying to make sense of a differential equation. An astute reader can find within such an expression meaning that is not at all obvious within a so-called *pure math* context.

Chapter 4

ELEMENTARY DEFINITIONS and THE KINEMATIC EQUATIONS

A.) Speed:

1.) Average speed (s_{avg}): A scalar quantity that denotes the *average distance traversed per unit time* (i.e., the average rate at which ground is covered). With units of *meters per second*, it is mathematically defined as:

$$s_{avg} = \frac{\Delta d}{\Delta t},$$

where Δd is the total-distance-traveled during a time interval Δt.

a.) Example: A running woman covers 100 meters in 15 seconds, then changes direction and hops 30 meters in 10 seconds (see Figure 4.1). What is her *average speed* for the overall motion?

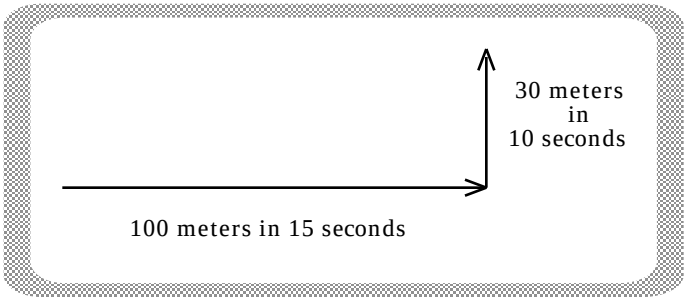


FIGURE 4.1

$s_{avg} = \Delta d / \Delta t$
$= (130 \text{ m}) / (25 \text{ sec})$
$= 5.2 \text{ m/s}.$

Note 1: What this average gives you is the SINGLE CONSTANT SPEED that will move the woman over the required distance (130 meters) in the required time (25 seconds). It does not tell you how fast she is actually traveling at any given instant. She could run the first 80 meters in 10 seconds, then stand panting for 2 seconds, then do the last 20 meters of the first leg in the remaining 3 seconds. *Average speed* tells you nothing about the actual motion; all it tells you is the single speed that would be required to go the distance at a uniform run in the allocated time.

Note 2: *Speed* is not a quantity physicists use very much. It is being presented here as a preamble to more interesting and useful variables to come.

2.) Instantaneous speed (s): A measure of an object's *distance traveled per unit time* (i.e., its rate of travel), measured at a particular point in time.

a.) Example: The running woman in Example 1a is found to be moving with a speed of 8 m/s as she passes the 15 meter mark, three tenths of a second into the race. Her *instantaneous speed* at the 15 meter mark is, therefore, 8 m/s.

b.) Mathematically, *instantaneous speed* (referred to simply as *speed* from here on) is defined as:

$$s = \text{limit}_{\Delta t \Rightarrow 0} \left(\Delta d / \Delta t \right).$$

Note: Translation: At a particular point in time, an object's *instantaneous speed* is equal to its *average speed* calculated over a very tiny time interval (i.e., as Δt approaches zero). Although this is technically a Calculus problem (we are actually looking at the *time derivative* of the distance function), it will not be written in that form. Again, the idea of *speed* is useful as a concept only. We will rarely use it as a mathematical entity.

B.) Velocity–Magnitude and Direction:

1.) Average velocity ($\mathbf{v}_{avg}$): A *vector* quantity that denotes the *average displacement* (i.e., the *net resultant* change of position) *per unit time* over some large time interval. With units of *meters per second* (normally written as m/s), it is mathematically defined as:

$$\mathbf{v}_{avg} = \frac{\Delta \mathbf{r}}{\Delta t},$$

where $\Delta \boldsymbol{r}$ is the NET DISPLACEMENT of the object during a time interval Δt. The direction of $\boldsymbol{v}_{avg}$ is the same as the direction of $\Delta \boldsymbol{r}$, (i.e., that of the *net displacement*).

a.) Example: A woman covers 100 meters in 15 seconds, then changes direction and hops 30 meters in 10 seconds (see Figure 4.2). What is her *average velocity* for the overall motion?

The magnitude:

$$\left|\mathbf{v}_{avg}\right| = \left|\frac{\Delta \mathbf{r}}{\Delta t}\right|$$
$$= (104.4 \text{ m})/(25 \text{ sec})$$
$$= 4.176 \text{ m/s}.$$

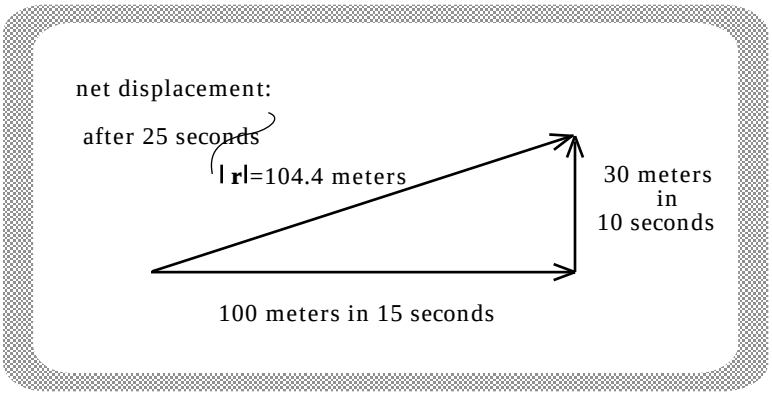


FIGURE 4.2

The direction (using trig. and the sketch):

$$\phi = \tan^{-1}[(30 \text{ m})/(100 \text{ m})]$$
$$= 16.7^{o}.$$

As a vector:

$$\mathbf{v}_{avg} = 4.176 \text{ m/s} \angle 16.7^{o}.$$

Note 1: This average value gives you the constant number of meters-per-second, moving DIRECTLY from the initial to the final position, required to effect the net displacement in the allotted time. As was the case with average speed, it does not reflect the *actual* velocity of the woman at any particular instant.

Note 2: *Average velocity* is not a quantity physicists use very much, but *instantaneous velocity* is!

2.) Instantaneous velocity (**v**): A measure of an object's *displacement per unit time* as measured at a particular point in time.

a.) Mathematically, *instantaneous velocity* (referred to as *velocity* from here on) is defined as:

$$\mathbf{v} = \text{limit}_{\Delta t \Rightarrow 0} \left(\Delta \mathbf{r} / \Delta t\right)$$

where the direction of $\Delta \boldsymbol{r}$ is the direction of motion at a given instant. As this is the definition of a derivative, we can write the relationship as:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

b.) Example: A body's position function is $\boldsymbol{r}(t) = [(7k_1t^3)\boldsymbol{i} - (4k_2/t)\boldsymbol{j}]$ *meters* (the *k's* are added for the sake of units--we will ignore them from here on--i.e., set their magnitudes at *one*). What is $\boldsymbol{v}(t)$?

Solution:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= \frac{d\left[(7t^3)\mathbf{i} - (4/t)\mathbf{j}\right]}{dt} \ \frac{\text{meters}}{\text{sec}}$$

$$= \left[(21t^2)\mathbf{i} + (4/t^2)\mathbf{j}\right] \text{ m/s.}$$

Note: THIS IS IMPORTANT. The *sign* of the velocity tells you the direction of motion of the body at a particular point in time. That is, if the velocity is *-3**j** m/s*, the body is moving in the *-y direction*.

3.) *Velocity* and the *POSITION vs. TIME* Graph:

a.) Consider the *POSITION vs. TIME* graph shown in Figure 4.3 to the right. *The slope of the tangent to the curve at a time t_1* gives us *the change of position with time at that point* (i.e., the velocity at that point). By definition, that slope equals the derivative of the position function (*dx/dt*) evaluated at t_1.

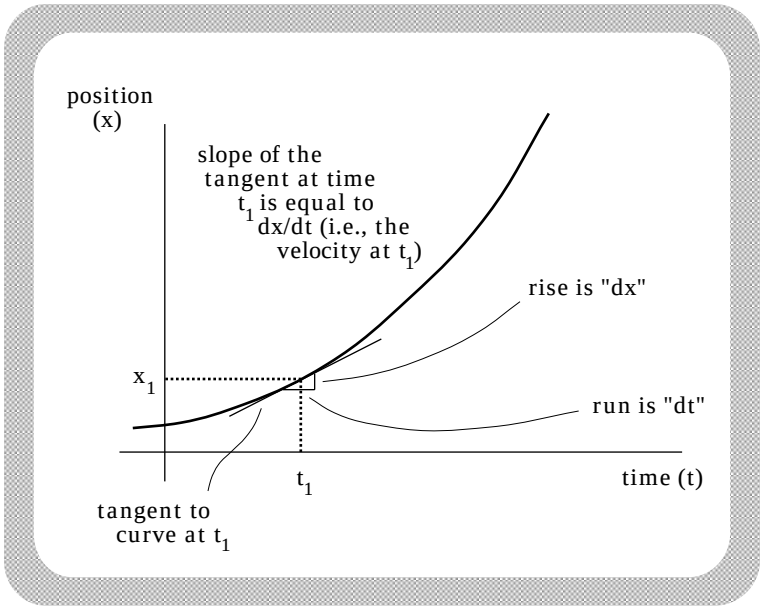


FIGURE 4.3

b.) Bottom line on *POSITION vs. TIME* graphs: To get the instantaneous velocity of a body whose POSITION vs. TIME graph is given but whose position *x(t)* is not explicitly known, draw a tangent to the curve at the time of interest, then determine the slope of that tangent. The slope will numerically equal the velocity of the body at that point in time.

4.) *Displacement* and the *VELOCITY vs. TIME* Graph:

a.) As was discussed in Chapter 3, the area under a *Velocity vs. Time* graph equals the *net displacement* of a body over the time interval in question (see Figure 4.4).

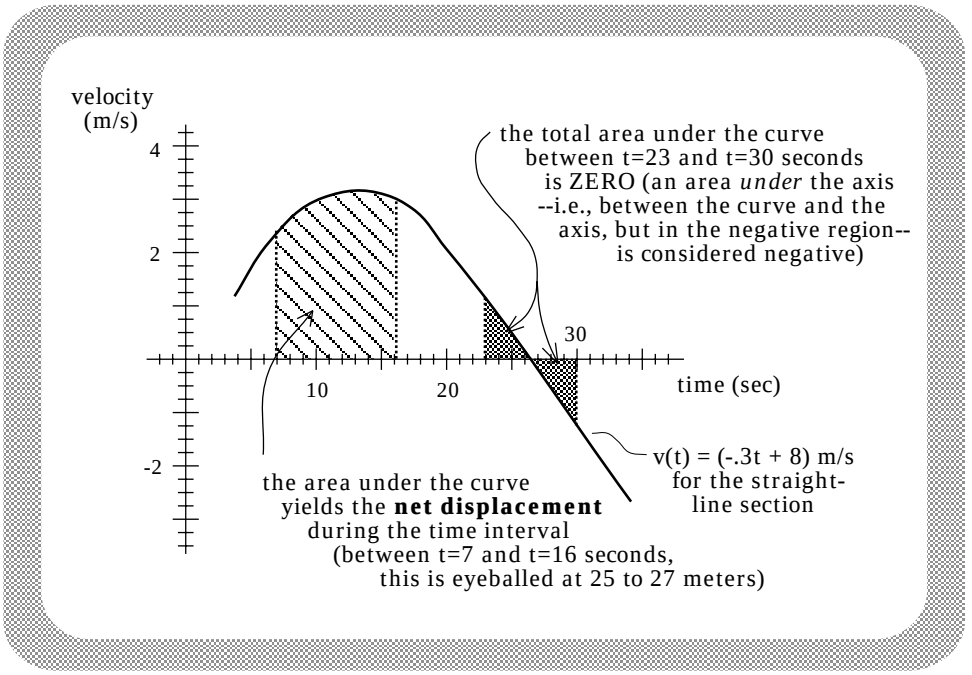

velocity (m/s)
4
2
-2
10
20
30
time (sec)
the total area under the curve between t=23 and t=30 seconds is ZERO (an area *under* the axis --i.e., between the curve and the axis, but in the negative region-- is considered negative)
v(t) = (-.3t + 8) m/s for the straight-line section
the area under the curve yields the **net displacement** during the time interval (between t=7 and t=16 seconds, this is eyeballed at 25 to 27 meters)

FIGURE 4.4

b.) In general, if we are given a velocity curve without an explicit function for the velocity (i.e., *v(t)*), we can find the *distance traveled* (i.e., Δx) by eyeballing the area under the curve over the time interval.

Note: Velocities under the axis (i.e., in the negative region) denote motion in the negative direction. That means a displacement associated with an area found *under the axis* is associated with *negative* displacement.

c.) If one knows the velocity function *v(t)*, the area under the curve (i.e., Δx) can be determined by integrating the velocity function over the time interval (i.e., by executing $\int$(v)dt).

d.) Example from the graph: Approximating the velocity function that defines the *straight-line section* of the graph as presented below:

$$v(t) = (-.3t + 8) \text{ m/s},$$

what is the area under the curve between times *t = 23 seconds* and *t = 30 seconds*? That is the same as asking, "What is the *net displacement* of the body during the time interval?"

Note: If our graph is to be believed, the answer to this question had better be zero! How so? The displacement between *t = 23 seconds* and *t = 26.5 seconds* is positive (that is, the *velocity* is positive so the body's *displacement* is in the *positive x direction*). On the other hand, the displacement between *t = 26.5 seconds* and *t = 30 seconds* is negative. Due to the symmetry of the situation, the two areas had better add up to zero.

i.) Integrating to determine the area under the curve between the times t_1 and t_2, we get:

$$\begin{aligned}
\Delta x &= \int_{t=23}^{30} (-.3t + 8)dt \\
&= \int_{t=23}^{30} (-.3t)dt + \int_{t=23}^{30} (8)dt \\
&= \left[-.3\left(\frac{t^2}{2}\right) + 8t \right]_{t=23\ \text{sec}}^{30} \\
&= \left[-.3\left(\frac{(30)^2}{2}\right) + 8(30) \right] - \left[-.3\left(\frac{(23)^2}{2}\right) + 8(23) \right] \\
&= 105 - 104.65 \\
&= 0.35 \\
&\approx 0.
\end{aligned}$$

C.) Acceleration--Magnitude and Direction:

1.) Average acceleration ($\mathbf{a}_{avg}$): A *vector* quantity that denotes the average *change-of-velocity per unit time* over some large time interval. Its units are *meters per second per second* (usually written as m/s^2--see Note #2 below), and its mathematical definition is:

$$\mathbf{a}_{avg} = \frac{\Delta \mathbf{v}}{\Delta t},$$

where $\Delta \boldsymbol{v}$ is the net *change of velocity* during a time interval Δt.

The *direction* of $\boldsymbol{a}_{avg}$ is the same as that of $\Delta \boldsymbol{v}$.

Note 1: Although it may not be obvious now, the *sign* of an acceleration value tells us information that is not immediately obvious. Explanation later!

Note 2: (concerning acceleration's units): The fraction (1/3)/3 can be re-written as $\frac{(1/3)}{(3/1)}$. Bringing the denominator up into the numerator by flipping it over and multiplying, we get (1/3)(1/3), or $1/3^2$. By the same token, as acceleration measures the rate at which velocity changes per unit time, its units are the ratio (m/s)/s. Being analogous to (1/3)/3, this can be written as m/s^2.

a.) Example: A man has velocity $\boldsymbol{v_1} = (3\ m/s)\boldsymbol{i}$ when at x_1. Three seconds later, he is at x_2 moving with velocity $\boldsymbol{v_2} = (9\ m/s)\boldsymbol{i}$ (see Figure 4.5). What is his *average acceleration*?

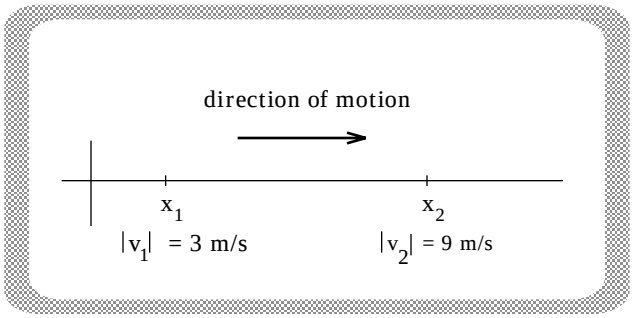


FIGURE 4.5

$$\begin{aligned}\mathbf{a}_{avg} &= \Delta\mathbf{v} / \Delta t \\ &= (\mathbf{v}_2 - \mathbf{v}_1) / (\Delta t) \\ &= (9\ m/s - 3\ m/s)\mathbf{i} / (3\ sec) \\ &= (2\ \mathbf{i})\ m/s^2.\end{aligned}$$

2.) Instantaneous acceleration (**a**): A measure of an object's *change-of-velocity per unit time* at a particular point in time.

a.) Mathematically, instantaneous acceleration (referred to as *acceleration* from here on) is defined as:

$$\mathbf{a} = \text{limit}_{\Delta t \Rightarrow 0} \left(\Delta\mathbf{v} / \Delta t \right),$$

where the direction of $\Delta \boldsymbol{v}$ is the direction of the net force acting on the body at a given instant. As this is the definition of a derivative, we can write the relationship as:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}.$$

b.) Example: If a body's velocity is $\boldsymbol{v}(t) = [(21t^2)\boldsymbol{i} + (4/t^2)\boldsymbol{j}]\ m/s$, (ignoring units-constants) what is the body's acceleration $\boldsymbol{a}(t)$?

Solution:

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d\left[(21t^2)\mathbf{i} + (4/t^2)\mathbf{j}\right]}{dt}\left(\frac{m/s}{s}\right) \\ &= \left[(42t)\mathbf{i} - (8/t^3)\mathbf{j}\right] m/s^2.\end{aligned}$$

3.) *Acceleration* and the *VELOCITY vs. TIME* Graph:

a.) Consider the *VELOCITY vs. TIME* graph shown in Figure 4.6 to the right. The *slope of the tangent to the curve* at time t_1 is the *change of velocity with time* at that point (i.e., the acceleration at that point). By definition, that slope equals the derivative of the velocity function (dv/dt), evaluated at t_1.

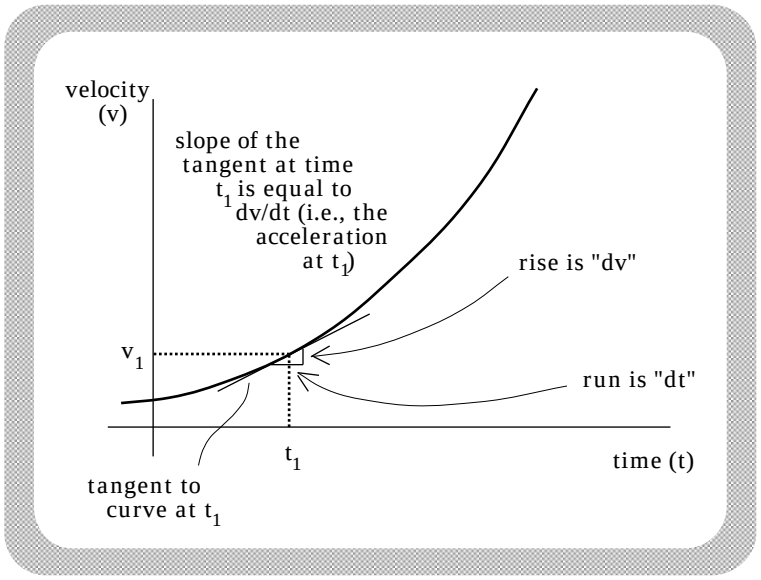


FIGURE 4.6

b.) Bottom line on *VELOCITY vs. TIME* graphs: To get the *instantaneous acceleration* of a body whose *VELOCITY vs. TIME* graph is given but whose velocity function *v(t)* is not explicitly known, draw a tangent to the curve at the time of interest, then determine the *slope of that tangent*. The slope will be numerically equal to the acceleration of the body at that point in time.

Note: If we have an *ACCELERATION vs. TIME* graph, the area under the graph between times t_1 and t_2 equals the *velocity change* during that time interval, and the *slope of the tangent* to the graph defines the *rate of change of acceleration with time*. This latter quantity is called *the jerk* of the motion (tough to believe, but true).

D.) Sign Significance for VELOCITY and ACCELERATION:

1.) Sign of the *VELOCITY* vector:

a.) It has already been noted that the direction of the *velocity* vector is the same as the direction of motion. A quick example follows:

b.) Consider an object moving along the *x axis*. It is initially found at $x_3 = 3$ *meters*; three seconds later it is found at $x_4 = -7$ *meters* (see Figure 4.7). What is the *average velocity* of the motion?

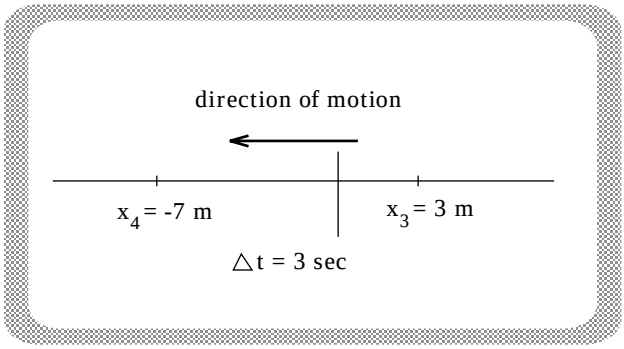


FIGURE 4.7

Solution:

$$\mathbf{v}_{avg} = \Delta\mathbf{r} / \Delta t$$
$$= \Delta\mathbf{x} / \Delta t$$
$$= (x_4 - x_3)\mathbf{i} / \Delta t$$
$$= [(-7\text{ m}) - (+3\text{ m})]\mathbf{i} / (3\text{ sec})$$
$$= (-3.33\ \mathbf{i})\text{ m/s}.$$

Note: It is important to include *negative signs* when dealing with position variables.

c.) When the body is moving in the *-x direction*, the direction of the velocity vector is, indeed, in the *-**i** direction*.

Note: Even though -30 is smaller than +2 on a number line, the *sign of a velocity* has nothing to do with *magnitude* (i.e., how fast you are going)--all it tells you is *which way* you are going. (To see this: which would you prefer-- to be hit by a car moving with a velocity of +2 m/s or -30 m/s?)

2.) Sign of the *ACCELERATION* vector:

Note: Warning! You are about to find that the information wrapped up in the *sign* of an *acceleration* quantity is considerably more complicated than the information wrapped up in the sign of a *velocity* quantity.

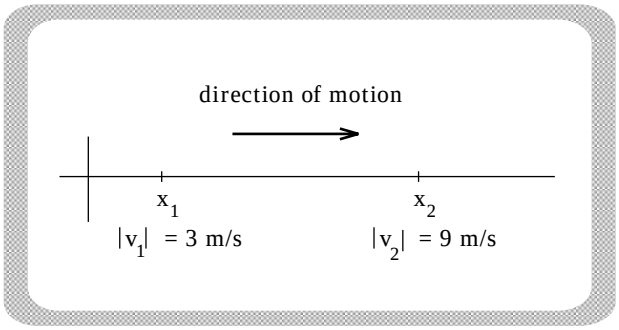


FIGURE 4.8

a.) A woman finds she is moving in the *+x direction* with velocity $v_1 = 3$

m/s. Three seconds later, she is moving with velocity $v_2 = 9$ *m/s* (see Figure 4.8 on the previous page). What is her *average acceleration*?

Note: Because we are working in one dimension only, we will not bother carrying the unit vector ***i*** along in the calculation.

$$\begin{aligned}\mathbf{a}_{avg} &= \Delta\mathbf{v} / \Delta t \\ &= (v_{sec\ pt} - v_{first\ pt}) / (\Delta t) \\ &= (v_2 - v_1) / (\Delta t) \\ &= (9\ m/s - 3\ m/s)/(3\ sec) \\ &= +2\ m/s^2.\end{aligned}$$

Observation 1: An individual *speeding up* while moving in the *+x direction* has a *POSITIVE acceleration*.

b.) A woman finds she is moving in the *+x direction* with velocity $v_3 = 9$ *m/s*. Three seconds later, she is moving with velocity $v_4 = 3$ *m/s* (see Figure 4.9). What is her average acceleration?

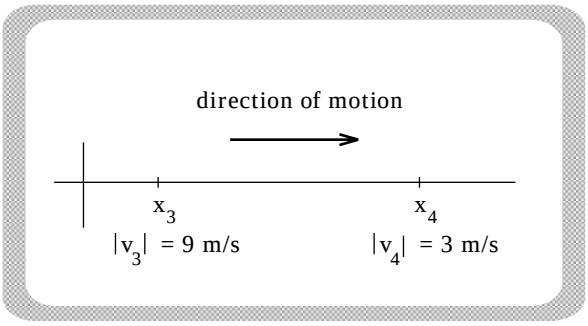


FIGURE 4.9

$$\begin{aligned}\mathbf{a}_{avg} &= \Delta\mathbf{v} / \Delta t \\ &= (v_4 - v_3) / (\Delta t) \\ &= (3\ m/s - 9\ m/s)/(3\ sec) \\ &= -2\ m/s^2.\end{aligned}$$

Observation 2: An individual *slowing down* while moving in the *+x direction* has a *NEGATIVE acceleration*.

Note: The combination of Observations 1 and 2 normally leads people to believe that *speeding up* is associated with positive acceleration (often referred to simply as acceleration) and *slowing down* is associated with negative acceleration (often called deceleration). THIS IS NOT ALWAYS THE CASE, as will be shown below.

c.) A woman moves in the *-x direction* with velocity $v_5 = 3$ *m/s*. Three seconds later, she is found to be moving at velocity $v_6 = 9$ *m/s*. What is her average acceleration?

Note: THERE IS SOMETHING RADICALLY WRONG WITH THE STATEMENT OF THIS PROBLEM. Can you find the error?

The problem should be stated: A woman finds she is moving in the *-x direction* with velocity $v_5 = -3$ *m/s*. Three seconds later, she is found to be moving at velocity $v_6 = -9$ *m/s* (see Figure 4.10). What is her *average acceleration*?

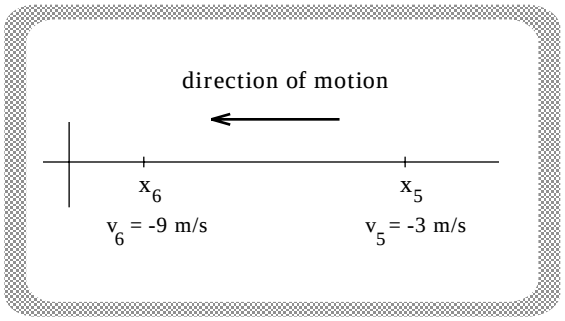


FIGURE 4.10

Note: The "RADICAL PROBLEM" alluded to above has to do with signs. BE CAREFUL WITH YOUR SIGNS; YOU WILL RARELY IF EVER WORK WITH TRUE, SIGNLESS MAGNITUDES. The sign of the *velocity* of an object moving in the *-x direction* is *negative*!

Solving the problem:

$$\begin{aligned} \mathbf{a}_{avg} &= \Delta \mathbf{v} / \Delta t \\ &= (v_6 - v_5) / (\Delta t) \\ &= [(-9 \text{ m/s}) - (-3 \text{ m/s})]/(3 \text{ sec}) \\ &= -2 \text{ m/s}^2. \end{aligned}$$

Observation 3: Here we have a NEGATIVE ACCELERATION, but the woman isn't slowing down--she's *speeding up*.

Likewise, if the woman is moving in the *-x direction* with velocity $v_7 = -9$ *m/s* and, three seconds later, finds herself moving at $v_8 = -3$ *m/s*, her *acceleration* will be calculated as $+2$ m/s^2. This is a POSITIVE acceleration associated with a *slow-down*.

d.) Bottom line:

i.) For $+x$ motion (i.e., positive velocity):

+ avg. acc. ⇒ increase of speed
- avg. acc. ⇒ decrease of speed.

ii.) For $-x$ motion (i.e., negative velocity)

+ avg. acc. ⇒ decrease of speed
- avg. acc. ⇒ increase of speed.

e.) Conclusion? When an object's *velocity* and *acceleration* have the same sign (i.e., are in the same direction), the body will physically speed up. When an object's *velocity* and *acceleration* have different signs, the body will slow down.

i.) In a way, this makes perfect sense. Acceleration comes only when a *net force* is applied to a body (that is, acceleration and *net force* are proportional to one another). A positive force (i.e., a *net force* directed in the *positive direction*) produces positive acceleration no matter what the velocity is. By the same token, negative force always produces negative acceleration.

ii.) In other words, if a body moving in the *-x direction* has a positive force applied to it (see Figure 4.11a to the right), we would expect the body to *slow down*. This is exactly what a POSITIVE ACCELERATION does. Likewise, you would expect the body to speed up if a negative force, hence negative acceleration, were applied (see Figure 4.11b).

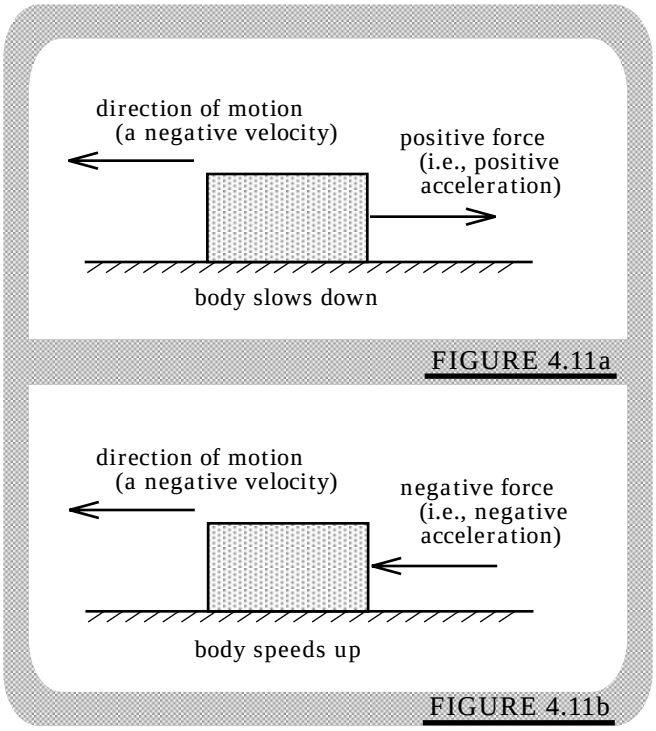


FIGURE 4.11a

FIGURE 4.11b

iii.) Bottom line: In both cases, our "like-directions-cause-speed-up, unlike-directions-cause-slow-down" observation is reasonable.

E.) The Kinematic Equations:

1.) To this point, we have dealt with general *position*, *velocity*, and *acceleration* functions. A special case occurs when a body is constrained to move with a CONSTANT *acceleration*.

Note: There are many constant-acceleration systems within nature. As an example: The gravitational freefall of an object near the earth's surface.

2.) With a constant acceleration, there are a number of equations that can be written that make problem-solving much easier. Collectively, these relationships are called *the kinematic equations*. They are summarized below for one dimensional motion with explanations and derivations to follow:

a.) $x_2 = x_1 + v_1 \Delta t + (1/2)a(\Delta t)^2$:

i.) This states that after a time period (Δt) of constant acceleration a, an object's coordinate position x_2 equals:

ii.) Its initial position x_1 (i.e., its position at the beginning of the time interval--this initial time is usually called t_1), plus;

iii.) The *change* of position $v_1 \Delta t$ due to the fact that the body has an initial velocity (i.e., v_1) at the beginning of the time interval, plus;

iv.) The additional position change $(1/2)a(\Delta t)^2$ that occurs due to the body's acceleration.

b.) $v_2 = v_1 + a \Delta t$:

i.) This states that a body's velocity v_2 after an interval Δt of constant acceleration a equals:

ii.) The body's velocity v_1 (i.e., its velocity at the beginning of the period), plus;

iii.) The increase or decrease of velocity $a \Delta t$ due to the body's acceleration.

c.) $(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1)$:

i.) This states that the square of a body's velocity at time t_2 (i.e., v_2) after an interval of constant acceleration a equals:

ii.) The square of the body's velocity v_1 at the beginning of the time interval (i.e., at t_1), plus;

iii.) 2 *times* the acceleration (a) *times* the change of position Δx.

d.) $x_2 = x_1 + v_{avg}\Delta t$:

i.) This states that an object's coordinate position x_2 after a period (Δt) of motion during which the average velocity has been v_{avg} equals:

ii.) The body's initial position x_1 (i.e., its position at the start of the time interval at time t_1), plus;

iii.) The additional displacement $v_{avg}\Delta t$ due to the body's motion during the time interval.

e.) $v_{avg} = (v_1 + v_2)/2$:

i.) Assuming the velocity function is linear (i.e., the acceleration is a constant), the average velocity v_{avg} between times t_1 and t_2 will simply equal the sum of the end velocities (v_1 plus v_2) divided by two.

2.) Why do we want these equations? There are times when we know, say, a body's *final velocity*, *acceleration*, and *time of acceleration*, and would like to know its *initial velocity*. We could use Calculus on the problem, but why go to all the bother when we have a kinematic equation ($v_2 = v_1 + a\Delta t$) that has all the variables we know along with the variable we are trying to determine? In other words, there are circumstances when we can short-cut the Calculus by simply using the CONSTANT ACCELERATION equations that follow from the calculus (you'll see how they follow shortly).

Note concerning the following material: The following derivations are provided so that you will have some clue as to what the variables in the kinematic equations stand for and why they relate to one another as they do. You will be expected to understand the concepts outlined below, but you will <u>not</u> be asked to reproduce the derivations. In other words, skim this material.

3.) Derivation of $v_2 = v_1 + a\Delta t$:

a.) Assume a body moves in *one-dimensional motion* under the influence of a *constant* acceleration a (as this is a one-dimensional situation, we will drop the unit vector notation). Additionally, assume that:

i.) At some initial point in time t_1, the body is positioned at x_1 and is found to be moving with velocity v_1; and

ii.) Later, at some arbitrary time t_2, the body is positioned at x_2 and is found to be moving with velocity v_2.

b.) We know the *acceleration* is the *time derivative of the velocity* function, which means we can write:

$$a = \frac{dv}{dt}$$
$$\Rightarrow dv = a(dt).$$

c.) This essentially says that a *differential* ("differential" meaning *very small*)*velocity change dv* over a *differential time interval dt* will equal the constant *rate at which the velocity changes* (i.e., the acceleration *a*) times the *time interval dt* over which the change occurs.

We can sum the velocity changes (i.e., integrate) between times t_1 and t_2. Noting that the acceleration a is a constant and, hence, can be pulled outside the integral, we can write:

$$\int_{v_1}^{v_2} dv = a\int_{t_1}^{t_2} dt$$
$$\Rightarrow v\Big|_{v_1}^{v_2} = a\big[t\big]_{t_1}^{t_2}$$
$$\Rightarrow v_2 - v_1 = a(t_2 - t_1)$$
$$\Rightarrow v_2 = v_1 + a(\Delta t) \qquad \text{(Equation A).}$$

Note 1: It is not unusual to find this expression written in physics books as $v_2 = v_1 + at$, where the Δt has mysteriously become, simply, t. This bit of magic is justified as follows:

If the clock begins at $t_1 = 0$ and proceeds to some arbitrary time $t_2 = t$, the *change in time* is $\Delta t = (t_2 - t_1) = (t - 0) = t$. When this is incorporated into our equation, $v_2 = v_1 + a\,\Delta t$ becomes $v_2 = v_1 + at$.

Observation: This is *very sloppy* notation, using what looks like a particular *point in time t* in place of the time interval that belongs in the equation. Nevertheless, that is the way most physics books write it.

The moral? Be aware of what symbols mean so as not to be led astray.

Note 2: This equation is often presented as:

$$a = \frac{v_2 - v_1}{\Delta t}.$$

Note 3: This is the definition of the *average acceleration*. That makes sense. If the acceleration is constant in a system, the *average acceleration* and the *instantaneous acceleration* will be numerically equal.

4.) Derivation of $x_2 = x_1 + v_1 \Delta t + (1/2)a(\Delta t)^2$:

a.) If we assume our clock *starts* at t_1 (i.e., $t_1 = 0$), and if we set t_2 equal to an arbitrary time t so that $v_2 = v(t)$, we can rewrite *Equation A* as:

$$v(t) = v_1 + at.$$

b.) We know that the velocity $v(t)$ is the *time derivative* of the displacement function (dx/dt), which means we can write:

$$v(t) = \frac{dx}{dt}.$$

c.) Combining the two equations above we get:

$$\frac{dx}{dt} = v_1 + at.$$

d.) Briefly manipulating (i.e., not showing all the steps) yields:

$$\begin{aligned}
&\frac{dx}{dt} = v_1 + at \\
&\quad \Rightarrow dx = (v_1 + at)dt \\
&\qquad \Rightarrow \int_{x_1}^{x_2} dx = \int_{t=0}^{t} (v_1 + at)dt \\
&\qquad\qquad = \int_{t=0}^{t} (v_1)dt + \int_{t=0}^{t} (at)dt \\
&\quad \Rightarrow \; x_2 - x_1 = v_1 t + \frac{1}{2} a t^2 \\
&\qquad \Rightarrow \quad x_2 = x_1 + v_1 t + \frac{1}{2} a t^2.
\end{aligned}$$

e.) Remembering that the t variable is really a Δt and $x_2 - x_1$ is Δx, this equation has an interesting graphical link. Knowing that the area under a *VELOCITY vs. TIME* graph is related to the distance traveled Δx during a given time interval, we can determine Δx using the geometry of the constant acceleration polygon shown in Figure 4.12. Doing so yields an expression that is exactly the same as the kinematics expression derived above.

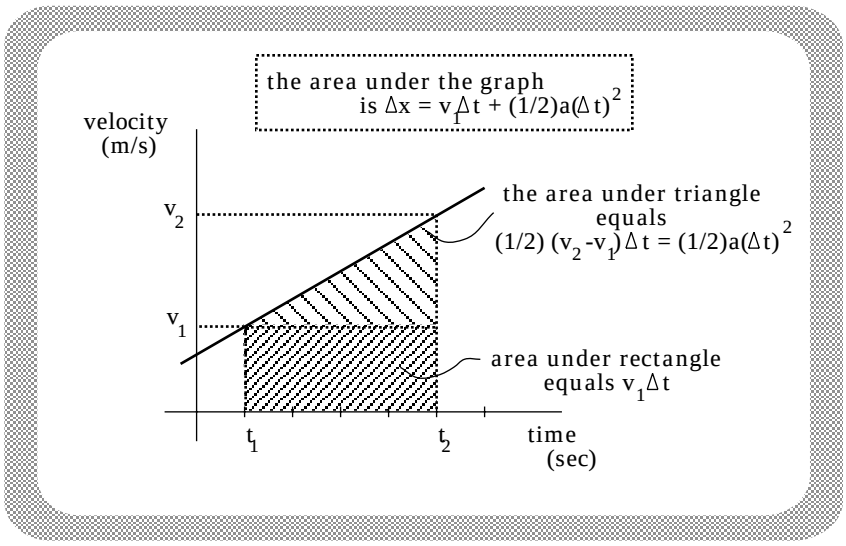


FIGURE 4.12

5.) Derivation of the expression $v_{avg} = (v_1+v_2)/2$:

a.) Looking at the graph in Figure 4.13, it can be seen that if the acceleration is constant (i.e., the velocity is a linear function), the average velocity of the body over a time interval Δt defined by the expression $\Delta t = t_2 - t_1$ is

$$v_{avg} = \frac{v_1 + v_2}{2},$$

where v_1 and v_2 are the initial and final velocities over the interval. <u>This expression is *very rarely used*</u> but will be useful in a derivation that follows.

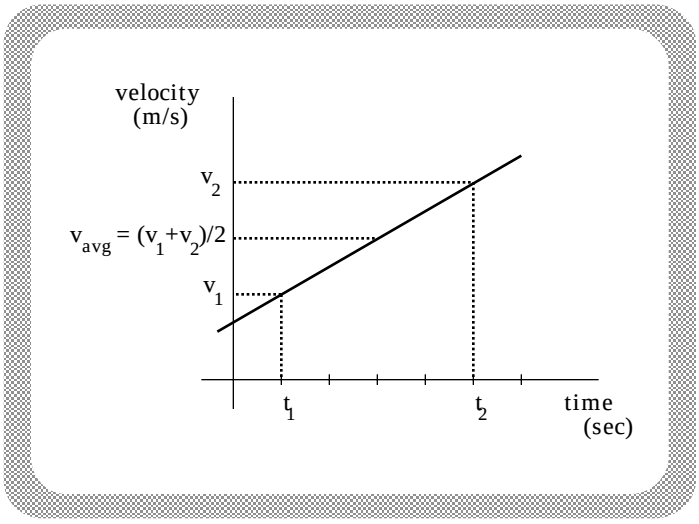


FIGURE 4.13

6.) Derivation of $\Delta x = v_{avg} \Delta t$:

a.) This is the old, "distance equals rate times time" equation you learned in the sixth grade with the *distance* term expressed as Δx and the *rate* term expressed as v_{avg}. Written in this notation, we get:

$$\Delta x = v_{avg} \Delta t.$$

Note: As is the case with all expressions having v_{avg} in them, this equation is very rarely used in the context of problem-solving.

7.) Derivation of $v_2^2 = v_1^2 + 2a \Delta x$:

a.) We can eliminate v_{avg} from $x_2 = x_1 + v_{avg} \Delta t$ using $v_{avg} = (v_2+v_1)/2$. Doing so yields:

$$x_2 = x_1 + [(v_2+v_1)/2]\Delta t.$$

b.) Using $v_2 - v_1 = a \Delta t$, we can solve for Δt, finding:

$$\Delta t = (v_2 - v_1)/a.$$

c.) Putting the equations from *Parts a* and *b* together, we get:

$$x_2 - x_1 = [(v_2+v_1)/2][(v_2 - v_1)/a].$$

d.) Putting $x_2 - x_1 = \Delta x$ and manipulating, we can reduce this to:

$$v_2^2 = v_1^2 + 2a\Delta x.$$

8.) A re-statement of the kinematic equations is presented below:

$(x_2 - x_1) = v_1 \Delta t + (1/2)a(\Delta t)^2$ (used often)

$a = (v_2 - v_1)/\Delta t$ or $v_2 = v_1 + a \Delta t$ (used often)

$(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1)$ (used often)

$(x_2 - x_1) = v_{avg} \Delta t$ or $v_{avg} = (x_2 - x_1)/\Delta t$ (rarely used)

$$v_{avg} = (v_2 + v_1)/2 \qquad \text{(rarely used).}$$

Note 1: Be careful about signs. As an example, an object moving from $x_1 = -3$ *meters* to $x_2 = -5$ *meters* does *not* have a displacement (i.e., Δx) of 2 meters. Following the math, we get $\Delta x = (x_2 - x_1) = [(-5) - (-3)] = -2$ *meters*.

Displacement is a vector; *negative* displacement means the body is moving to the left. Signs matter! Be careful with them. (The same is true whenever using velocity parameters in the equations!)

Note 2: Be sure your use of the kinematic equations is legitimate. If you are not sure whether the *acceleration* is constant, don't use them.

F.) The Kinematic Equations--Some One-Liners:

1.) A Porsche whose initial velocity is 20 m/s accelerates at 5 m/s^2 for three seconds. What is its velocity at the end of that time period?

Solution: We know the initial velocity, the constant acceleration, and the time interval over which the acceleration occurred. The equation that includes the variables we know along with the variable we need is $a = (v_2 - v_1)/\Delta t$. Using it, we get:

$$a = (v_2 - v_1)/\Delta t,$$

or $$(5\ m/s^2) = [v_2 - (20\ m/s)] / (3\ sec)$$

$$\Rightarrow \quad v_2 = 35\ m/s.$$

2.) When our Porsche is *20 meters* to the left of a stop sign (i.e., on the *negative* side of an axis placed at the sign), it is moving with velocity of *30 m/s*. If it accelerates at a rate of $-20\ m/s^2$, how fast will it be going when at $x = -10\ m$?

Solution: We know the initial and final positions and velocities. The relationship that will do it for us is $(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1)$. Using it yields:

$$(v_2)^2 = (v_1)^2 + 2a[x_2 - x_1]$$

$$(v_2)^2 = (30\ m)^2 + 2(-20\ m/s^2)\,[(-10\ m) - (-20\ m)]$$

$$\Rightarrow \quad v_2 = \pm 22.36\ m/s.$$

Note: This kinematic equation doesn't understand *how* your Porsche is slowing down. One possibility is that you hit the brakes when at $x = -20$ *meters*

(i.e., 20 meters to the left of the origin) and slide with a positive velocity (i.e., a velocity that moves to the right) through the *x = -10 meters* point. Another possibility is that you put the Porsche into reverse while moving in the positive direction and floor it. (This is a really dumb way to slow a car down, but it will do the trick provided you don't blow your transmission in the process.) In that case, you will slide through the *x = -10 meters* point on your way to a dead stop. The difference is that the car will then begin to move backwards in the negative direction passing through the *x = -10 meters* again but with a *negative* velocity.

The kinematic relationship you are using can't differentiate between any of these scenarios, so it deals with the problem from a purely mathematical standpoint and solves for the car's *two* potential velocities (one positive, one negative) at the *x = -10 meters* point.

In short, it is up to you to recognize the physical constraints of the problem and decide which sign is appropriate. Additionally, because there is no time parameter in $(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1)$ . . . hence no way to eliminate this ambiguity . . . this is the only kinematic relationship that does not give you the *for sure* correct sign as a part of the velocity term.

3.) In problem 2 above, how long will it take our Porsche to go from *x = -20 meters* to *x = -10 meters*?

Solution: Given the *Note* above, you'd expect *two* possible times to arise. The relationships $(x_2 - x_1) = v_1 \Delta t + (1/2)a(\Delta t)^2$ give us that. Using it yields:

$$[\ (x_2 - x_1\ )] = v_1\ \Delta t + (1/2)\ a\ (\Delta t)^2$$
$$[(-10\text{ m}) - (-20\text{ m})] = (30\text{ m/s})\ t + .5\ (-20\text{ m/s}^2)\ t^2$$
$$\Rightarrow \quad t = .38 \text{ seconds and } 2.62 \text{ seconds.}$$

4.) A dragster capable of accelerating at 12 m/s^2 is given a running start at the beginning of a 400 meter race (i.e., it is allowed an initial velocity v_1). With this initial velocity, it is able to make its run in 6 seconds. What was v_1?

Solution: We know the acceleration, the distance traveled $(x_2 - x_1)$, and the time of travel. To determine the initial velocity:

$$(\ x_2 - x_1) = v_1\ \Delta t + (1/2)\ a\ (\Delta t)^2$$
$$(400\text{ m} - 0) = v_1(6\text{ sec}) + (1/2)(12\text{ m/s}^2)(6\text{ sec})^2$$
$$\Rightarrow \quad v_1 = 30.7\text{ m/s.}$$

5.) A dragster accelerates from rest to 110 m/s in 350 meters. What is its acceleration?

Solution: We know the initial and final velocities and the distance traveled *(x_2 - x_1)*. To get the acceleration, we could use:

$$(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1)$$
$$\Rightarrow \quad a = [\ (v_2)^2 - (v_1)^2] / [2(\ x_2 - x_1)]$$
$$= [(110\ m/s)^2 - (0)^2] / [2(350\ m - 0)\ ]$$
$$= 17.29\ m/s^2.$$

G.) One More One-Dimensional Kinematics Problem--Freefall:

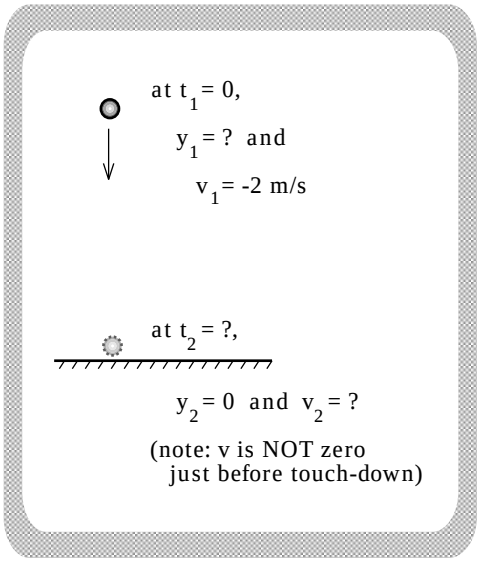


FIGURE 4.14

1.) A ball is thrown downward with an initial velocity of -2 m/s. It takes three seconds to hit the ground (see Figure 4.14). We want to determine: a.) *How high* above the ground was the ball released, and b.) *how fast* was it moving just before it hit the ground?

Solution: We know the initial velocity, the time of flight, and the acceleration (the acceleration of gravity near the earth's surface is ALWAYS approximated as *-g*, or -9.8 m/s^2):

a.) To determine $y_2 - y_1$:

$$(y_2 - y_1) = \quad v_1 \quad \Delta t \quad + (1/2) \quad a \quad (\Delta t)^2$$
$$(\ 0\ - y_1) = (-2m/s)(3\ sec) + (1/2)(-9.8\ m/s^2)(3\ sec)^2$$
$$= -50.1\ meters$$
$$\Rightarrow \quad y_1 = +50.1\ meters.$$

Note 1: Why +50.1 meters instead of -50.1 meters? Because we've placed our coordinate axis so that ground level is *y = 0*. If we had put the axis where the ball became free, our final position would have been *y = -50.1 meters*.

Note 2: Notice how helpful a sketch can be in visualizing a problem. Get into the habit of using sketches whenever you can.

Note 3: A temptation might have been to use $(v_2)^2 = (v_1)^2 + 2a(y_2 - y_1)$. That would be a bad move as the *velocity at ground level* is unknown (no, it is *not* zero--it is equal to the velocity *just before* touchdown).

b.) To determine v_2: With the information we now have, we could determine the velocity just before touchdown in either of two ways:

The first way:
$$(v_2)^2 = (v_1)^2 + 2a(y_2 - y_1)$$
$$= (-2 \text{ m/s})^2 + 2(-9.8 \text{ m/s}^2)(0 \text{ m} - 50.1 \text{ m})$$
$$= 986 \text{ m}^2/\text{s}^2.$$
$$\Rightarrow \quad v_2 = 31.4 \text{ m/s}.$$

Note 1: As the *velocity* quantities are squared in this equation, all *negative signs* are lost in the math and the calculated value of v_2 will be a *magnitude only*. As the velocity is actually directed downward, v_2 as a vector should be written *(31.4 m/s)(-**j**)*, or *(-31.4 m/s)(**j**)*.

Note 2: It is important to notice that the particular kinematic equation used above will always yield VELOCITY MAGNITUDES ONLY.

The second way:
$$v_2 = v_1 + a\,\Delta t$$
$$= (-2 \text{ m/s}) + (-9.8 \text{ m/s}^2)(3 \text{ sec})$$
$$= -31.4 \text{ m/s}.$$

Note 3: As this particular kinematic equation does not square its velocity terms, it yields both *magnitude* and appropriate *sign*. As a vector, the final solution for v_2 using this approach is *(-31.4 m/s)(**j**)*.

H.) Kinematics in Two Dimensions--Projectile Motion:

1.) Background: The net acceleration of a body moving in two dimensions can be written as $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j}$, where the acceleration components a_x and a_y may or may not be the same but are assumed to be constants. A general expression for the body's instantaneous velocity can be written $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$, and a vector defining the body's position can be expressed as $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$.

Having formally defined these quantities, common sense tells us that a net force F_x in the *x direction* (hence an acceleration in the *x direction*) will only affect a body's motion in the *x direction*. As F_x will *not* affect the body's motion in the *y direction*, *x* and *y-type* motion must be independent of one another and must, consequently, be treated as separate entities.

2.) With the observation made above, consider a cannon positioned as shown in Figure 4.15. Its muzzle is oriented at a known angle $\theta = 30^o$ above the horizontal, and its muzzle velocity is known to be $v_1 = 100\ m/s$ (the *muzzle velocity* denotes the magnitude of the projectile's velocity as it leaves the cannon). If the cannonball becomes free at a known height $y_1 = 2\ meters$, and if it lands on a plateau whose height is $y_2 = 80\ meters$, determine:

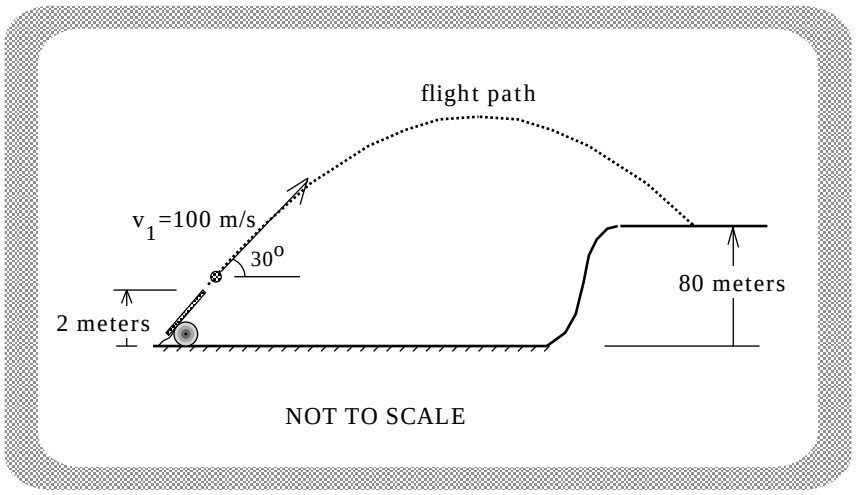


FIGURE 4.15

2a.) The time of flight Δt;

2b.) The final horizontal position x_2 of the ball at touchdown;

2c.) The velocity $\boldsymbol{v}_{top}$ of the cannonball at the top of its flight;

2d.) The cannonball's maximum height y_{top}; and

2e.) The velocity $\boldsymbol{v}_2$ of the cannonball just before its touch-down on the plateau.

3.) Solutions:

a.) Preliminary TIME OF FLIGHT note: Let's assume you have been sent to a point down-range of the cannon (see Figure 4.16a for your positioning). You have been provided with a special flight-sensing-scope that allows you to watch the cannonball's motion as it comes out of the cannon and

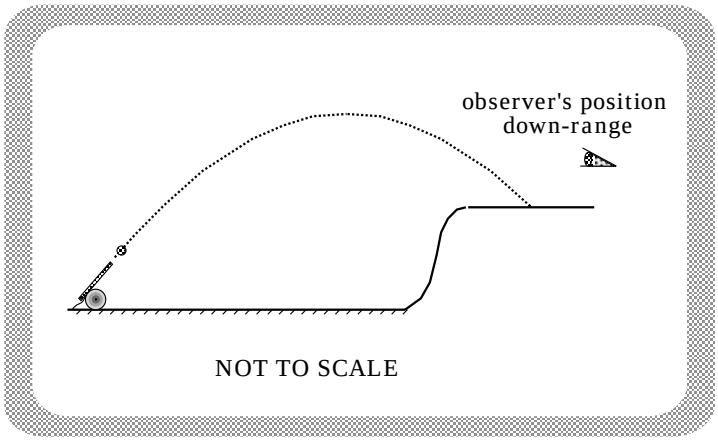


FIGURE 4.16a

proceeds on its path (you are obviously far enough away so you won't get hit by the projectile when it comes down). Additionally, let's assume that the device ruins your depth-perception (that is, you can see the ball but you don't get the feeling that it is coming toward you). From your perspective, how will the cannonball's motion look?

Reflection suggests that the cannonball will appear to rise straight upward, reach some maximum height, stop for a moment, then proceed back down toward the ground (see Figure 4.16b). Further consideration suggests the ball's initial velocity will equal the *y component* of the ball's muzzle velocity *($v_1 \sin\theta = 100 \sin 30 = 50$ m/s)*.

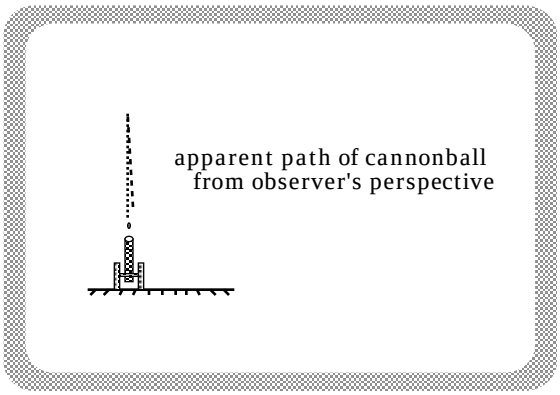


FIGURE 4.16b

From a different perspective, the cannonball's motion will exactly mimic that of a basketball thrown from *y=2 meters* directly upward with velocity 50 m/s released just as the cannonball leaves the muzzle (see Figure 4.16c).

We know how to use our kinematic equations to analyze the one-dimensional motion of a basketball thrown directly upward; we can use those same equations to determine the time-of-flight Δt required for either the basketball or the cannonball.

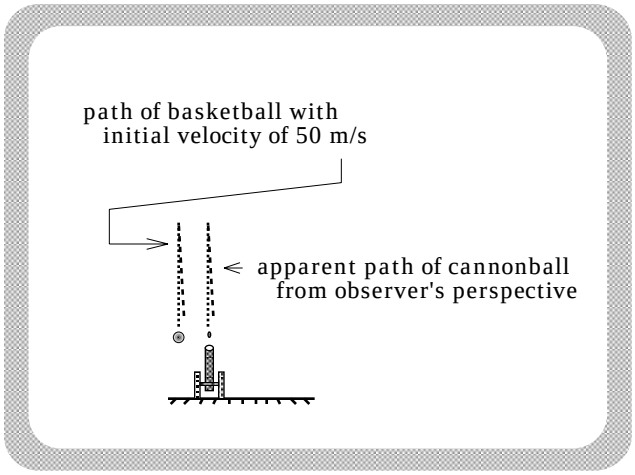


FIGURE 4.16c

Specifically:

i.) Both balls begin at $y_1 = 2$ *meters*;

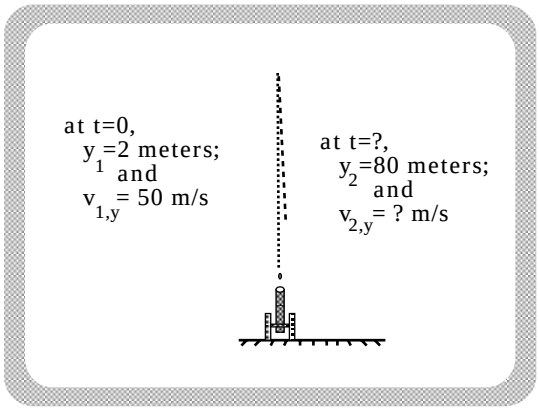
ii.) Both balls rise, then fall back to $y_2 = 80$ *meters*;

iii.) The initial velocity upward is $v_{1,y} = v_1 \sin\theta = (100 \text{ m})(\sin 30^o) = +50$ *m/s*; and

iv.) The touchdown velocity is $v_{2,y} = ?$;

v.) Acceleration in the *y direction* is due to gravity, or $a_y = -g = -(9.8\ m/s^2)$; and

vi.) The time of flight is $\Delta t = ?$



FIGUR E 4.17

b.) To determine Question 2a--*Time of Flight*: The kinematic equation that will allow us to solve for the time-of-flight Δt, given the initial and final *y* positions, the *y* acceleration, and initial *y* velocity (see Figure 4.17), is:

$$(y_2 - y_1) = v_{1,y} \Delta t + (1/2)\, a_y (\Delta t)^2$$
$$(y_2 - y_1) = (v_1 \sin\theta)\Delta t + (1/2)(-g)(\Delta t)^2$$

$$\Rightarrow (80\ m - 2m) = (100\ m/s)(\sin 30^o)\Delta t + .5(-9.8\ m/s^2)(\Delta t)^2.$$

Replacing Δt with *t* for simplicity, we get:

$$4.9t^2 - 50t + 78 = 0.$$

Using the quadratic formula (the solution for *t* in $at^2 + bt + c = 0$ is $t = [-b \pm (b^2 - 4ac)^{1/2}] / 2a \ldots$), we get

$$t = \{-(-50) \pm [\,(-50)^2 - 4\,(4.9)\,(78)\,]^{1/2}\,\} / 2\,(4.9)$$

$$= 1.92 \text{ and } 8.28 \text{ seconds.}$$

Note: There is nothing wrong with the fact that the quadratic equations yields two solutions to this problem. The ball will be at height $y_2 = 80$ *meters* twice--once as it moves upward toward its maximum height and once on its way back down. We're interested in the time it takes to come back down to $y_2 = 80$ *meters*, so we will take the larger time of 8.28 seconds.

c.) Preliminary note to DISTANCE TRAVELED problem: Let us assume you have been placed in a helicopter and stationed high above the cannon range looking down over it. Again, you have the special flight-sensing-scope, and again you have no depth-perception when using it. From this new perspective, how will the *cannonball's* motion look (assuming you can ignore parallax problems)?

In this case, it will appear to be moving along a straight line in the *x direction*, and it will appear to be moving with a *constant velocity*. This makes sense. There are no forces acting in the horizontal which means there will be nothing to accelerate the body in the *x direction* (we are assuming there is no air-friction or wind in the system). The cannonball's *velocity* in the *x direction* will always be the *x component* of the muzzle velocity ($v_1 cos\ \theta = 100\ cos\ 30 = 86.6\ m/s$). In fact, the cannonball's motion will exactly mimic that of a car driving at a constant velocity of 86.6 m/s along the side of the range. Using our kinematic equations for the projectile's *x-type* motion, we know that

i.) $x_1 = 0$;

ii.) $x_2 = ?$

iii.) The initial velocity will be the *x component* of the muzzle velocity, or $v_{1,x} = v_1 \cos\theta = (100\ m)(cos\ 30) = +86.6\ m/s$; and

iv.) $v_{2,x} = v_{1,x} = 86.6\ m/s$ (i.e., *x* velocity doesn't change)

v.) The *x direction* acceleration will be $a_x = 0$.

vi.) The time of flight (from *Part a*) will be $t = 8.28\ seconds$.

d.) To determine Question 2b--Horizontal Displacement: The kinematic equation we will use is the same one used in the first question, but evaluated for *x-type* motion instead of *y-type* motion:

$$(x_2 - x_1) = v_{1,x}\ \Delta t + (1/2)a_x(\Delta t)^2$$
$$(x_2 - x_1) = (v_1 \cos\theta)\Delta t + (1/2)a_x(\Delta t)^2$$
$$\Rightarrow\ (x_2 - 0) = (86.6\ m/s)\Delta t\ + .5\ (0)(\Delta t)^2$$
$$\Rightarrow\ x_2 = (86.6\ m/s)\ \Delta t$$
$$\Rightarrow\ x_2 = (86.6\ m/s)(8.28\ sec)$$
$$= 717\ meters.$$

Note: How would the approach have differed if the first and second questions had been switched?

The equation $(x_2 - x_1) = v_{1,x}\Delta t + (1/2)a_x(\Delta t)^2$ would still have worked for the *x* motion, yielding $x_2 = v_1 \cos\theta\ \Delta t$, but the time of flight Δt would have been unknown. To get Δt, the equation $(y_2 - y_1) = v_{1,y}\Delta t + (1/2)a_y(\Delta t)^2$ would have

had to have been evaluated for the body's *y-motion*. In other words, you would have used the same equations, but you would have written them down in the opposite order.

e.) To determine Question 2c--Velocity at Maximum Height (see Figure 4.18): In general, all velocities have components that can be written as $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$. At the top of the flight-path, $\boldsymbol{v}_{top} = v_{x,top}\boldsymbol{i} + v_{y,top}\boldsymbol{j}$.

We know that at the cannon-ball's peak:

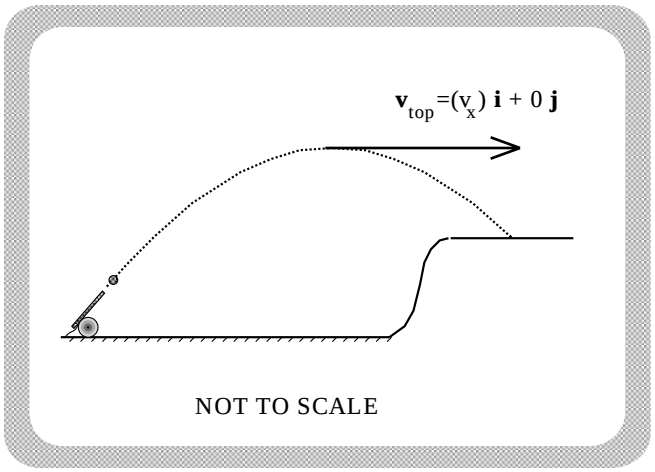


FIGURE 4.18

i.) The ball will have no vertical motion at all (that is what it means to be at the top of the path). Conclusion: $v_{y,top} = 0$.

ii.) The cannonball's horizontal (*x-type*) velocity will be as always, hence $v_{x,top} = 86.6\ m/s$.

iii.) Putting it all together, $\boldsymbol{v}_{top} = (86.6\ m/s)\boldsymbol{i} + 0\boldsymbol{j}$.

f.) Solution to Question 2d--Maximum Height: The cannonball's distance above the ground (its height) is related solely to its *y-type motion*. We have already noticed that $v_{y,top} = 0$ (i.e., the ball stops in the vertical when it reaches the top of its flight). That, coupled with the fact that we know that $a_y = -g = -9.8\ m/s^2$ and $v_{y,1} = +50\ m/s$, allows us to use $(v_{y,top})^2 = (v_{1,y})^2 + 2a_y(y_{max} - y_1)$ to solve for y_{max}. Doing so yields:

$$(v_{y,top})^2 = (v_{1,y})^2 + 2\ a_y\ (y_{max} - y_1)$$
$$(0)^2 = (50\ m/s)^2 + 2\ (-9.8\ m/s^2)\ (y_{max} - 2\ m)$$
$$\Rightarrow\ y_{max} = 129.6\ \text{meters}.$$

g.) Solution to Question 2e--Velocity Just Before Touch-down: The velocity of the cannonball just before touchdown will have a form $\boldsymbol{v} = v_{2,x}\boldsymbol{i} + v_{2,y}\boldsymbol{j}$ (see Figure 4.19). From all we've said to this point:

i.) It should be obvious that the *x component* will be the same as always--86.6 m/s.

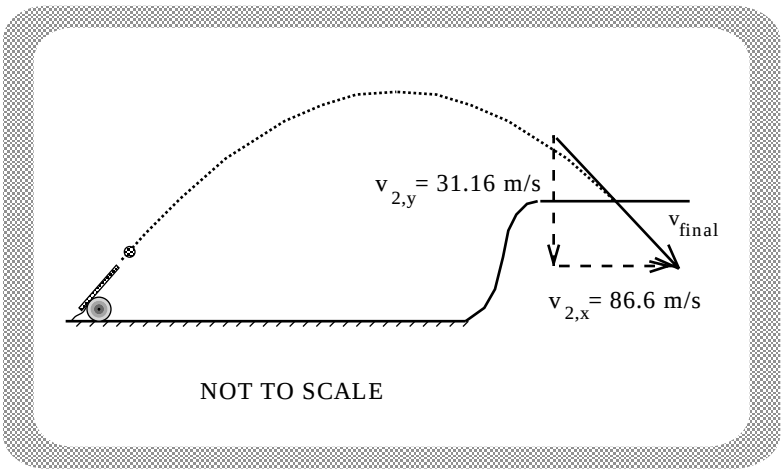


FIGURE 4.19

ii.) The *y component* of the velocity will require the use of the equation $(v_{2,y})^2 = (v_{1,y})^2 + 2\, a_y\, (y_2 - y_1)$ evaluated for *y* motion between $y_1 = 2$ *meters* and $y_2 = 80$ *meters*. Doing so yields:

$$\begin{aligned}(v_{2,y})^2 &= (v_{1,y})^2 + 2\, a_y\, (y_2 - y_1)\\ &= (50\ \text{m/s})^2 + 2\,(-9.8\ \text{m/s}^2)\,(80\ \text{m} - 2\ \text{m})\\ &= 971.2\ \text{m}^2/\text{s}^2\\ \Rightarrow\ v_{2,y} &= 31.16\ \text{m/s}.\end{aligned}$$

iii.) Conclusion: $\boldsymbol{v}_2 = (86.6\ \text{m/s})\boldsymbol{i} + (-31.16\ \text{m/s})\boldsymbol{j}$.

Note: The equation used to determine $v_{2,y}$ yields velocity magnitudes only. You have to put the *negative sign* in manually after noticing that the *y* motion should be downward at the point of interest (see Figure 2.31).

4.) Bottom line on two-dimensional motion:

a.) Treat each direction as an entity with its own set of kinematic equations;

b.) When asked for "distance traveled in the *x direction*," think

$$(x_2 - x_1) = v_{1,x}\Delta t + (1/2)a_x(\Delta t)^2$$

with $a_x = 0$. Use this in conjunction with the same equation evaluated in the *y direction*. The time variable will allow you to link the two equations (it takes the same amount of time to go the horizontal distance as it does to go up, then down to the final vertical position).

c.) When asked to determine maximum height, think vertical motion and the equation

$$(v_{2,y})^2 = (v_{1,y})^2 + 2\, a_y\, (y_{max} - y_1)$$

with the y_{max} velocity (i.e., $v_{2,y}$) equal to zero.

d.) Be careful not to confuse *x-type* acceleration with *y-type* acceleration, especially for freefall problems (one is ZERO while the other is *-g*).

QUESTIONS

Note: Don't get hung up on Question #4.2. Understanding how to think about graphical information is important, but not as important as knowing the basic definitions and learning how to use the kinematic equations.

4.1) A sprinter runs around a 440 meter circular track in 49 seconds.

a.) What is her average speed?

b.) What is her average velocity?

c.) Can you tell anything about her instantaneous velocity 5 seconds after the start?

4.2) A student turns in the graph shown in Figure I without bothering to label the vertical axis. The graph is related to the motion of a tricycle, but all you know for sure is that at t = 1 second the trike is moving with an approximate velocity of -1 m/s. Is the graph a *position versus time* graph, a *velocity versus time* graph, or an *acceleration versus time* graph? Explain briefly.

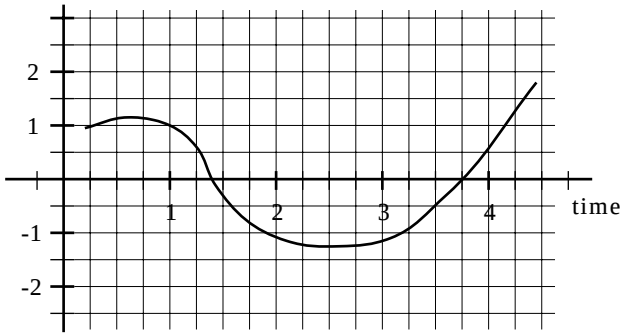


FIGURE I

4.3) Figure II is a *velocity versus time* graph for the motion of an ant moving in one dimension across the floor. Assuming you don't explicitly know the velocity function:

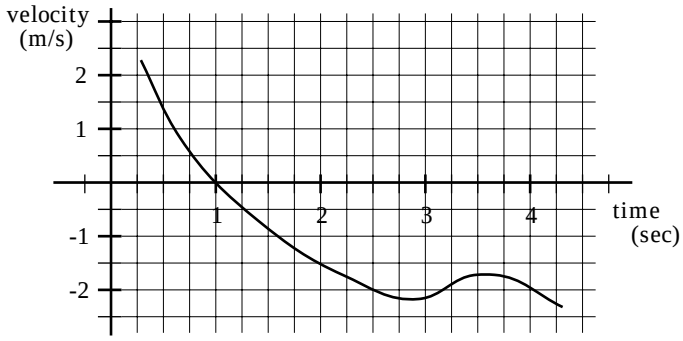


FIGURE II

a.) What is the ant's approximate displacement between times *t = .5 second* and *t = 3 seconds* (eyeball it off the graph--this is not a Calculus problem!)?

b.) What is the ant's average velocity between times *t = .5 seconds* and *t = 3 seconds*? (This is a bit off-the-wall, more of a definition/use-your-head question; if you don't see it, don't spend a lot of time on it.)

c.) What is the ant's velocity at *t = .5 seconds*? . . . at *t = 3 seconds*?

d.) What is the ant's acceleration at *t = .5 seconds*? . . . at *t = 3 seconds*?

e.) When is the ant moving in the *+x direction*?

f.) When is the ant standing still?

g.) When is the ant's acceleration approximately zero?

h.) When is the change of the ant's acceleration zero?

4.4) A body moves under the influence of a velocity function given as:

$$\mathbf{v}(t) = (3e^{-1.5kt}\mathbf{i} - 4k_1 t\mathbf{j}) \text{ m/s.}$$

Assuming both k and k_1 have magnitudes of *one* and the appropriate units:

a.) Determine the velocity of the body at $t_1 = 2$ *seconds*.

b.) Determine a general expression for the acceleration of the body as a function of time.

c.) Determine the acceleration of the body at time $t_1 = 2$ *seconds*.

d.) Could you have used kinematics to do any of the above problems? Explain your response.

e.) Determine a general expression for the position of the body as a function of time. As this will have components, call it ***r(t)***. (Hint: Do this in pieces; that is, use $v_x(t)$ to determine an expression for $x(t)$, then use $v_y(t)$ to do the same for $y(t)$).

f.) Determine the displacement of the body between times $t_1 = 2$ *seconds* and $t_2 = 3.5$ *seconds*.

g.) Assuming the body is at $x_1 = -.1$ *meters*, $y_1 = -8$ *meters* at time $t = 2$ *seconds*. *Without* using the displacement function $r(t)$ derived above, what is its position coordinate at time $t = 3.5$ *seconds*?

h.) For the amusement of it, determine *the jerk* of the system.

i.) What are the units of k?

4.5) Bats at Carlsbad Caverns leave the cave at dusk in search of food. When they return at dawn, they fly over the cliff face that supports the cave entrance, fold their arms and legs, then plummet like rocks until a few meters above the floor of the cave entrance where they spread out the skin membranes between their arms and legs and pull out of the dive. Assuming they drop from a height of 100 meters and do not open their leg/wings until they are 3 meters above the floor:

a.) Ignoring air friction, how fast (i.e., the magnitude of their velocity) are they moving by the time they pull out of the freefall?

b.) If their vertical velocity essentially drops to zero as they move from 3 meters to 1 meter above the floor (i.e., during the time period in which they pull out of the freefall), what is their vertical "pull-out" *acceleration*?

c.) How long does it take them to execute their pull-out?

4.6) A particle moving in one dimension has a position function defined as:

$$x(t) = bt^4 - ct.$$

Assuming $b = 6\ m/s^4$ and $c = 2\ m/s$:

a.) At what point in time does the particle change its direction along the x axis?

b.) In what direction is the body traveling when its acceleration is 12 m/s^2?

4.7) A stunt-woman freefalls from rest. She is observed to be moving 25 m/s at a particular point in time (call her *position* at that point in time *Point A*).

a.) How far will she have fallen 2 seconds after passing *Point A*?

b.) How fast will she be moving 2 seconds after passing *Point A*?

4.8) One car moving with a constant velocity of 18 m/s passes a second car initially moving at 4 m/s. As it does, the second car begins to accelerate at a rate of 6 m/s^2.

a.) How long does it take the second car to catch the first car?

b.) How far do the cars travel during the time interval required for the second car to catch the first car?

c.) What is the second car's velocity as it passes the first car?

d.) What is the second car's average velocity during the period required for it to pass the first car?

e.) How long will it take the second car to reach 100 m/s?

4.9) A falling rock takes .14 seconds to pass from the top to the bottom of a 1.75 meter tall window in a multi-story building.

a.) What is the velocity of the rock when at the top of the window?

b.) Assuming the rock is given an initial downward velocity of 7 m/s when released at the top of the building, what is the distance between the top of the building to the *bottom* of the window?

c.) If the rock were *not* given an initial velocity of -7 m/s but instead started from rest, how would its acceleration as it passed by the top of the window have changed from the originally stated problem?

4.10) You are driving a car that can accelerate at 3 m/s^2 (it's a Nash Rambler) and can brake at 3 m/s^2. You approach an intersection that is 18

meters wide. The light turns yellow. It stays yellow for 1.2 seconds before turning red. If you accelerate, you must make it through the intersection before the light turns red to be safe. If you brake, you must stop before reaching the cross-walk-restraining-line to be safe.

Tough as it may be to believe, there is a range of distances between which you will neither be able to successfully accelerate nor brake and still be safe. Assuming you are moving 40 m/s (about 80 mph--ouch), and assuming your *reaction time* is zero (that is, you accelerate or brake just as the light turns yellow), the following will allow you to determine that range.

a.) Pedal to the metal, what is the farthest you can be from the restraining line and still be able to accelerate through the intersection before the light turns red?

b.) Braking like mad, what is the closest you can be to the restraining line and still be able to come to a grinding halt before going over the restraining line? (Note that the *slide time* does not have to be 1.2 seconds--you can still be sliding after the light turns red just as long as you don't ultimately go over the restraining line.)

c.) In conclusion, what are the *you're going to die no matter what* limits?

4.11) A 3-meter-tall elevator accelerates at a rate of 1.5 m/s^2 when it's working properly. After a shaky start, it is found to be moving with a velocity of 3.4 m/s just as its *floor* passes a point (call this *Point A*) 4 meters above the ground. As it passes this point, a bolt in the ceiling of the elevator comes loose and freefalls to the elevator's floor.

a.) Determine the bolt's maximum height above the ground during its freefall.

b.) How long did it take for the bolt to meet the floor?

c.) What was the bolt's net displacement during the freefall?

d.) What was the bolt's velocity just before striking the floor?

4.12) A batter strikes a baseball 1.3 meters above the plate. The ball leaves the bat at an angle of 50^o with a velocity of 41 m/s.

a.) How long will it take for the ball to touch down in the outfield?

b.) How far (horizontally) will the ball travel before touch down?

c.) How high will the ball travel during the flight?

d.) What will the ball's velocity be just before touch down?

Chapter 5

NEWTON'S LAWS

A.) Newton's Three Laws:

1.) Newton's First Law: In an inertial frame of reference, bodies in motion tend to stay in motion in a straight line, and bodies at rest tend to stay at rest, unless impinged upon by a net external force.

a.) Example: When an object in space is given a quick push and then left alone, it will move with a constant velocity in the direction of the push until an outside (external) force accelerates it into another kind of motion (i.e., makes it go faster, slower, change directions, or some combination thereof).

Until a net external force is applied, the constant-velocity, straight-line motion of a body will continue unchanged.

2.) Newton's Second Law: The *acceleration* $\boldsymbol{a}$ of a body (as a vector) is proportional to the *net force* $\boldsymbol{F}$ (also a vector) acting on the body.

a.) Mathematically, this can be stated as

$$\mathbf{F} = m\mathbf{a},$$

where the proportionality constant m is the *mass of the object being accelerated* (see section on "What is MASS" at the end of this chapter).

b.) As a force in the *x direction* will not make a body accelerate in the *y direction*, we can break both the *net force* and *acceleration* into their component parts and write three direction-related force equations:

$$F_{net,x} = m\,(a_x),$$
$$F_{net,y} = m\,(a_y),$$
$$F_{net,z} = m\,(a_z).$$

Note 1: These look like scalar equations because we haven't included unit vectors to designate direction, but each nevertheless represents a vector

quantity. As such, we need to be careful of *direction* as denoted by *positive* and *negative* signs.

Note 2: Newton's Second Law is the work-horse of the three laws. It is a very powerful tool for analyzing situations in which forces and accelerations are involved. There is a technique involved in its use, outlined in *Part B*. LEARN THE APPROACH. Understand it. You should become so familiar with it that you can use it to evaluate problems *you have never seen before* using nothing more than it and your head.

3.) Newton's Third Law: For every force in the universe, there exists somewhere an *equal* and *opposite* reaction force.

Note: This idea of an *action/reaction* pair is somewhat misleading as it suggests that one force follows the other. As the examples will show, that is not the case. We will use the terminology because it is standard, not because it is an intelligent way of characterizing the interaction.

a.) Example 1: Imagine hitting a door with your hand. Doing so applies a force to the door which, if great enough, will break the door. In turn, the door applies a force to you which, if great enough, will break your hand. The size and direction of the force you apply must, according to N.T.L., be equal and opposite the force it applies to you.

i.) In the case of the door, the *action* is characterized by the statement, "You apply a force to the door" while the *reaction* is characterized by the statement, "The door applies a force to you." Notice that the wording is almost identical with the exception of reversed noun and/or pronoun order. This is always the case with the characterization of action/reaction pairs.

ii.) Don't be misled by the apparent personification of "the door." It obviously hasn't made a conscious decision to hit you back, even though the characterization implies that such be the case. Newton's Third Law is a commentary on the *structural reality* of the universe. The only way forces can be generated are in action/reaction pairs. By implication, if you try to apply 180 pounds of force to a table that can only provide 150 pounds of reaction force back on you, you will never succeed. The table will give way as soon as you exceed the 150 pounds that it can apply to you.

b.) Example 2: If the *earth* applies a gravitational force on *you*, what is the reaction force that must, according to N.T.L., exist?

i.) Answer: The reaction force is the gravitational force that *you* exert on the *earth*. Again, the same rearranged language is used to state the *reaction force* as was used to state the *action force*.

c.) Example 3: A truck runs head-on into a train. The train is ten times more massive than the truck. How do the forces experienced by each compare?

i.) The answer is, THEY ARE THE SAME! The train applies a force to the truck and, as a consequence, experiences an acceleration. The truck applies a force to the train and, as a consequence, experiences an acceleration. The two accelerations are different but, according to N.T.L., the two forces *must be* equal. Figures 5.1a and 5.1b depict this situation along with the fact that as *ma* equals the force on each body, that *product* must be the same for each body.

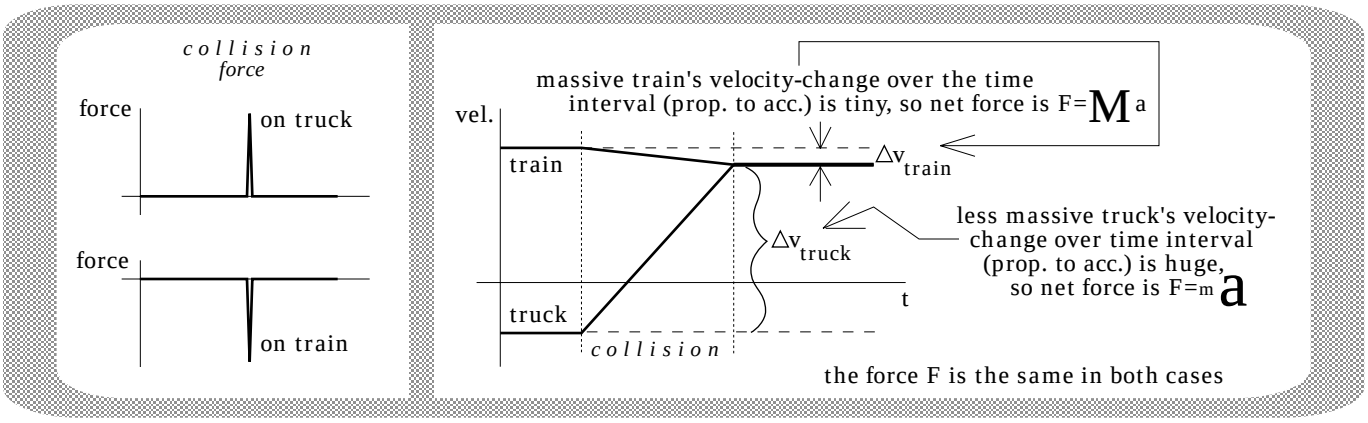


FIGURE 5.1a

FIGURE 5.1b

B.) Newton's Second Law and Types of Forces:

1.) *Newton's Second Law* is intimately related to force acting on a body. A brief preliminary discussion of the five kinds of forces you will work with follows in *Parts 2* through *6*:

2.) Gravitational force:

a.) Gravity is a force of attraction between any two massive objects.

b.) When the earth is one of the two bodies involved:

i.) The gravitational force felt by the second object (you, for instance) while positioned on the earth's surface will always be directed toward the earth's center; and

ii.) The gravitational force on the body will have a magnitude equal to the product *mg*, where *m* is the mass of the object in question and *g* is the gravitational acceleration near the earth (9.8 m/s^2 in the MKS system of units; 980 cm/s^2 in the CGS system of units; 32.2 ft/s^2 in the English (i.e., our) system of units).

Note: There are two metric systems of units (the MKS and the CGS systems) and one non-metric system (our own--the "English" system):

The MKS system uses *meters* for length, *kilograms* for mass, and *seconds* for time (hence, MKS). Force is in *newtons* and energy is in *joules*.

The CGS system uses *centimeters* for length, *grams* for mass, and *seconds* for time (hence, CGS). Force is in *dynes* and energy is in *ergs*.

The English system will never be used in this book (a base-10 version is used in engineering applications, but physicists generally stay away from it). For completeness, though, it uses *feet* for length, *slugs* for mass, and *seconds* for time. Force is in *pounds* and energy is in *foot-pounds*.

NEVER MIX UNIT-SYSTEMS. If you find yourself with mass information in grams (the CGS system) and you want to know the body's weight, do not multiply the mass by 9.8 m/s^2--the MKS value for the acceleration of gravity. Either convert the mass to kilograms (divide by 1000) or use the CGS value for *g* (i.e., 980 cm/s^2).

iii.) The gravitational force will be equal to the body's weight (that's right, folks, when you step on a scale and it measures your weight, you are really measuring *mg*--the force of attraction between you and the earth). As weight is a *force*, its units in the MKS system are *kilogram·meter/second*^{2}, or *newtons*.

c.) Example: Assuming we neglect air friction, a freely falling object has only one force acting on it--gravity (see Figure 5.1c).

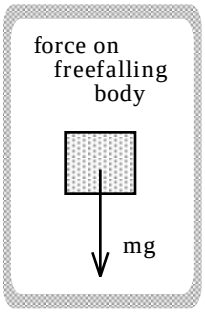


FIGURE 5.1c

3.) Normal force:

a.) A normal force is a force of support provided to a body by a surface in which the body is in contact;

b.) Normally characterized by *N*, a *normal force* is always directed *perpendicular to* and *away from* the surface providing the support;

c.) Example 1: In Figure 5.2a, a book is supported by a table. Figure 5.2b depicts all the forces acting on the book (this is called a *free body diagram*, or *f.b.d.*). Notice that the *normal force* acting on the book is *perpendicular to* and *away from* the table's top (the *gravitational force* is also shown).

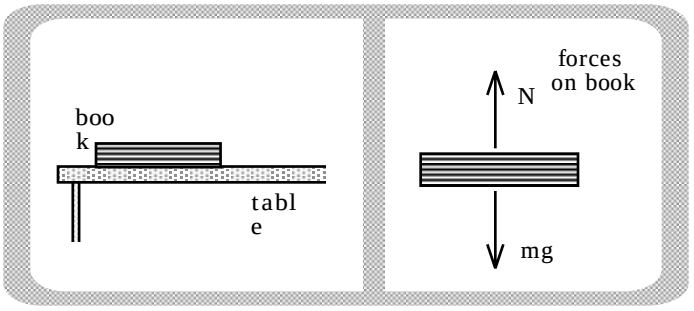


FIGURE 5.2a FIGURE 5.2b

d.) Example 2: In Figure 5.3a, a book rests on an incline plane. Figure 5.3b shows the f.b.d. for the forces acting on the book. As was the case in Example 1, the *normal force* acting on the book is *perpendicular to* and directed *away from* the incline's top, even though the top is not in the horizontal!

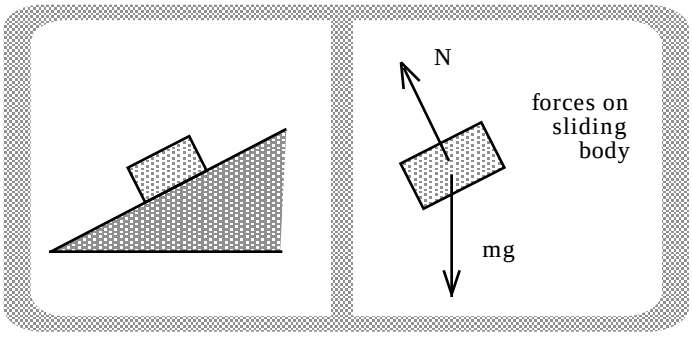


FIGURE 5.3a FIGURE 5.3b

Note: The *normal force* acts on the underside of the book, as shown in Figure 5.4. Though technically dubious, *normal forces* are usually presented as in Figure 5.3b. This is done for convenience (you will see why shortly). Either positioning is acceptable.

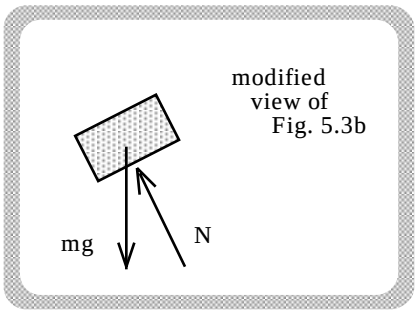


FIGURE 5.4

4.) Tension force:

a.) Tension force is applied to a body by a rope, string, or cable.

b.) Normally characterized by T, tension forces are always applied *along the line of the cable* and *away from* the body in question.

c.) Example: A string is attached to a hanging mass, threaded over a pulley, and attached to a second mass sitting on a table (see Figure 5.5a). The forces on the *hanging mass* are shown in Figure 5.5b; the forces on the *tabled mass* are shown in Figure 5.5c. In both cases, the *tension force* is directed *away from the body* and *along the line of* the string.

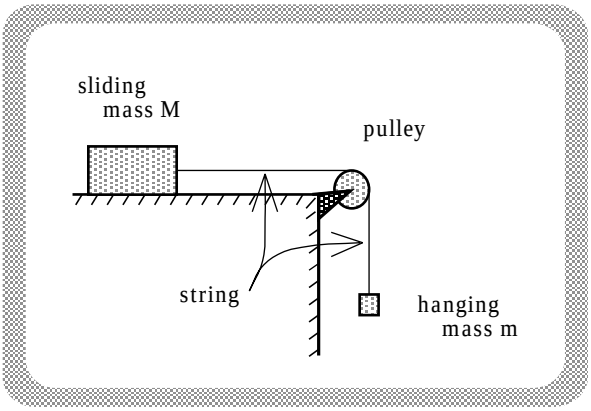


FIGURE 5.5a

Note: If the pulley is one of those mythical "frictionless, massless" jobs assumed-into-existence by physics departments across the country, the *massless* nature of the pulley will allow the *magnitude of the tension force* on either side of the pulley to be the same. In other words, an ideal pulley changes the DIRECTION of the tension force (in this case, from the horizontal to the vertical) but does not change the force's magnitude.

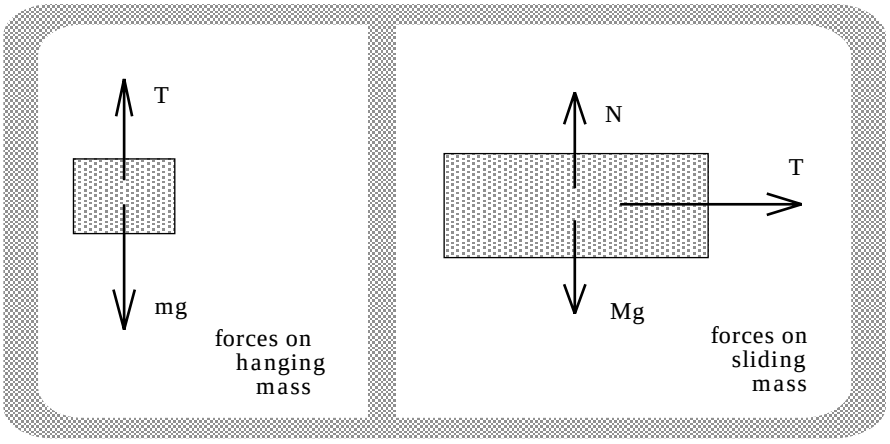


FIGURE 5.5b

FIGURE 5.5c

5.) Frictional force:

a.) Friction is produced by the atomic interaction between two bodies as they either slide over one another (this is called *kinetic friction*) or sit motionless in contact with one another (this is called *static friction*).

Note: When two bodies come very close (i.e., rest against one another), there is a weak atomic bonding that occurs between the electrons of the one structure and the protons of the other (and vice versa). It is almost as though the atoms of the two bodies have melded to some degree. When the bodies try to move over one another, this bonding has to be sheared. That shearing is what produces the retarding effects we call friction.

b.) Kinetic friction: Sometimes called *sliding friction*, kinetic friction occurs when one body slides over or against a second body. The direction of *kinetic friction* is always *opposite* the direction of motion.

From experimentation, it has been observed that the amount of *kinetic friction* f_k a body experiences is proportional to the size of the *normal force N* exerted on the body by the structure it slides against. Mathematically, these two parameters (f_k and *N*) are related as:

$$f_k = \mu_k N,$$

where μ_k is a proportionality constant called the *coefficient of kinetic friction*.

As μ_k is normally given in a problem, all that is required to calculate the magnitude of the *kinetic frictional force* on a body is the *normal force* exerted by the supporting structure.

Note: The statement *kinetic friction is a function of normal force only--surface area has nothing to do with it* is true ONLY as long as you are dealing with two rigid bodies that are sliding relative to one another.

The tires of a dragster do NOT fit this bill. Dragster drivers spin their tires before a race to make them sticky. As such, the friction analogy for race tires is more like the dragging of a piece of Scotch tape across a desk than the sliding of rigid bodies--the bigger the surface area, the greater the stickiness and traction. My gratitude to Jim Malone (Mercersburg Academy) for contacting Mike Trinko of Goodyear's Race Group for this clarification.

c.) Example of *kinetic friction*: A block slides down a frictional surface (see Figure 5.6a). The *coefficient of kinetic friction* μ_k is known. Figure 5.6b shows all the external forces acting on the block.

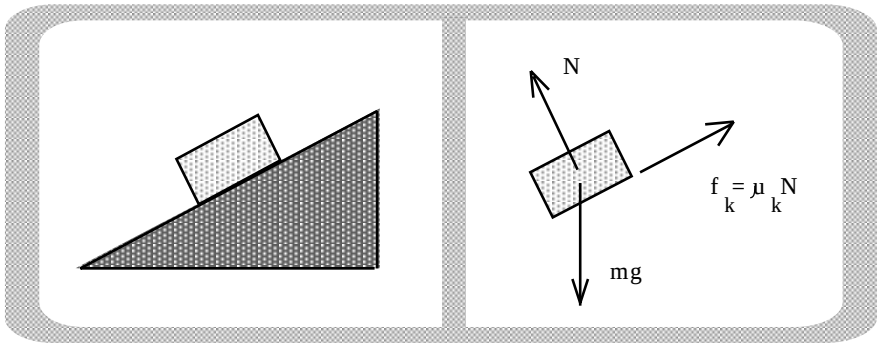


FIGURE 5.6a

FIGURE 5.6b

Note: Notice that the normal force *N* sometimes equals the weight *mg* of the sliding body (an example of this can be seen in Figure 5.2b), BUT NOT ALWAYS. For the incline plane depicted above, *N* and *mg* are not even in the same direction.

d.) Static friction: If a book, for instance, sits on a table and a tiny external force is applied to the book *parallel to the table top*, the force may or may not move the book. There will be a force-of-opposition in this case provided by *static friction*. Its cause is the same as the partial atomic bonding that produces *kinetic friction* with one exception; static frictional bonding is stronger (with the two surfaces stationary relative to one another, the atoms meld more deeply requiring a greater shearing force to separate them).

If successively harder and harder forces are applied, the static frictional force will counter each push with a force of equal and opposite magnitude until the *applied force* is great enough to shear the partial bonding between the surfaces. At that point, the book will break loose and begin to slide.

It has been experimentally observed that this *maximum* static frictional force $f_{s,max}$ is proportional to the *normal force N* applied to the book by the table. Mathematically, these two parameters ($f_{s,max}$ and N) can be related as:

$$f_{s,max} = \mu_s N,$$

where μ_s is called the *coefficient of static friction*.

Note: Most physics books do not write the *maximum static frictional force* as $f_{s,max}$. Instead, they simply write f_s. For simplicity, we will do the same. Be clear, though. If you know f_s, you know only *one of the infinitely many* possible static frictional forces that could be exerted between the two bodies. Which force do you know? The *maximum static frictional force*.

On the other hand, if you know f_k you know the single, CONSTANT *kinetic frictional force* that exists between two bodies sliding relative to one another. No matter what their velocity (assuming heating doesn't change the characteristics of the two surfaces and alter the *coefficient of kinetic friction*), the *frictional force* f_k will always be the same.

The two quantities--f_s and f_k-- look similar as far as notation goes, but they tell us two different things!

6.) Push-me, pull-you force:

a.) Any force that does not fall into one of the above categories falls into this one. Ex: A shove from a friend. The magnitude of such a force is usually characterized by an F, possibly with a subscript.

C.) Newton's Second Law--APPROACHES:

1.) There are two ways to deal with Newton Second Law problems.

a.) The first is the formal, technically kosher way to proceed. It has specific steps and works on even the most convoluted force/acceleration problems. These include situations in which forces do not act in the same direction as the motion (e.g., centripetal situations), situations in which several objects within the system experience non-equal accelerations, and, with slight modification, situations in which some of the forces involved are velocity dependent.

b.) The second approach is a simplified version of the first. When applicable, it makes the evaluation of N.S.L. problems considerably quicker and easier (this can be very useful when dealing with multiple choice AP questions). As such, your initial temptation may be to focus on that approach to the exclusion of the first. That would be a serious mistake. Getting the answer isn't important, here. Understanding the approaches is. To that end, both approaches will be presented in the evaluation of the example problems in the next section.

Note: The *formal approach* is the only approach presented in the Solutions section at the back of the book. If you have correctly used the simplified version on a given problem, you can still use the book-end solutions to check your final answer.

2.) The simplified approach: When dealing with the simplified approach, the question you want to ask is, "What are the forces that motivate or retard acceleration in the system?" Determine those forces, equate them to ma, where m is the total mass of the system, and you're done.

3.) The formal approach: The following is a step-by-step outline for the use of the formal approach associated with Newton's Second Law as a problem-solving tool. Read through quickly, then continue on to see how the technique is used in the Example Problems section. Once you have read through that section, use this outline as a reference whenever you tackle complex Newton's Second Law problems.

a.) Look at the problem, blanch, think *I couldn't possibly do this*, then begin *the approach* starting with *Step b*.

b.) Draw a sketch of the entire system if one is not provided.

c.) Take <u>one body</u> in the system and draw a *free body diagram* (f.b.d.) for it.

i.) A *free body diagram* is a sketch of the body-in-question (it is normally depicted as a box) showing *all the forces acting on the body*. These forces are depicted as arrows. They don't have to be drawn to scale but do have to be directionally accurate (i.e., if the force is in the vertical, don't draw it halfway between the vertical and the horizontal).

Note: F.B.D.'s show only the forces that act *on the body-in-question*. They do *not* show forces that the body applies to other bodies.

ii.) When drawing an f.b.d., be sure the *orientation* of the box representing the body is the same as the actual orientation of the body in the problem. That is, if the object is a crate on an incline, the f.b.d. should depict the object as a box *on a slant*, not as a box in the horizontal--see Figure 5.6.

d.) Choose *x* and *y axes* and place them on your f.b.d. One axis must be in *the direction of the acceleration you are trying to determine*.

Note: This means your axes will not always be in the vertical and horizontal. Think about the direction of acceleration of the crate sketched in Figure 5.6.

e.) If there are forces on the f.b.d. that are not along the *x* and *y* directions, replace them with their *x* and *y components*.

f.) Using N.S.L., sum the forces in the *x* direction and set them equal to the product ma_x. If an additional relationship (equation) is needed, sum the forces in the *y* direction and set them equal to the product ma_y (assuming you are interested in the variables involved).

g.) Repeat the above process for all the bodies in the system *or until you have enough equations to solve for the unknowns you are interested in* in the system.

WARNING: ASSUME NOTHING that can be derived using an f.b.d. and N.S.L.

D.) Example Problems:

The easiest way to become comfortable with both the simplified and formal approaches outlined above is to try a problem. There are two in this section: one that is relatively simple and a second that is more complex. Both have within them potential pitfalls. Look for the sticking points. Understanding

these difficulties will make it easier for you to do other N.S.L.-type problems later on your own (this is especially true of the formal approach).

Note: While you are reading, remember that your goal is to generate equations that relate the *acceleration* of one or more of the masses within the dynamic (i.e., moving) system to the forces acting on that mass.

1.) Example 1--The Run-Away Car using the Formal Approach: Somebody gives your car a shove, then exits. The car rolls freely on a horizontal road toward a cliff. You try to stop the car by applying a known force $\boldsymbol{F}$ at a known angle θ (see Figure 5.7). If the car's mass is m, derive a general algebraic expression for the *acceleration* of the car and, for the fun of it, the *normal force* applied to the car by the ground. We'll use *the formal approach* first.

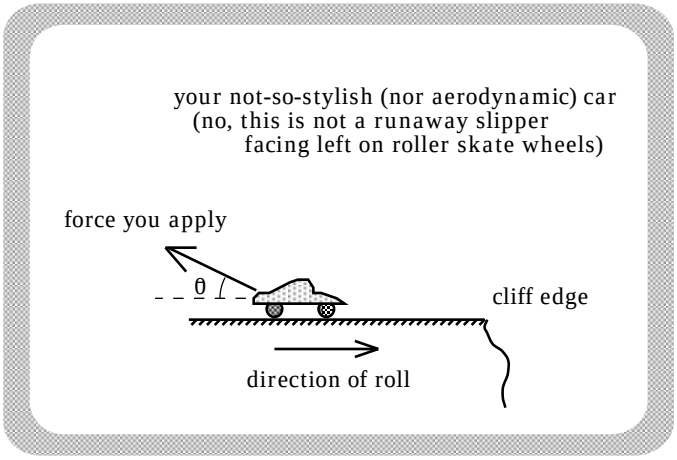


FIGURE 5.7

--Solution and Approach *By-the-Numbers*:

--Step a: The sketch is seen in Fig. 5.7.

--Step b: A *free body diagram* (f.b.d.) of the forces acting on the car is shown in Figure 5.8.

Notice that the *direction of the car's motion* (i.e., the direction of its velocity vector) is not the same as the *direction of its acceleration*. That means, simply, that the car is slowing down.
Notice also that the *velocity vector* is not found on the f.b.d. Only FORCES go on free body diagrams.

--Step c: Figure 5.9 shows the placement of appropriate coordinate axes. One is in the horizontal along the car's *line of acceleration*, and the other is *perpendicular* to the first.

Note: Although it is customary to make horizontal axes *"+" to the right* and

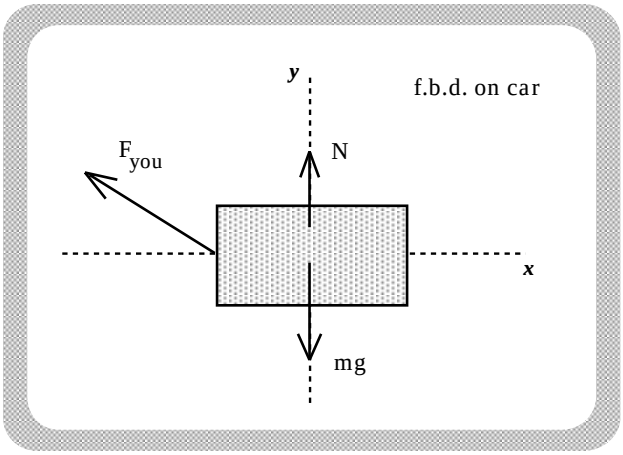


FIGURE 5.9

vertical axes *"+" in the upward direction*, you can define any direction as positive.

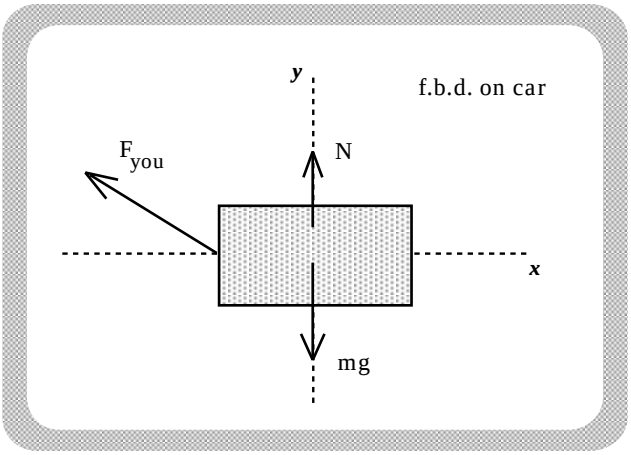


FIGURE 5.9

--Step d: There is one force that is off-axis--the pulling force *F*. Figure 5.10 shows a revised f.b.d. with *F* replaced by its components along the two axes.

Note: In Figure 5.10, the vertical component of ***F*** (i.e., $F \sin\theta$) is positioned out away from the body, but it actually acts at the same point as does ***F***. It has been drawn as part of a right triangle to make its determination easier. Having made that determination, you could re-position the vector as shown in Figure 5.11. There is nothing wrong with doing this, though it does seem a waste of time if its original positioning doesn't confuse you.

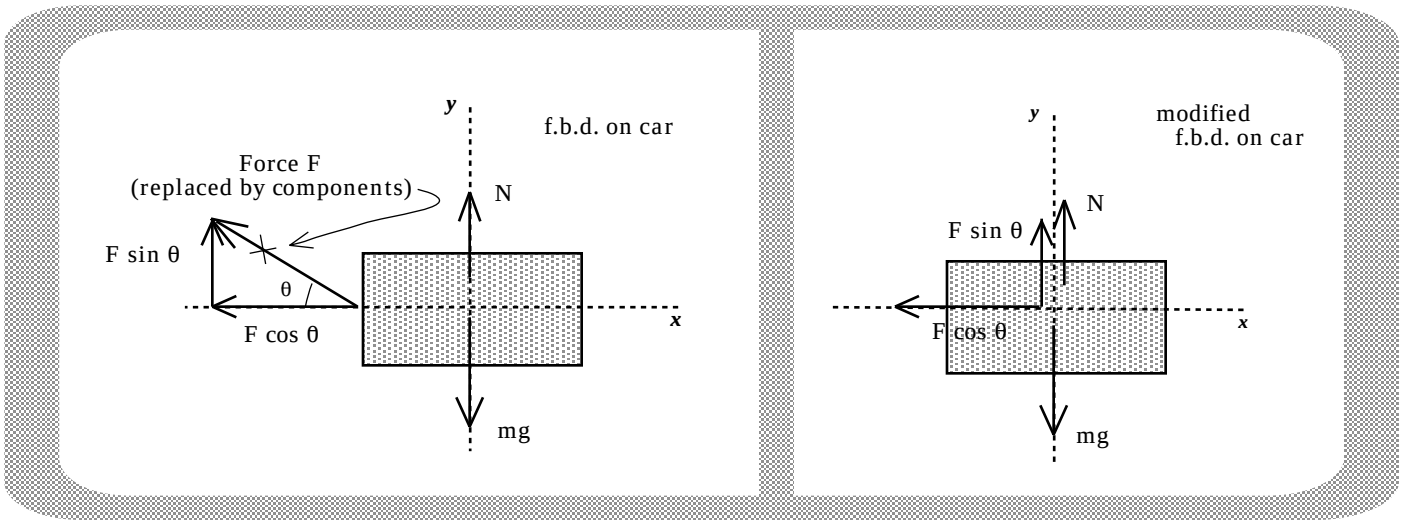


FIGURE 5.10

FIGURE 5.11

--Step e: The *sum of the forces* in the *x* and *y* directions is shown below. Notice that each has been preceded by a blurb that denotes what is about to happen. That is, the "$\sum F_x$" notation alerts the reader that you are beginning a *sum-of-the-forces-equals-"ma"* process in a given direction. *You will be expected to use this same notation* as a preamble whenever you use Newton's Second Law.

$\underline{\sum F_x}$:

$-F(\cos\theta) = -m(a_x)$

$\Rightarrow \quad a_x = F(\cos\theta) / m.$

Note 1: Notice the *negative sign* in front of the *component of F along the x axis* term? If that force component had been in the opposite direction, its magnitude would have remained *F (cos θ)*, but the sign in front of the component would have been *positive*. In other words, the expression *F (cos θ)* stands for the *magnitude* of the force. When writing out the N.S.L. expression, you have to account for the *direction* of a force by <u>manually</u> inserting the appropriate *positive* or *negative* sign in front of its magnitude value.

Note 2: Just as was the case with the force component discussed in Note 1, the acceleration term a_x can be treated in one of two ways. We can either define it to be a vector, leaving its sign embedded within the symbol, or we can define it to be a *magnitude*, unembed its sign, and manually put that sign in front of *ma*. For reasons that will become obvious later, the latter is preferable.

Note 3: This *unembedding the sign* isn't as scary as it looks. If the assumed direction for the acceleration is incorrect, we will simply end up with an acceleration value that has a negative sign in front of its magnitude when numerically calculated. As magnitudes cannot be negative, the negative sign simply means we have assumed the *wrong direction* for the acceleration.

--back to Step e:

In the *y direction*:

$\underline{\sum F_y}$:

$+F(\sin\theta) + N - mg = +m(a_y)$ (= 0 as $a_y = 0$).

As shown, the acceleration a_y is zero (the car is not hopping up and down off the road). As such, this expression can be re-written as:

$$N = -F(\sin\theta) + mg.$$

Note: The normal force exerted by the ground on the car is *not equal to the weight of the car*. This shouldn't be surprising. The car is being pulled

upward by the vertical component of ***F***, requiring less *normal force* from the ground to keep the car from falling through the road.

--Step f: We have enough equations to solve for the parameters in which we are interested.

Using the derived equations with a car mass of 1000 kg, we find that if the angle is 30^o and the pulling force is 800 nts (you *are* a goooorilla):

$$a_x = F(\cos\theta)/m$$
$$= (800\text{ nt})(\cos 30^o)/(1000\text{ kg})$$
$$= .69\text{ m/s}^2$$

and

$$N = -F(\sin\theta) + mg$$
$$= -(800\text{ nt})(\sin 30^o) + (1000\text{ kg})(9.8\text{ m/s}^2)$$
$$= 9400\text{ nts.}$$

Note: One *newton* is equal to about a quarter of a pound.

2.) Example 1--The Run-Away Car using the Simplified Approach: Somebody gives your car a shove, then exits. The car rolls freely on a horizontal road toward a cliff. You try to stop the car by applying a known force *F* at a known angle θ (see Figure 5.7). If the car's mass is *m*, derive a general algebraic expression for the *acceleration* of the car.

a.) The question to ask is, "What are the forces that motivate the car to accelerate?" In this case, the only force acting along the line of acceleration is $F(\cos\theta)$. Putting that equal to ma_x yields:

$$F\cos\theta = ma_x$$
$$\Rightarrow \quad a_x = (F\cos\theta)/m.$$

b.) Although this undoubtedly seems wickedly easy in comparison to the more plodding *formal approach* presented in the previous section, remember what we are doing. We are examining relatively simple problems with an eye to developing techniques that will serve you well when you are asked to take apart more complex scenarios. Be patient. As hard as it may be to believe, the *formal approach* is definitely the way to go in many situations.

3.) Example 2--Incline and Pulley using the Formal Approach: A string attached to a known hanging mass m_1 is threaded over a *frictionless, massless pulley* and attached at its other end to a second known mass m_2 that is supported by a *frictional* incline plane. The mass m_1 is additionally wedged *frictionlessly* against the incline by a known force F (see Figure 5.12). Both the angle of the incline θ_2 and the angle θ_1 at which F assaults the block are known. The *coefficient of kinetic friction* μ_k between m_2 and the incline is also known. Assuming the string is inextensible (that is, it isn't acting like a rubber band) and mass m_2 is moving *up* the incline when first viewed, determine the *magnitude* of the *acceleration* of the system.

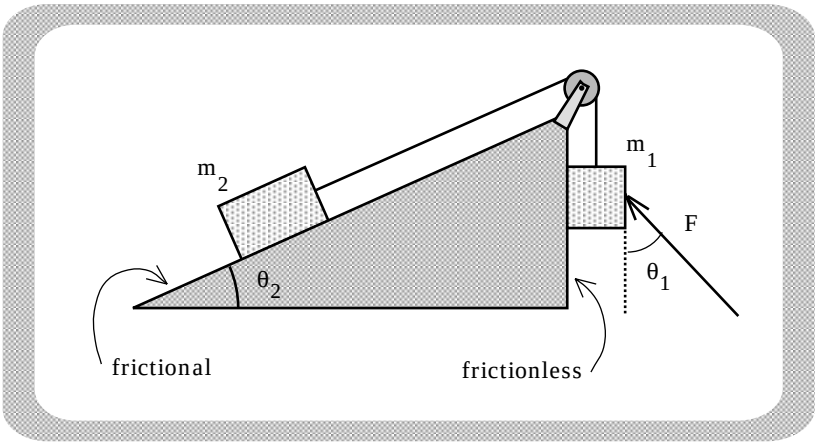


FIGURE 5.12

Note 1: I've done this analysis in steps to allow you to see how each of the formal steps alluded to in the previous section plays out in a problem. I'm doing it this way ONLY because the approach is new to you and I want you to see and understand what each formal step actually does. At the end of the presentation, I will redo the problem the way I would expect you to proceed if such a problem were given on a test.

Note 2: As the string is inextensible, the "magnitude of the acceleration of the system" will be the *magnitude* of the acceleration of either of the two masses. If we know a for one body, assuming a is a magnitude, we know it for the other.

--Solution and Approach *By-the-Numbers*:

--Step b: The sketch of the system has already been shown in Figure 5.12.

--Step c: We will begin with mass m_1. An f.b.d. of the forces acting on m_1 is shown in Figure 5.13.

Note: The *frictional force* between any two objects sliding relative to one another is $f_k=\mu_k N$. If there had been a *frictional force* acting between m_1 and the incline's vertical wall, we would need to determine N_1 before we could

determine f_k. In this case, there is no frictional force along that surface so there is no need to determine N_1. Nevertheless, to be complete, our *free body diagram* MUST INCLUDE N_1 and all other horizontal forces acting on m_1.

Bottom line: By definition, a *free body diagram* includes ALL THE FORCES ACTING ON THE BODY FOR WHICH IT IS DRAWN.

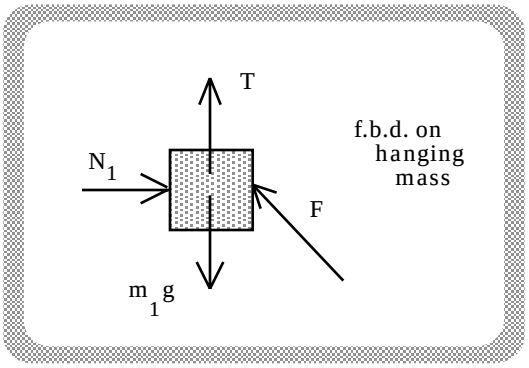


FIGURE 5.13

--Step d: Figure 5.14 shows the off-axis-forces broken into their components.

--Step e: The *sum of the forces* in the *v direction* is shown below.

$$\underline{\Sigma F_v}:$$

$$T - m_1 g + F\cos\theta_1 = -m_1(a_v)$$
$$= -m_1 a.$$

(*a* is defined to be the *magnitude* of the acceleration of both m_1 and m_2). This implies:

$$T = m_1 g - F\cos\theta_1 - m_1 a \qquad \text{(Equation A)}.$$

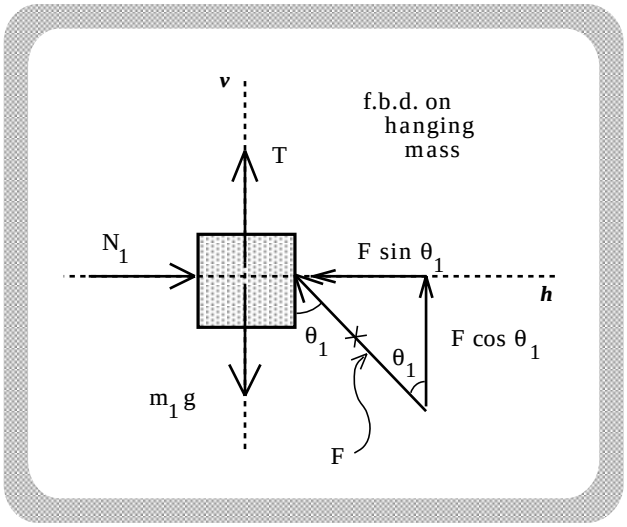


FIGURE 5.14

Note 1: We know that the initial velocity of m_1 is *downward* (m_1 has to be moving downward if m_2 is moving *up* the incline), but we really don't know whether the *acceleration* on m_1 is *up* (i.e., m_1 is slowing) or *down* (i.e., m_1 is increasing speed). Not knowing the relative sizes of m_1 and m_2 makes it impossible to tell which is happening.

As can be seen from the work done above, we are doing this problem on the assumption that the acceleration of m_1 is downward, hence the *negative* sign in front of the $m_1 a$ term on the right-hand side of N.S.L. If we are wrong, it won't matter as long as we are *consistent throughout the problem*. That is, if we assume m_1 accelerated *downward,* we must also assume that m_2 accelerates *up the incline*.

As was noted some sections ago, when the mass values are put into the final equation, the *sign* of the calculated acceleration magnitude will either be *positive* or *negative*. If it is positive, it means we have assumed the correct

direction for the acceleration; if it is negative, it means we have assumed the wrong direction. The magnitude will be the same in either case.

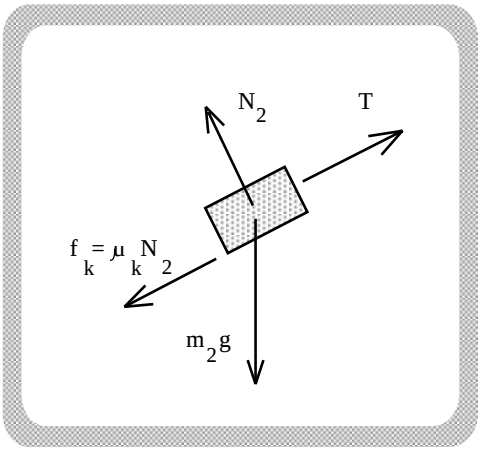


FIGURE 5.15

--Step f: We have one equation (Equation A) and two unknowns (T and a). We need another equation, which means we must look at m_2*'s* motion. Starting the approach over again on m_2:

For m_2:

--Step b: The sketch of the system is still shown in Figure 5.12.

--Step c: A *free body diagram* of the forces acting on m_2 is shown in Figure 5.15. Notice the frictional force acts *down* the incline--opposite the *upward* motion of m_2.

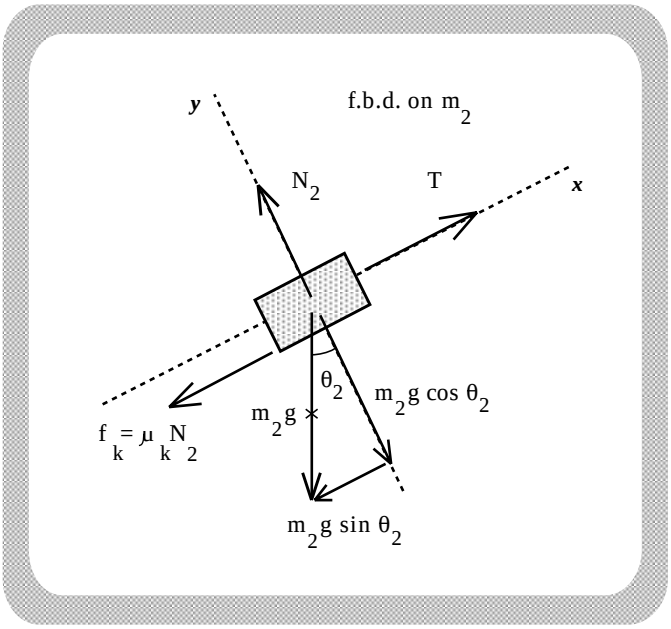


FIGURE 5.16

--Step d: Figure 5.16 shows the placement of appropriate coordinate axes. The *line* of m_2*'s* acceleration is along the incline, so one of the axes (labeled x) is placed along that line with the second axis (labeled y) *perpendicular* to the first.

--Step e: Summing the forces in the *y direction* will allow us to solve for N_2:

$\sum F_y$:

$$N_2 - m_2 g (\cos \theta_2) = m_2 (a_y)$$
$$= 0 \qquad (\text{as } a_y = 0)$$
$$\Rightarrow \quad N_2 = m_2 g (\cos \theta_2) \qquad \text{(Equation B).}$$

Now that we know N_2, we can sum the forces in the *x direction*:

$\underline{\sum F_x}$:

$$T - m_2 g (\sin \theta_2) - \mu_k N_2 = m_2(a_x)$$
$$= m_2 a,$$

(remember, a was defined as the magnitude of the acceleration of both m_1 and m_2). Substituting *Equation B* in for N_2 yields:

$$T - m_2 g(\sin \theta_2) - \mu_k [m_2 g(\cos \theta_2)] = m_2 a.$$

Substituting *Equation A* in for T yields:

$$[m_1 g - F\cos \theta_1 - m_1 a] - m_2 g(\sin \theta_2) - \mu_k [m_2 g(\cos \theta_2)] = m_2 a$$

$$\Rightarrow \quad a = [m_1 g - F\cos \theta_1 - m_2 g(\sin \theta_2) - \mu_k m_2 g(\cos \theta_2)] / (m_1 + m_2).$$

--Step 6: Finito!

4.) <u>Example 2: Incline and Pulley using the Formal Approach as expected ON A TEST:</u>

<u>For m_1</u>: (f.b.d., axes, and components shown in Figure 5.14)

$\underline{\sum F_v}$:

$$T - m_1 g + F\cos \theta_1 = -m_1(a)$$
$$\Rightarrow \quad T = m_1 g - F\cos \theta_1 - m_1 a \quad \text{(Equation A).}$$

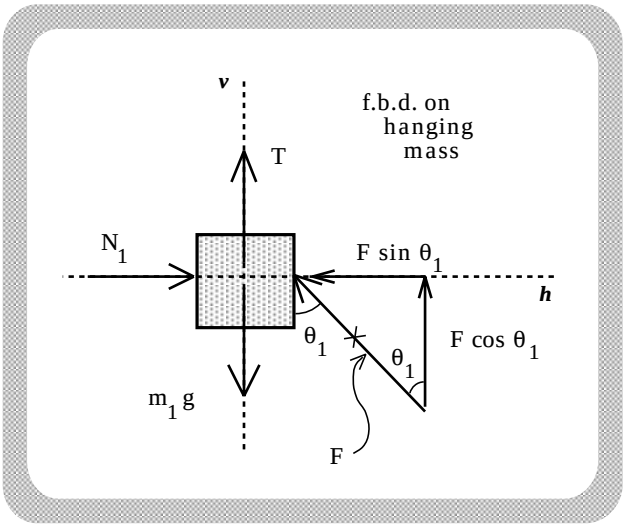


FIGURE 5.14

<u>For m_2</u>: (f.b.d., axes, and components shown in Figure 5.16)

$\underline{\sum F_y}$:

$$N_2 - m_2 g (\cos \theta_2) = m_2 (a_y) \qquad (= 0 \text{ as } a_y = 0)$$
$$\Rightarrow \quad N_2 = m_2 g (\cos \theta_2) \qquad \text{(Equation B).}$$

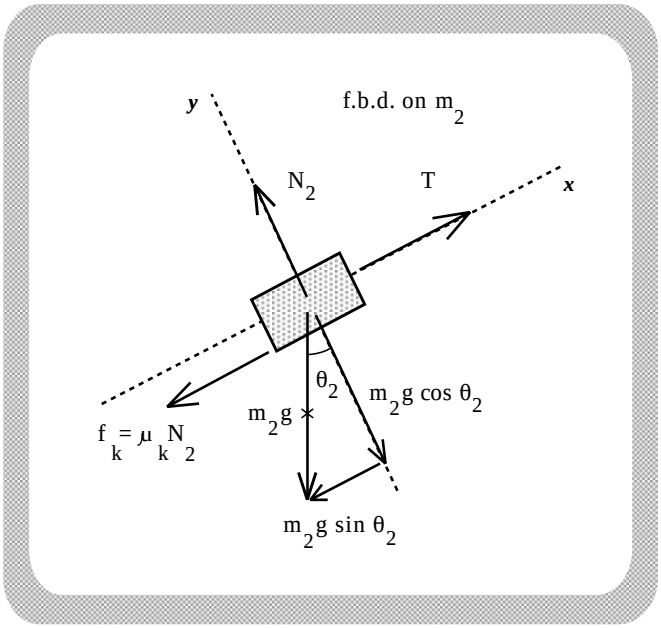


FIGURE 5.16

$\underline{\sum F_x}$:

$$T - m_2 g (\sin \theta_2) - \mu_k N_2 = m_2(a_x)$$
$$= m_2 a,$$

Using *Equation B* to eliminate N_2 and *Equation A* to eliminate T, we can write:

$$T - m_2 g(\sin \theta_2) - \mu_k N_2 = m_2 a$$
$$[m_1 g - F\cos \theta_1 - m_1 a] - m_2 g(\sin \theta_2) - \mu_k [m_2 g(\cos \theta_2)] = m_2 a$$

$$\Rightarrow \quad a = [m_1 g - F\cos \theta_1 - m_2 g(\sin \theta_2) - \mu_k m_2 g(\cos \theta_2)] / (m_1 + m_2).$$

5.) Observations about *Example Problem 2*:

a.) Let's try some numbers: Assume $m_1 = 5$ *kg*, $m_2 = 8$ *kg*, $\mu_k = .4$, the applied force $F = 3$ *newtons*, $\theta_1 = 50^o$, and $\theta_2 = 30^o$. Re-ordering, remembering that the magnitude of the *acceleration of gravity* $g = 9.8$ m/s^2, and omitting units to save space, the calculation becomes:

$$a = [[m_1 g - m_2 g(\sin \theta_2) - \mu_k m_2 g(\cos \theta_2)] - F\cos \theta_1] / (m_1 + m_2)$$
$$= [g [m_1 - m_2(\sin \theta_2) - \mu_k m_2(\cos \theta_2)] - F\cos \theta_1] / (m_1 + m_2)$$
$$= [(9.8) [(5) - (8) \sin 30^o - (.4) (8) \cos 30^o] - (3) \cos 50^o] / (5 + 8)$$
$$= -1.48 \text{ m/s}^2.$$

Note: We have a *negative sign* in front of the calculated acceleration. Why? In the problem, the hanging mass m_1 was stated to be traveling downward (that was the way the problem was set up). Additionally, we assumed that m_1 was *accelerating* downward (we didn't know for sure--that was our

guess). What the *negative sign* means is that, given the mass-values used, m_1*'s acceleration* was not *downward* but *upward*. Evidently, the hanging mass moved downward, *slowing in the process*.

b.) If m_1 had been moving *upward*, what would have been different? The *free body diagram* for m_1 would have looked identical to that shown in Figure 5.13, but m_2*'s* f.b.d. would have been different (see Figure 5.17). We still wouldn't have known the acceleration's direction, so we still would have had to assume one. If we had taken it to be *up the incline*, the final acceleration expression derived from Newton's Second Law would have been:

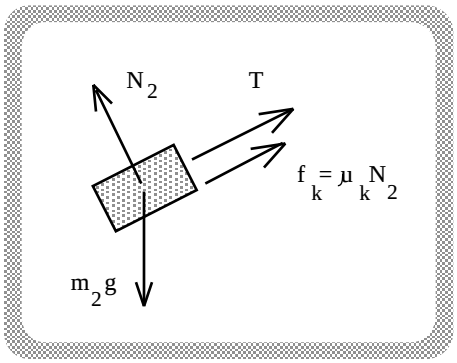


FIGURE 5.17

$$\begin{aligned} a &= [[m_1 g - m_2 g(\sin\theta_2) + \mu_k m_2 g(\cos\theta_2)] - F\cos\theta_1] / (m_1 + m_2) \\ &= [g\,[m_1 - m_2(\sin\theta_2) + \mu_k m_2(\cos\theta_2)] - F\cos\theta_1] / (m_1 + m_2) \\ &= [(9.8)\,[(5) - (8)\sin 30^o + (.4)(8)\cos 30^o] - (3)\cos 50^o] / (5 + 8) \\ &= +2.69\ m/s^2. \end{aligned}$$

Notice the acceleration value in this case turns out to be positive, implying that the assumed direction-of-acceleration for m_1 (<u>downward</u>) was correct (given the mass values we've used in the evaluation). Evidently, in this case, the hanging mass moved upward, *slowing in the process*.

Conclusion: In both cases, friction was great enough to slow the motion.

6.) <u>Example 2--Incline and Pulley using the Simple Approach:</u> A string attached to a known hanging mass m_1 is threaded over a *frictionless, massless pulley* and attached at its other end to a second known mass m_2 that is supported by a *frictional* incline plane. The mass m_1 is additionally wedged *frictionlessly* against the incline by a force F (see Figure 5.12). Both the angle of the incline θ_2 and the angle θ_1 at which F assaults the block are known. The *coefficient of kinetic friction* μ_k between m_2 and the incline is also known. Assuming the string is inextensible (that is, it isn't acting like a rubber band) and mass m_2 is moving *up* the incline when first viewed, determine the *magnitude* of the *acceleration* of the system.

a.) As before, the question to ask is, "What are the acting forces that motivate the system as a whole to accelerate?"

b.) In this case, there are several forces motivating the total mass of the system (i.e., $m_1 + m_2$) to accelerate. They are:

i.) The force of gravity on m_1: Its magnitude is $m_1 g$ and its effect via the string is to attempt to accelerate m_2 UP the incline.

Note: If we are talking about gravity on m_1, why do we care about m_2? We need some way to assign positiveness or negativeness to forces acting on *the SYSTEM* when, in fact, those forces may well be acting on different bodies oriented in entirely different ways from one another. The easiest way to do that is to pick a single body in the system and determine how a given force will ultimately affect that body. In this case, gravity is pulling m_1 down. This will tend to pull m_2 UP the incline, so I will arbitrarily identify that effect to be associated with a positive force.

ii.) The component of F that pushes m_1 upward: Its magnitude will be $F \cos\theta_1$. Its effect is to attempt to accelerate m_2 DOWN the incline, so we will call it a negative force.

iii.) The component of gravity that pushes m_2 DOWN the incline: Its magnitude is $m_2 g \sin\theta_2$ (this is the component of gravity *along the line of the incline*), and we will call it a negative force.

iv.) The force of friction on m_2: With the normal force equal to $m_2 g \cos\theta_2$, friction's magnitude is $f_k = m_k N = m_k(m_2 g \cos\theta_2)$. The direction of the frictional force will be opposite the direction of motion, or DOWN the incline, making it a negative force.

v.) The temptation might be to add in the tension force. In this case, tension is an internal force--a force that exists as a consequence of the interaction of pieces of the system. Internal forces will not affect the overall motion of the system. That is, if we add the tension force on m_1 (this will be upward on m_1 motivating m_2 to accelerate down the incline) to the tension force on m_2 (this will motivate m_2 to accelerate up the incline), the net effect will be zero. As such, tension IN THIS CASE is ignored.

c.) Summing up all of the forces and putting them equal to the total mass in the system, we get:

$$m_1 g - F(\cos\theta_1) - m_2 g(\sin\theta_2) - \mu_k(m_2 g\cos\theta_2) = (m_1 + m_2)a$$
$$\Rightarrow \quad a = [m_1 g - F\cos\theta_1 - m_2 g(\sin\theta_2) - \mu_k m_2 g(\cos\theta_2)] / (m_1 + m_2).$$

Note: AGAIN, don't get too excited about all of this. The simplified approach isn't always going to work. Get to know both procedures!

E.) Friction and Free Fall:

1.) Consider a body of mass *m* free falling in a fluid from rest under the influence of gravity.

a.) For pure free fall with no fluid-produced friction, theory predicts that the body's acceleration will be equal to *g* and the body's velocity at any point in time will (according to the kinematic equations) be equal to $v = -g\Delta t$ (this comes from $v = v_1 + (-g)\Delta t$, with $v_1 = 0$ and Δt being the *time of flight*).

b.) The situation changes dramatically when friction is taken into account.

2.) Friction in a free fall situation is caused by molecules colliding with the body as it falls through the fluid. The faster the object travels, the more molecules are hit *per unit time* and the more retarding force is generated. When this *frictional force* completely counterbalances gravity, the net force equals zero and the body *stops accelerating* (i.e., its velocity does not change from then on). Once the body reaches that state, it is said to be traveling at "terminal velocity."

3.) Assume a frictional force that is *proportional to the magnitude of the velocity* of the body (i.e., $f \propto v$, which is to say $f = kv$, where k is a proportionality constant related to the density of the material through which the body travels). We want to determine two things: the body's *terminal velocity* and the body's *velocity as a function of time*.

Minor Note: If we had been dealing with *air friction*, the frictional force would have been better approximated as $f = kv^2$.

4.) The first of the two questions is relatively easy:

a.) Figure 5.18 shows the situation. The body accelerates until the frictional force is exactly equal and opposite the force of gravity. At that point, the acceleration becomes zero and we can use N.S.L. to state:

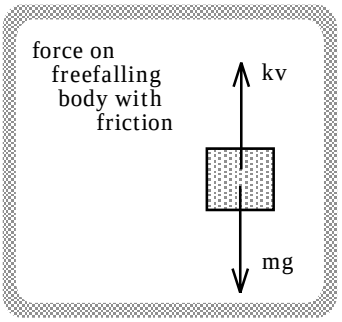


FIGURE 5.18

$$\underline{\Sigma F_y}:$$
$$kv_{term} - mg = m(a)$$
$$= 0 \quad \text{(as "a = 0")}$$

Dividing both sides by *k* yields the maximum velocity the body will attain. This *terminal velocity* is $v_{term} = (mg)/k$.

5.) Determining the body's *velocity as a function of time* is a considerably more difficult process. (Note that for simplicity, *v(t)* will be written from here on as *v*).

a.) Begin by writing N.S.L. in its most general form for the forces acting on the body at an arbitrary point in time *t*:

$$\underline{\Sigma F_y}:$$
$$kv - mg = -m(a)$$
$$= -m[dv/dt].$$

IMPORTANT Note: The sign selection in front of the *ma* term isn't as obvious as one might expect. <u>READ *Section I*</u> at the end of the chapter to understand the problem and its solution.

b.) We need to manipulate this equation to get both velocity-related terms (i.e., both *v* and *dv*/*dt*) on the same sides of the equal sign. By dividing both sides by *-m*, then rearranging, we get:

$$\frac{dv}{dt} = -\left[\frac{k}{m}v - g\right].$$

c.) We need to link the *v* and *dv* term by multiplication or division (the reason for this will become evident shortly). To do so, we rewrite our equation as:

$$\frac{dv}{dt} = \left(-\frac{k}{m}\right)\left[v - \frac{mg}{k}\right].$$

Note: No, you probably wouldn't have thought to do this if you were on your own. Then again, if you already knew the tricks of the trade you would not need to take this class!

d.) The above expression states: The *time rate of change of velocity* equals a *mass related constant* times a *velocity related quantity*.

e.) We are now in the position to divide both sides by $v - mg/k$ and multiply both sides by dt (ah, the sound of mathematicians groaning everywhere--it's OK, though, the equation that follows is true and good if the equation above is assumed). Doing so yields:

$$\frac{dv}{\left[v - \frac{mg}{k}\right]} = \left(-\frac{k}{m}\right)dt.$$

f.) The above equation states: A *differential change in velocity* dv divided by a *velocity dependent quantity* evaluated at time t (where t is the time about which dt straddles), is equal to a *constant* times the size of the *differential time interval* dt over which the velocity change occurred.

Fortunately for those not into semantics, what the equation *says* is not nearly as important as what we can do with it.

g.) If we assume the body's velocity at $t = 0$ is $v = 0$, we can integrate both sides of this equation (i.e., summing the *differential velocity-related-quantities* over the time interval, etc.) between $t = 0$ and t, yielding:

$$\int_{v=0}^{v(t)} \frac{dv}{\left[v - \frac{mg}{k}\right]} = \left(-\frac{k}{m}\right)\int_{t=0}^{t} dt.$$

h.) The integral $\int dt$ simply equals t evaluated at $t = 0$ and t. The integral $\int dv/(v + c)$, where c is a constant, equals the *natural log* of the absolute value of $v + c$, evaluated at the limits. As such, we can write:

$$\int_{v=0}^{v(t)} \frac{dv}{\left[v - \frac{mg}{k}\right]} = \left(-\frac{k}{m}\right)\int_{t=0}^{t} dt$$

$$\Rightarrow \quad \ln\left|v - \frac{mg}{k}\right|_{v=0}^{v(t)} = \left(-\frac{k}{m}\right)t.$$

i.) Noting that *v(t)* is always smaller in magnitude than the terminal velocity of *mg/k*, the argument *v - mg/k* will always be negative. That means we can remove the *absolute value* sign in the *natural log* expression as long as we re-write the argument as *mg/k - v(t)* (this will make that difference *always positive*, as required by the *absolute value sign* we wanted to remove). Doing so, then evaluating, yields:

$$\ln\left|v-\frac{mg}{k}\right|_{v=0}^{v(t)} = \left(-\frac{k}{m}\right)t$$

$$\Rightarrow \quad \ln\left(\frac{mg}{k}-v\right)_{v=0}^{v(t)} = \left(-\frac{k}{m}\right)t$$

$$\Rightarrow \quad \left[\ln\left(\frac{mg}{k}-v(t)\right)-\ln\left(\frac{mg}{k}-0\right)\right] = \left(-\frac{k}{m}\right)t.$$

j.) The expression *ln (a) - ln (b)* equals *ln (a/b)*. As such, our expression can be re-written as:

$$\ln\left[\frac{\left(\frac{mg}{k}-v(t)\right)}{\frac{mg}{k}}\right] = \left(-\frac{k}{m}\right)t.$$

k.) We need to get the velocity variable out from under the *"ln" function*. As $e^{ln\,(a)} = a$, we can use both sides of the equation as exponents of *e* and write:

$$e^{\ln\left[\frac{\left(\frac{mg}{k}-v(t)\right)}{\left(\frac{mg}{k}\right)}\right]} = e^{\left(-\frac{k}{m}\right)t}$$

$$\Rightarrow \quad \left[\frac{\left(\frac{mg}{k}-v(t)\right)}{\left(\frac{mg}{k}\right)}\right] = e^{\left(-\frac{k}{m}\right)t}$$

$$\Rightarrow \quad \frac{mg}{k}-v(t) = \left(\frac{mg}{k}\right)e^{\left(-\frac{k}{m}\right)t}$$

$$\Rightarrow v(t) = \frac{mg}{k}\left[1-e^{\left(-\frac{k}{m}\right)t}\right].$$

l.) Notice that our expression makes sense at the extremes. That is:

i.) At $t = 0$, the $e^{-(k/m)t}$ expression becomes "1" (i.e., $e^0 = 1$) and the right-hand side of the equation becomes ZERO. That is exactly what our *velocity* was supposed to be at $t = 0$.

ii.) At $t = \infty$, the $e^{-(k/m)t}$ expression becomes zero (i.e., the expression $e^{-(k/m)t}$ equals $1/e^{(k/m)t}$, which equals $1/e^{\infty} = 0$ at $t = \infty$) and the right-hand side of the equation becomes mg/k. That, if you remember, was the *terminal velocity* of the body. In short, the extremes seem to fit our expression nicely.

6.) Why do we care? The *Advanced Placement Test* has, on occasion, had questions like: *A body accelerates from rest under the influence of a given velocity dependent force. Determine the body's velocity as a function of time (i.e., what is v(t)).* The functions are not always as stinky as the one we have just examined, but the *approach-for-solving* is similar!

F.) Centripetal Force:

1.) A net force acting on a body produces an acceleration, and an acceleration implies a *change of velocity*. So far, the only accelerations we have dealt with have been those that change the *magnitude* of a body's velocity--that make objects *speed up* or *slow down*. There is another way an acceleration can change a body's motion: it can change the direction of the body's *velocity* vector.

2.) Example--the M.O.B. maneuver:

a.) In days of yore when rock-n-roll was in and '55 Chevys were the *hot rod* of choice, there was a technique the execution of which guaranteed, if not romance, at least close contact between a guy and his date. It was called the infamous M.O.B. maneuver.

The Situation: A boy picks up his date (we will assume the boy is driving). The two like one another. The vehicle is a 1955 Chevy without seat-belts (the norm for 1950's cars). It sports a bench seat (also the norm for 1950's cars). She gets in and sits next to the passenger-side door. He'd like her to sit next to him but is too bashful to ask her to move over. She'd *like* to sit next to him but is too shy to do it on her own.

Solution: the M.O.B. maneuver.

The Maneuver: Traveling at 30 miles per hour, the boy casually approaches an appropriate intersection (see Figure 5.19).

Upon reaching the corner, he makes a hard right turn without warning.

Consequence: she finds herself seated next to him in short order.

How so?

He is attached to the turning car via the steering wheel. As the car turns, the steering wheel exerts a force on him that makes him move on the *same semi-circular path* the car follows. She, on the other hand, is *not* attached to the car (the maneuver works best if the seat has been waxed beforehand). As there are no forces acting on her, she does exactly what any force-free body will do--she continues to move in a *straight line* (Newton's First Law--bodies in motion tend to stay in motion in a straight line unless impinged upon by a force). Doing so insures that he and she will, indeed, meet somewhere around the BINGO in Figure 5.20.

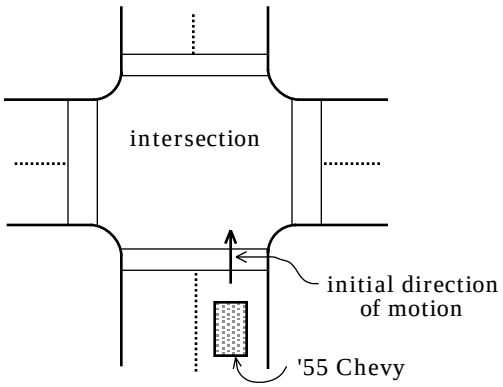


FIGURE 5.19

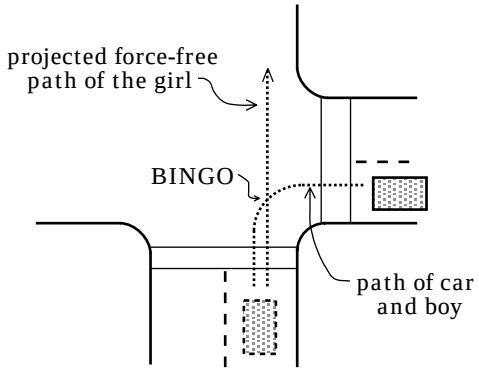


FIGURE 5.20

Note: You might wonder why it is called an MOB maneuver? Although it is probably *politically* and *socially* improper today, in the 1960's it stood for *Move Over, Baby*.

b.) The Point: An acceleration, hence a net force, is needed to *change the direction of a moving body* (hence, the *direction* of the body's *velocity* vector). That force will be perpendicular to the direction-of-motion, which is to say perpendicular to the direction of the velocity vector.

Note 1: The boy entered the curve at 30 mph, and he left the curve at 30 mph. The *magnitude* of his velocity didn't change, but the *direction* did. That would not have been the case if the force provided by the steering wheel had been in any direction other than *perpendicular to the boy's path*.

Note 2: Assuming this direction-changing force is constant in magnitude, it will always make the body move along a circular path about a fixed center. Forces that do this are often referred to as "center-seeking forces."

Note 3: The phrase *center-seeking* is a label only. In a given situation, the combination of forces that collectively qualify for that moniker must exist naturally within the system. They can be *normal* forces, *tension* forces, *gravitational* forces, *frictional* forces, *push-me-pull-you* forces, or some combination thereof.

Note 4: The word "centripetal" means *center-seeking*. It is the <u>*combination*</u> *of forces that is center-seeking* that is labeled the net *centripetal force*. Again, this is a label only. We are not talking about a new *kind* of force--only a new kind of *situation* in which the same old forces might be applied.

Note 5: Students often become confused when dealing with *centripetal forces* generated by friction between a car's tires and the road. For clarification, consider the car example above with one modification: assume the car is driving on a *dirt* road. What happens when the car's wheels are cranked into a hard, tight turn (see Figure 5.21)?

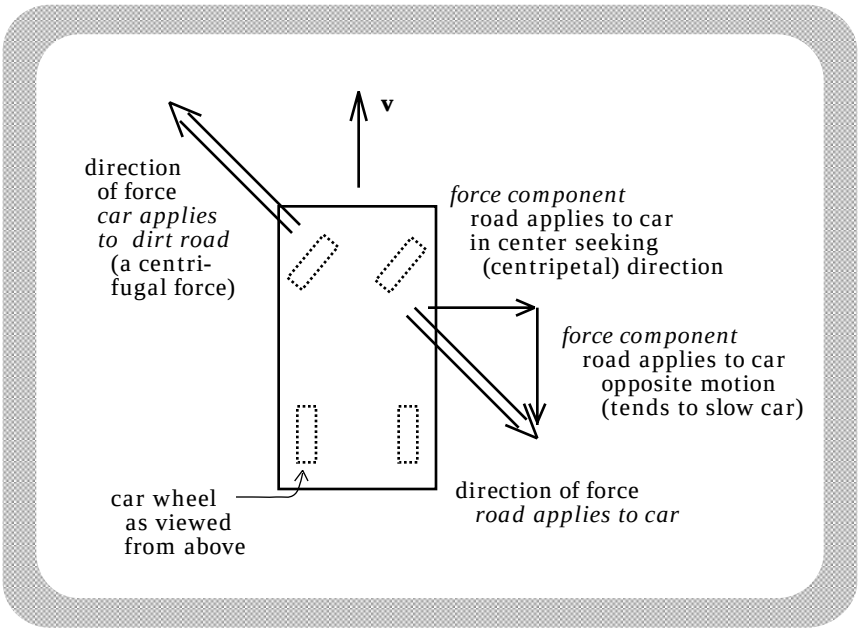

v
direction of force car applies to dirt road (a centrifugal force)
force component road applies to car in center seeking (centripetal) direction
force component road applies to car opposite motion (tends to slow car)
car wheel as viewed from above
direction of force road applies to car

FIGURE 5.21

The car's tires apply a *frictional force* to the ROAD and end up throwing dirt outward away from the car's ultimate path. The force the *car applies to the road* is, therefore, <u>directed outward</u> relative to the ultimate path of the car. On the other hand, the *road* applies a frictional force f_{car} to the car which, according to Newton's Third Law, is *equal in magnitude* and *opposite in direction* to the force the car applies to the road.

This force, f_{car}, has one component that is oriented opposite the direction of the car's motion and another component that is perpendicular to the direction of motion. The latter component pushes the CAR inward, making it follow a curved path. This is the center-seeking (*centripetal*) force in the system.

When using Newton's Second Law to analyze the motion of the car, the only forces we are interested in are the forces *acting on the car*. As such, the force the car applies to the road has no place in the analysis.

3.) Other Examples:

a.) Consider the moon orbiting the earth: The moon does not move off into space in a straight line because the earth exerts a *gravitational force* on the moon, pulling it away from the path it would have taken if it had been force-free (see Figure 5.22). Consequence: the moon is pulled into a nearly circular path around the earth.

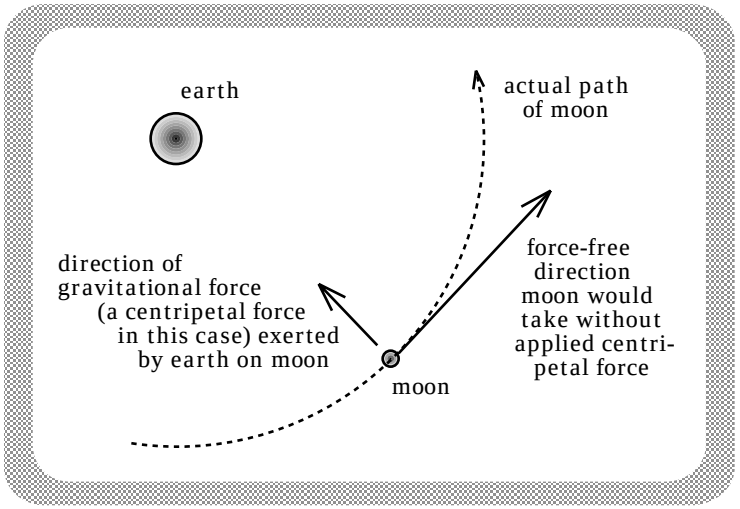

earth
actual path of moon
direction of gravitational force (a centripetal force in this case) exerted by earth on moon
force-free direction moon would take without applied centri-petal force
moon

FIGURE 5.22

In this case, *gravity* is the *center-seeking* (centripetal) force that naturally exists within the system.

Note: The moon's path isn't perfectly circular due to the initial conditions under which the moon's motion was generated.

b.) The *hammer throw* in Olympic competition is executed using a mass (a ball) attached to a chain. The contestant grasps the chain, swings it around and around until he or she reaches a relatively high spin-speed, and then lets the ball and chain go.

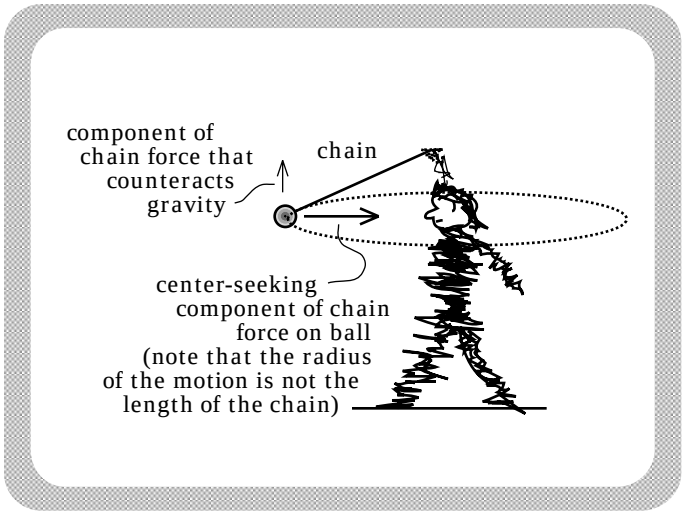

component of chain force that counteracts gravity
chain
center-seeking component of chain force on ball (note that the radius of the motion is not the length of the chain)

FIGURE 5.23

The proper technique does not suggest twirling the ball overhead in a horizontal plane, but let's assume old Joe is a bit off-form and does not know any better (see Figure 5.23). From whence does the *center-seeking* (centripetal) force on the ball come?

It comes from the *horizontal component* of the *tension* force provided by the chain. That component is the *centripetal force* in the system.

4.) Example Problem--Deadman's Curve: A car approaches a curve of known radius R. The *coefficient of static friction* between the tires and the road is μ_s. At what *maximum velocity* can the car take the curve without breaking traction and spinning out (Figure 5.24 shows situation from above)?

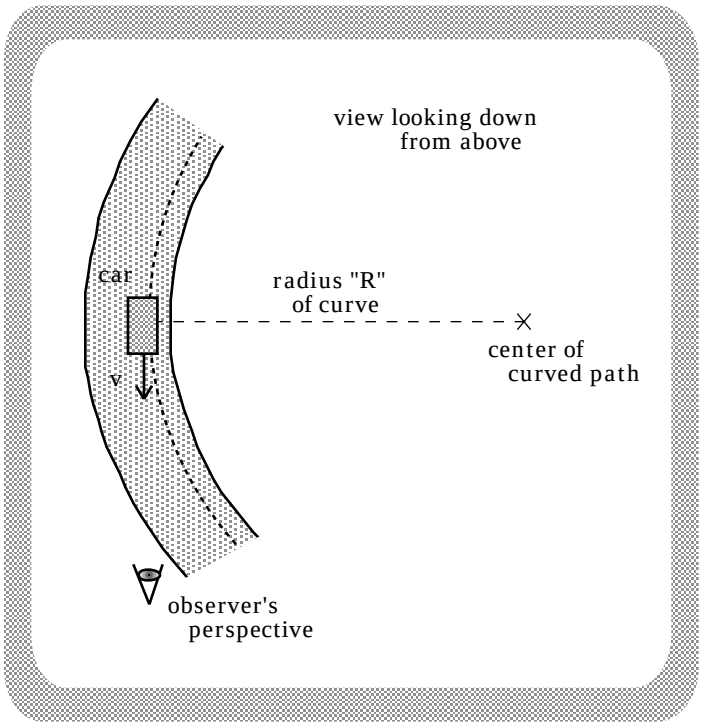


FIGURE 5.24

Note: Assume the wheels are turned so slightly that the component of f_s that opposes the car's motion is zero. That is, assume f_s is fully centripetal.

a.) This is a Newton's Second Law problem. We need to identify the important forces acting on the car, so we will use an f.b.d. perspective that views the car head-on (the view shown in Figure 5.25 depicts that perspective).

b.) For the vehicle to follow a circular path, we know there must be a force, hence acceleration, in the center-seeking direction (i.e., in the direction oriented *toward the center of the circle upon which the car is moving*).

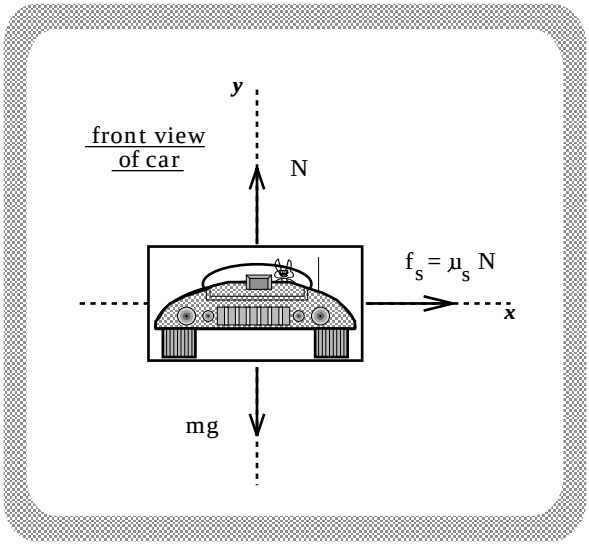


FIGURE 5.25

Note: The *centripetal direction* will be *perpendicular* to the direction of motion. As seen in our sketch, the car is coming *out of the page* which means the center-seeking direction is to the right in the plane of the page.

c.) Using N.S.L., we get:

$$\underline{\Sigma F_y}:$$
$$N - mg = ma_y.$$

As $a_y = 0$ (the car is not jumping up off the road), we get:

$$N = mg.$$

$$\underline{\Sigma F_{\text{center-seeking}}}:$$

$$\mu_s N = ma_c.$$

Substituting in $N = mg$ from above and solving for a_c, we get

$$a_c = \mu_s(mg)/m$$
$$= \mu_s g.$$

Note: This will be the maximum acceleration the road/tire contact can generate. How so? Because $\mu_s N$ is not just any static frictional force, it is the *maximum* static frictional force. It will be related to the maximum *centripetal acceleration* available to the car when it attempts to follow the curve, which in turn is related to the maximum *velocity* possible.

d.) If we had a constant acceleration whose direction was *tangent to the curve*, the velocity's magnitude would change with time and we could determine *v(t)* by solving the expression $a = dv/dt$. The problem? The acceleration a_c is *radially* directed. That means it is not oriented to change the velocity's magnitude--it is oriented to change the velocity's direction. As such, neither $a = dv/dt$ nor kinematics will help us.

We need to derive a relationship between the *velocity of the car* and the *acceleration* component that is changing the car's *direction* of motion (i.e., the centripetal acceleration). That derivation is at the end of the chapter (you'll never be asked to reproduce that derivation by yourself--it has been included so that you can convince yourself that nothing *dirty* has been done here).

e.) <u>The bottom line of that derivation:</u> if a body moving with *velocity magnitude v* is to move into a circular path of radius *R*, it must ex-

perience a center-seeking (centripetal) acceleration a_c numerically equal to v^2/R.

f.) With this information, we return to our problem. If $a_c = v^2/R$ and, from above,

$$a_c = \mu_s g,$$

we get

$$v^2/R = \mu_s g$$

$$\Rightarrow \quad v = (\mu_s R g)^{1/2}.$$

Note 1: This solution does make sense intuitively. Think about your own experience on the highways and byways of America: tight turns are taken slowly while wide turns are taken at higher speeds. That is exactly what our equation predicts. The larger the curve (i.e., the bigger *R*), the larger the predicted maximum velocity.

Note 2: The units for the variables under the radical had better work out to *meters per second*. Checking: the coefficient of static friction is unitless; *R* has the units of *meters*; and *g* the units of *meters per second squared*. The net units under the radical are *meters squared per seconds squared*, the square root of which is *meters per second*.

5.) Example Problem 2--Deadman's Curve with Banked Incline: Our car now approaches a banked curve of angle θ, radius *R*, and coefficient of static friction μ_s. What is the maximum velocity v_{max} with which the car can take the curve without flying up over the top of the embankment?

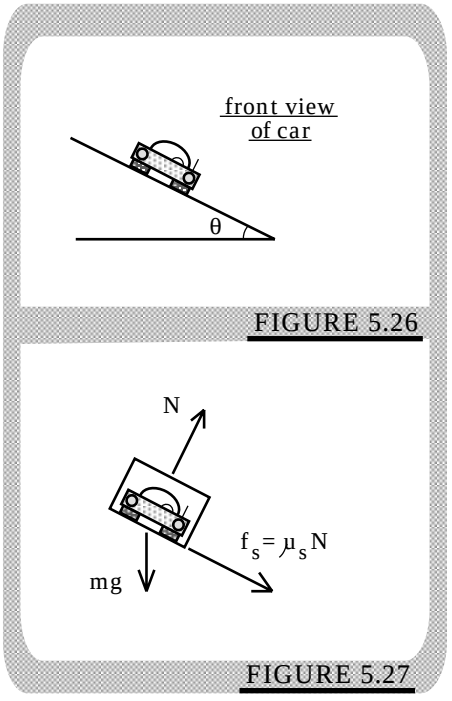


FIGURE 5.26

Note 1: As viewed from above, the car moves in a semi-circular path that looks exactly like the path taken by the car in the previous problem. We will, therefore, use Figure 5.24 to depict the overhead view here. A head-on perspective is shown in Figure 5.26. A *free body diagram* for the situation is shown in Figure 5.27.

FIGURE 5.27

Note 2: The *maximum velocity* will occur when the maximum static frictional force just holds the car from breaking loose and sliding up over the top of the bank. That frictional force must, therefore, be directed down the incline.

Note 3: One axis must be along the line of centripetal acceleration. That *center-seeking direction* is oriented along the line of the RADIUS of the car's motion. That direction is *not* along the line of the incline. That is, even though the car is physically tilted, the direction toward the CENTER of the car's circular path is in the *horizontal*. As such, one of the axes has been defined in that direction.

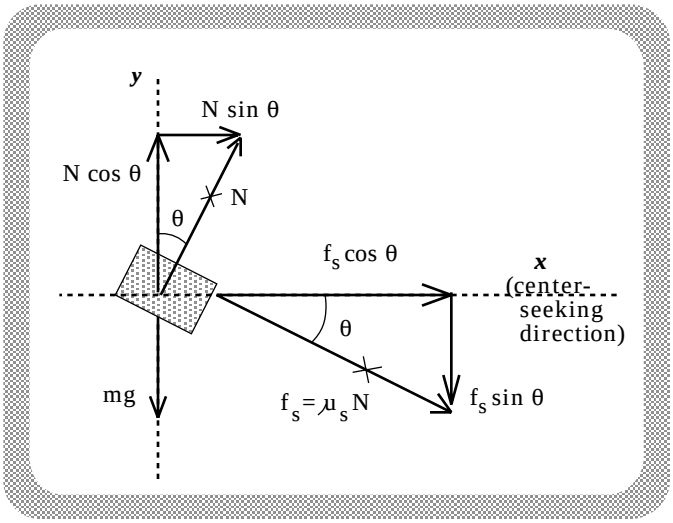


FIGURE 5.28

a.) Examining the modified *free body diagram* (modified in the sense that axes and force components have been included) shown in Figure 5.28, Newton's Second Law yields:

$$\underline{\Sigma F_y}:$$

$$N \cos\theta - mg - f_s \sin\theta = ma_y.$$

b.) As $a_y = 0$, and noting that $f_s = \mu_s N$, we get:

$$N\cos\theta - mg - (\mu_s N)\sin\theta = 0$$

$$\Rightarrow \quad N = (mg) / (\cos\theta - \mu_s \sin\theta).$$

c.) In the *center-seeking* direction:

$$\underline{\Sigma F_{\text{center-seeking}}}:$$

$$N \sin\theta + f_s \cos\theta = ma_c.$$

d.) Substituting in $f_s = \mu_s N$, $N = [(mg)/(\cos\theta - \mu_s \sin\theta)]$, and letting $a_c = v^2/R$, the equation

$$N \sin\theta + \mu_s N \cos\theta = ma_c$$

becomes

$$[(mg)/(\cos\theta - \mu_s \sin\theta)]\sin\theta + \mu_s[(mg)/(\cos\theta - \mu_s \sin\theta)] \cos\theta = mv^2/R.$$

e.) Dividing out the *m's* and rearranging, we get:

$$v_{max} = [R\,(g \sin\theta + \mu_s g \cos\theta)/(\cos\theta - \mu_s \sin\theta)]^{1/2}.$$

Note 1: When $\theta = 0$ (i.e., the same situation we had in the Example Problem 1), we get $v_{max} = (\mu_s Rg)^{1/2}$ as expected.

Note 2: For $\theta = 30^o$, $\mu_s = .25$ (this corresponds to a relatively slick roadway), and a radius of 50 meters, the calculated value of $v_{max} = 21.77$ *m/s*. This is right around 50 mph.

6.) Alternate Example--Problem 2: The *free body diagram* in Figure 5.29 presents another possibility. If the angle of the inclined road is great enough, it is possible that the maximum static frictional force on a stationary or slowly moving car would not be large enough to hold the car on the incline. Consequence: the car would break loose and slide *down* the incline.

In that case, a car could make it through the turn but it would have to be moving with a velocity above some minimum value. As the car would want to slide down the incline for velocities below that minimum, the static frictional force would be up the incline (the car wants to slide down).

Consider the car and embankment used in *Part 5* above. What is the *minimum velocity* v_{min} the car can take the curve without sliding down into the infield at the bottom of the embankment?

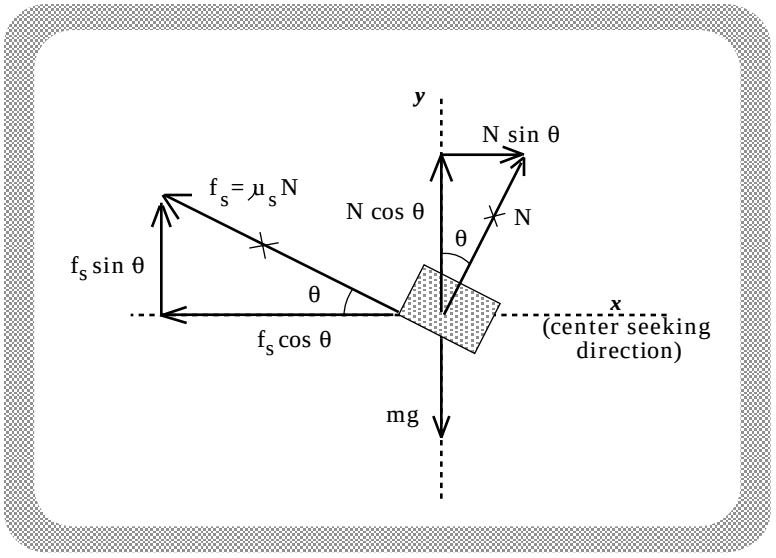


FIGURE 5.29

a.) The f.b.d. for this situation is shown in Figure 5.29:

b.) N.S.L. maintains:

$\underline{\sum F_y}$:

$$N \cos\theta - mg + f_s \sin\theta = ma_y.$$

As $a_y = 0$ and $f_s = \mu_s N$, we get:

$$N \cos\theta - mg + (\mu_s N) \sin\theta = 0$$

$$\Rightarrow \quad N = (mg) / (\cos\theta + \mu_s \sin\theta).$$

c.) In the *center-seeking* direction:

$$\underline{\Sigma F_{\text{center-seeking}}}:$$

$$N \sin\theta - f_s \cos\theta = ma_c.$$

d.) Substituting $f_s = \mu_s N$; substituting the quantity $[(mg)/(\cos\theta + m_s \sin\theta)]$ for N; and setting $a_c = v^2/R$, the equation

$$N \sin\theta - \mu_s N \cos\theta = ma_c$$

becomes

$$[(mg)/(\cos\theta + \mu_s \sin\theta)]\sin\theta - \mu_s[(mg)/(\cos\theta + \mu_s \sin\theta)]\cos\theta = mv^2/R.$$

e.) Dividing out the *m's* and rearranging, we get:

$$v_{min} = [R (g \sin\theta - \mu_s g \cos\theta)/(\cos\theta + \mu_s \sin\theta)]^{1/2}.$$

Note 1: When $\theta = 0$ (the situation we had in the Example Problem 1), we get:

$$v_{min} = (-\mu_s Rg)^{1/2},$$

an imaginary value. This implies that there is *no minimum speed* at which a car could take a flat curve and *not* make it through.

Note 2: For $\theta = 30^o$, $\mu_s = .25$ (again, this is a very slick roadway), and a radius of 50 meters, the calculated value of $v_{min} = 11.9$ *m/s*. This is around 25 mph. Below this speed, the car will not hold traction with the road and will slide down toward the bottom of the bank.

7.) Derivation of Centripetal Acceleration Expression: You will not be asked to duplicate this proof; it is included for the sake of completeness.

It was stated above that the magnitude of any *centripetal acceleration* acting on an object moving in a circular path is numerically equal to v^2/R, where v is the magnitude of the object's velocity and R the radius of the path. The derivation from which that claim comes follows:

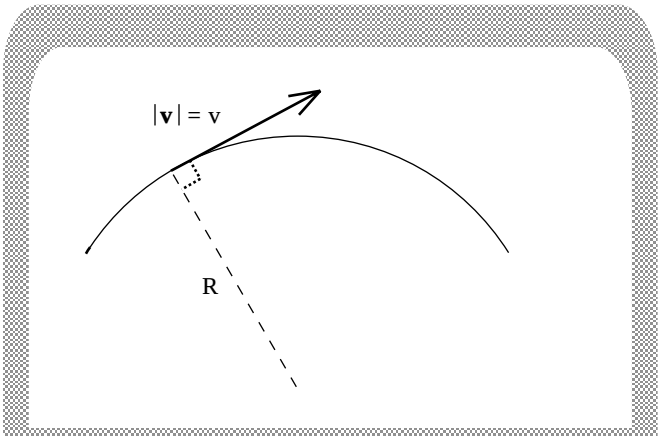


FIGURE 5.30a

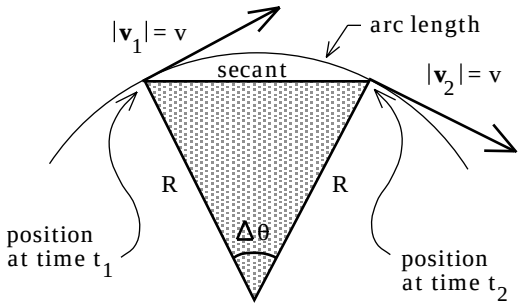


FIGURE 5.30b

a.) A body with a constant velocity-magnitude v moves on a circular path of radius R as shown in Figure 5.30a. Between times t_1 and t_2, the body moves through an angular displacement of $\Delta\theta$. Figure 5.30b shows this set-up along with a secant connecting the position points at times t_1 and t_2. Notice that:

i.) The triangle is isosceles (two of the sides are equal in length to the radius R of the motion);

ii.) The third side of the triangle (the secant) is identified as Δs;

iii.) Δs and the arc length subtended by $\Delta\theta$ are not equal as long as $\Delta\theta$ is large; and

iv.) The arc length is numerically equal to the velocity-magnitude v times the time interval (Δt) required to travel between positions 1 and 2, or:

$$\text{arc length} = v\ \Delta t.$$

b.) The vector difference between the velocity vectors at t_1 and t_2 yields a change of velocity vector $\Delta \boldsymbol{v}$ over that period of time. That vector subtraction is shown in Figure 5.31. Notice that:

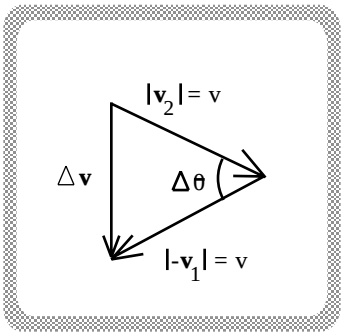


FIGURE 5.31

i.) The direction of the vector difference Δv is approximately oriented toward the center of the circular path on which the body moves;

ii.) The triangle created by the subtraction is isosceles (two of the sides are numerically equal to the magnitude of the constant velocity v); and

iii.) Because the velocity vectors are perpendicular to their respective radii, the isosceles triangle created by the radii and secant (Figure 5.30b) is *similar* to the triangle created by the velocity-vector-difference (Figure 5.31).

c.) With *similar* triangles, the ratio of any two sides of one triangle will be equal to the ratio of the corresponding sides of the second triangle. As such, we can write:

$$\Delta v/v = \Delta s/R.$$

d.) If the time interval is allowed to become very small (i.e., t_1 and t_2 approach one another), the arc length approaches Δs and we can write:

$$\text{arc length} = \Delta s \text{ (in the limit as } \Delta t \text{ goes to zero).}$$

But the arc length equals $v\Delta t$, which implies that as $\Delta t \rightarrow 0$,

$$\Delta s \rightarrow v\Delta t.$$

e.) Plugging this back into our ratio, we can re-write $\Delta v/v = \Delta s/R$ as:

$$\lim_{\Delta t \rightarrow 0} \left(\Delta v / v \right) = \left(v\Delta t / R \right).$$

Rearranging yields:

$$\lim_{\Delta t \rightarrow 0} \left(\Delta v / \Delta t \right) = \left(v^2 / R \right).$$

f.) The left-hand side of this expression is the definition of instantaneous acceleration. What acceleration? The acceleration whose direction is *center-seeking*.

Bottom line: The *magnitude* of the *centripetal acceleration* required to move a body whose velocity magnitude is v into a circular path of radius R is v^2/R.

G.) A Note About MASS:

1.) Although the material you are about to read will not be included on a test, it is important that you understand what *mass* really is. READ THIS SECTION ONCE. If nothing else, it will help you stave off massive confusion later on when we get to rotational motion.

2.) There are certain characteristics that are true of all material objects. For instance, all objects have *a tendency to resist changes in their motion.*

a.) Example: A rock placed in space will not suddenly, spontaneously accelerate for no reason. It will sit in its place until a force makes it move.

b.) The unwillingness of an object to spontaneously change its motion is called *inertia.* As the amount of inertia an object has is intimately related to how much *force* will be required to *accelerate* the object, quantifying the idea of inertia is important. Early scientists satisfied that need by defining an *inertia-related quantity* they called "inertial mass."

c.) The system they devised is simple. A platinum-iridium alloy cylinder, currently housed in a vault at the *Bureau of Weights and Measures* in Sevres near Paris, France, was defined as having *one kilogram* of inertial mass. All other inertial mass values are measured relative to that cylinder. That is, an object with the same amount of *resistance to changing its motion* as does the standard is said to have "one kilogram of *inertial mass*." An object with twice the *resistance to changing its motion* is said to have two kilograms of *inertial mass*; one-half the resistance implies one-half kilogram of *inertial mass,* etc.

In other words, the inertial mass of a body gives us a numerical way of defining how much inertia an object has RELATIVE TO THE STANDARD. Put still another way, *inertial mass is a relative measure of a body's tendency to resist changes in its motion.*

d.) Although France is a beautiful country, it would be terribly inconvenient for laboratory scientists around the world if they had to travel to France every time they wanted to determine an *inertial mass* value, so scientists further generated a laboratory technique for measuring *inertial masses.* It uti-

lizes an inertial balance--a tray mounted on two thin blades that allow the tray to vibrate back and forth. The more mass that is placed in the tray, the slower the tray vibrates. A simple formula relates the tray's vibratory rate (its *period* of motion) to the amount of *inertial mass* there is in the tray.

Although it works, using an inertial balance is a VERY CUMBERSOME and time consuming operation.

3.) There are other characteristics that are true of material objects. For instance, all massive objects are attracted to all other massive objects (at least according to Newton). We call this attraction *gravity*.

a.) A measure of a body's *willingness to be attracted to another body* is related to what is called the "gravitational mass" of an object.

b.) To provide a quantitative measure of *gravitational mass*, scientists have taken an agreed upon object as the standard against which all subsequent *gravitational mass measurements* are made (again, this standard is housed today in Sevres, France).

c.) The technique for measuring *gravitational mass* utilizes a *balance* or *electronic scale*. The object is placed on a scale which consists of a spring-mounted pan. The gravitational attraction between *the object and the earth* pulls the object toward the earth and compresses the spring in the process. The scale is calibrated to translate spring-compression into gravitational mass (assuming that is what the scale is calibrated to read--in some cases, such scales are calibrated to read force, hence American bathroom scales measure in pounds).

MEASURING GRAVITATIONAL MASS IS EASY.

4.) Somewhere down the line someone noticed a wholly unexpected and profoundly improbable relationship between gravitational and inertial mass. It was observed that if the same standard (that is, the same object) was used for both, a second object with twice the *gravitational mass* relative to the standard would also have twice the *inertial mass*.

a.) THIS DOES NOT HAVE TO BE THE CASE. There is no obvious reason why a body with twice the *resistance to changing its motion* (relative to the standard) should also have twice the *willingness to be attracted to other objects*. The two characteristics are completely independent of one another, yet they appear to parallel one another to a high degree of precision (in fact, the best comparisons to date have accuracy to around 10^9 with *no discrepancy found* even at that order of magnitude).

b.) Scientists could have called the units of *gravitational mass* anything they wanted, but because they knew the parallel between *gravitational mass* and *inertial mass* existed, they decided to give *gravitational mass* the units of "kilograms."

That means that as defined, a body with two kilograms of *gravitational mass* also has two kilograms of *inertial mass*.

c.) The beauty of this choice is obvious. Newton's Second Law relates a body's acceleration to the force applied to it. The proportionality constant--the mass term--is an *inertial mass* quantity (the body's resistance to changing its motion is the characteristic that governs how much acceleration the body will feel when a given force is applied). Determining *inertial masses* in the lab is a pain in the arse--inertial balances are not easy devices to set up or use. But *gravitational mass* is easy to measure. All you need is an electronic balance. So when you or I or any lab technician needs to know a body's *inertial mass*, all we have to do is measure the body's gravitational mass on a scale and we have what we want.

d.) Bottom line: Due to the parallel, most people no longer distinguish between *gravitational mass* and *inertial mass*. As the two are numerically interchangeable, people nowadays simply refer to a body's "mass" and leave it at that.

5.) There is an interesting consequence of this peculiar parallel between *gravitational mass* and *inertial mass* which is most easily observed by considering the following question: If a body with two kilograms of *gravitational mass* is attracted to the earth twice as much as a body with only one kilogram of gravitational mass, why doesn't the two kilogram mass free fall toward the earth faster than the one kilogram mass?

a.) The answer is simple. A body with twice the *gravitational mass* also has twice the *inertial mass*; it needs a greater force to overcome its greater inertia.

b.) In short, it does not matter how massive an object is, its *inertia* and its *willingness to be attracted to the earth* will always balance one another out making the object accelerate at the same rate as all other objects (assuming you ignore air friction, etc.).

H.) Fictitious Forces: Centrifugal And Others:

1.) Although the phrases look and sound similar, the ideas behind *centripetal force* and *centrifugal force* are patently different.

a.) The phrase *centripetal force* is a label used to identify forces that motivate a body out of straight line motion and into curved motion. These *center-seeking forces* must exist *naturally* within a system.

b.) The phrase *centrifugal force* is a label used to identify forces that must be ASSUMED TO EXIST if one is to use Newton's Second Law in analyzing problems from a non-inertial (i.e., accelerated) frame of reference. What does this mean? Read on.

2.) Consider the following example (a slight modification of the M.O.B. maneuver): You are driving in your car. You notice that the box sitting next to you slides *away from you* as you make a left turn (Figure 5.32). Does that mean there is a force acting to push the parcel toward the right door?

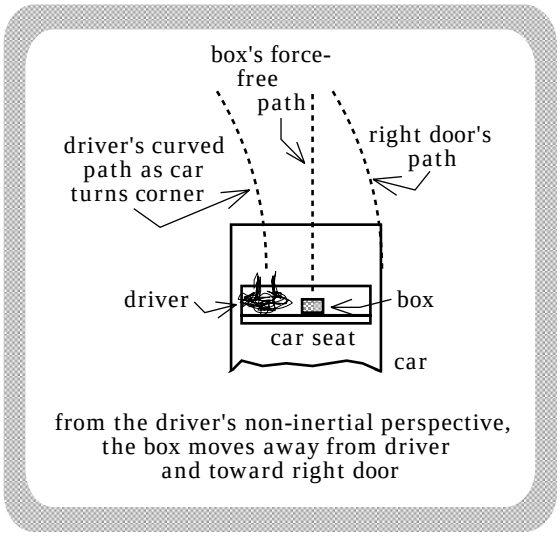

box's force-free path
driver's curved path as car turns corner
right door's path
driver
box
car seat
car
from the driver's non-inertial perspective, the box moves away from driver and toward right door

FIGURE 5.32

a.) The answer to the question is *no*. The only forces that are acting on the parcel are gravity and a normal force from the seat (we'll ignore friction). So what is going on?

b.) What is really happening is that *you* are moving away from the parcel (it continues moving in a straight line path, relative to the street) as you and the car are centripetally forced into the turn.

c.) The problem? If you wanted to analyze the parcel's motion from *your frame of reference* (i.e., from a frame attached to the centripetally accelerating car), Newton's Second Law shouldn't work. Why? Because N.S.L. only works in non-accelerated frames of reference.

d.) What's nice is that you *can* make N.S.L. work if you are clever. If you assume there exists a *fictitious force* acting to push the parcel away from you, Newton's Laws will work just fine. For circular motion, that force is called a *centrifugal force*. Its magnitude is equal to mv^2/R. In short, you don't *have* to analyze the problem from a non-inertial frame of reference, but you can if you are so disposed.

3.) After all that, reconsider the initial statement in *Part H-1a* and *b*:

a.) The phrase *centripetal force* is a label used to identify forces that motivate a body out of straight line motion and into curved motion. These *center-seeking forces* must exist *naturally* within a system.

b.) The phrase *centrifugal force* is a label used to identify forces that must be ASSUMED TO EXIST if one is to use Newton's Second Law in analyzing curved-path problems from a non-inertial (i.e., accelerated) frame of reference.

Big Note: What you are about to see is *for your own edification only*. You will not be tested on it. It has been included because the idea of fictitious forces is used a lot at the college level in analysing certain types of physics problems. If you ever need to understand the approach, the following should help.

4.) You are tethered to the center of a rotating platform (Figure 5.33a). You feel a frictional force that keeps you from sliding laterally across the platform, you feel gravity, you feel a normal force, you feel the force of the tether pulling you in, and you feel what appears to be a force pushing you outward. Let's asume you've measured the tension in the tether and found it to be *T*. How fast is the platform moving? Analyze this situation from both an inertial and a non-inertial frame of reference.

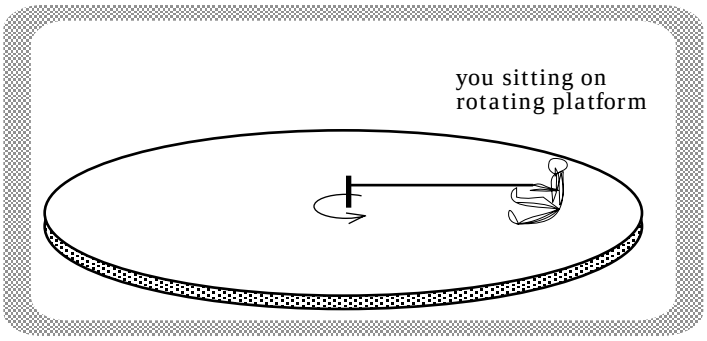


FIGURE 5.33a

a.) From an inertial frame of reference (i.e., a *stationary* observer looking from the platform's side), the forces acting on you are shown in Figure 5.33b. Using N.S.L., we can write:

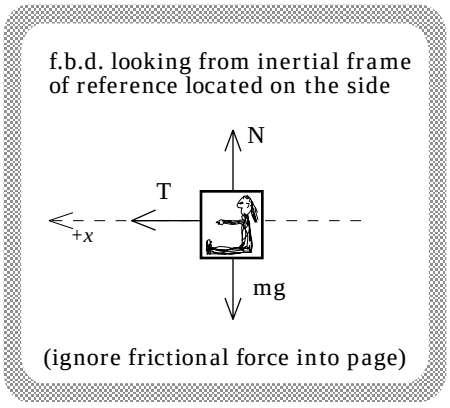


FIGURE 5.33b

$$\underline{\Sigma F_x}:$$

$$T = m(a_x)$$

$$= m(v^2/R)$$

$$\Rightarrow \quad v = (TR/m)^{1/2}.$$

b.) From a non-inertial frame of reference (i.e., someone sitting next to you on the platform), the forces acting on you are shown in Figure 5.33c. Noting that as far as you are concerned, you aren't accelerating at all (you don't seem to be moving along the line of the tether), we can use N.S.L. to write:

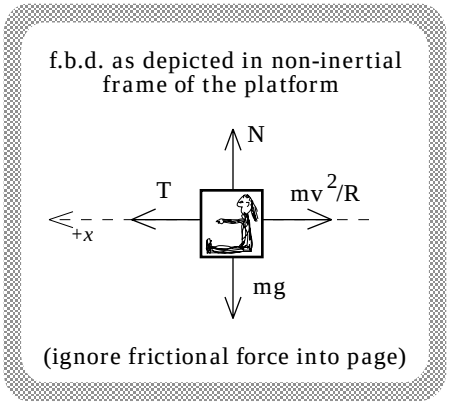


FIGURE 5.33c

$\underline{\sum F_x}$:

$T - m(v^2/R) = ma_x = 0$ (as $a_x = 0$ in this frame)

$\Rightarrow \quad v = (TR/m)^{1/2}$.

c.) The velocities determined in both frames are the same.

Note: The trick is to make the actual *ma* value, as observed in the inertial frame (this would be mv^2/R in this case), into a fictitious force in the non-inertial frame, then to look to see if there appears to be an apparent acceleration in the non-inertial frame. In the *car rounding the corner* example, the box appeared to be accelerating away from you, so in that case, there is an acceleration as viewed in the non-inertial frame of your car. In the above example, there doesn't appear to be acceleration in the non-inertial frame as you move around the circle, so $a = 0$ in that case. In both cases, though, you took the real, inertial acceleration (for these cases, v^2/R), multiplied it by m, and treated it as though it were a force.

5.) One more *fictitious force* example: A mass on a spring is placed in an accelerating rocket (Figure 5.34a) out in space (i.e., no gravity). The rocket's acceleration is known to be A. Assuming you also know the spring constant, by how much is the spring elongated from its equilibrium position (the variable y in the sketch)? Analyze the situation from both the inertial frame of someone watching from outside the rocket, and then from the non-inertial frame of someone sitting inside the rocket with the spring.

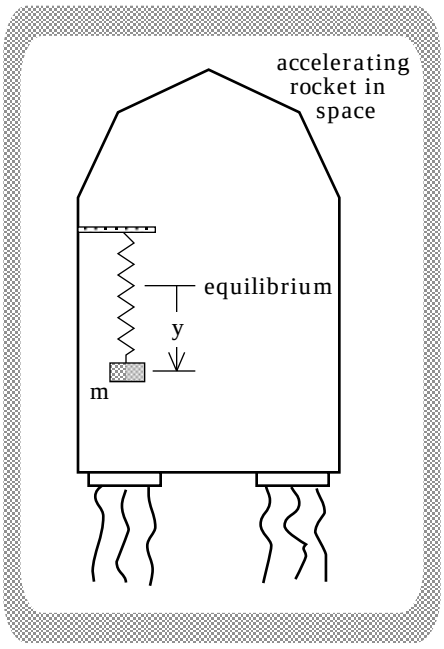


FIGURE 5.34a

a.) From an inertial frame of reference (i.e., a stationary observer outside the rocket), the only force accelerating m in this gravity free situation is ky (Figure 5.34b). Using N.S.L., we can write:

$\underline{\sum F_y}$:

$ky = m(a_y)$

$= m(A)$

$\Rightarrow \quad y = (mA/k)$.

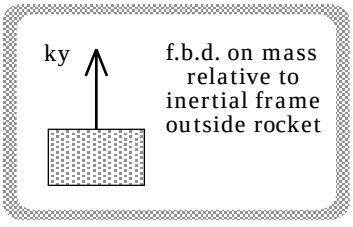


FIGURE 5.34b

b.) From a non-inertial frame of reference (i.e., inside the rocket), there are two apparent forces acting: *ky* pulling up and a fictitious force equal to *mA* pulling the spring down (Figure 5.34c). Noting that *m* is not accelerating relative to the rocket, N.S.L. yields:

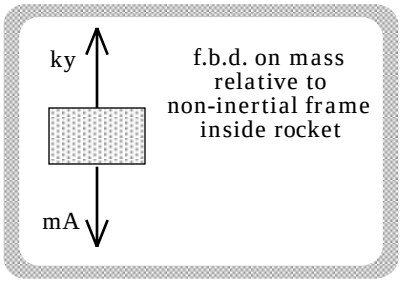


FIGURE 5.34c

$\underline{\Sigma F_y}$:

ky - m(A) = 0 (as $a = 0$ as viewed from inside the frame)

$\Rightarrow$ y = (mA/k).

c.) Huzzah! The *y* values calculated in both cases are the same.

d.) Now re-read the first sentence of the Note just above H-5.

I.) Determining A Time Dependent Velocity *v(t)* From N.S.L.:

1.) There is one class of N.S.L. problems in which unembedding acceleration signs can get you into serious trouble. As was the case in the *Friction and Free Fall* section (*Section E*), it is the situation in which a body's *velocity as a function of time--v(t)*--is being sought.

2.) Though it may not have been obvious at the time when we did the frictional free fall problem:

a.) We identified the frictional force as having a magnitude of *kv*, where *v* was the *magnitude* of the body's velocity vector (remember, we had to manually insert a positive or negative sign when writing out the N.S.L. expression for the situation).

b.) N.S.L. was used to relate the force quantity *kv* to the acceleration *dv*/*dt* yielding a differential expression we could solve.

c.) Though it was, again, probably not obvious, the *dv*/*dt* notation alluded to above *had* to be defined as the derivative of the velocity *MAGNITUDE*. Otherwise, the *v* and *dv*/*dt* terms would have been the mathematical equivalent of apples and oranges, one being a vector and the other a magnitude.

d.) Because the problem was set up as it was, we got away with being notationally sloppy without disastrous consequences. You are

about to see what those consequences could have been if we hadn't been so lucky, and how you can avert the problem altogether.

3.) To understand the difficulty, consider the following:

a.) A mass moves *to the right* on a frictionless surface. A force is applied to the mass *to the right* (see Figure 5.35). During the body's motion, the magnitude of the body's velocity *increases* and the *velocity's magnitude-change* is *positive*.

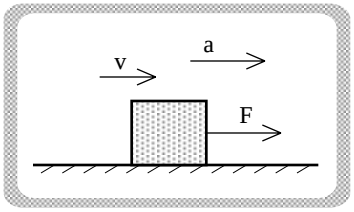


FIGURE 5.35

i.) Justification? If the body's velocity increases from 5 m/s to 7 m/s, the change of the velocity magnitude will be $\Delta v = (7\ m/s - 5\ m/s)$ . . . a positive quantity.

ii.) What about *dv*/*dt*? In this case, *dv*/*dt* is inherently positive (that is, just as Δv has a positive sign embedded within itself, so does *dv* and, hence, *dv*/*dt*).

iii.) So what does N.S.L. do for us? Summing the forces in the *x direction* yields:

$$F = m(dv/dt).$$

iv.) Kindly notice that this expression makes perfect sense as it stands. The force is inherently positive, *dv*/*dt* is inherently positive, the two sides of the equation are equivalent and all is well.

b.) Now, consider the same mass moving *to the left* on the same frictionless surface. The force is still being applied *to the right* (see Figure 5.36). During the body's motion, the magnitude of the body's velocity *decreases* and the *velocity's magnitude-change* is *negative*.

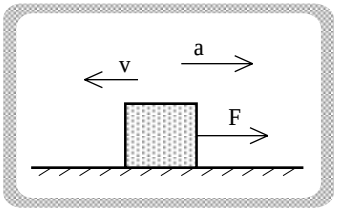


FIGURE 5.36

i.) Justification? If the body's velocity drops from 5 m/s to 3 m/s, the change of the velocity *magnitude* will be $\Delta v = (3\ m/s - 5\ m/s)$ . . . a negative quantity.

ii.) What about *dv*/*dt*? In this case, *dv*/*dt* is inherently negative (that is, just as Δv has a negative sign embedded within itself, so does *dv* and, hence, *dv*/*dt*).

iii.) So what does N.S.L. do for us? As the freebody diagram hasn't changed (velocities are not included on them), summing the forces in the *x direction* yields:

$$F = m(dv/dt).$$

iv.) Oops. We have a problem. The force is still inherently positive, but *dv*/*dt* is now inherently negative. The only way the two sides of the equation can equal one another is if a negative sign is inserted in front of the *ma* term. This insertion would NOT be the consequence of the acceleration (as a vector) being negative. In fact, if you take the derivative of the velocity *vector*, you will get a positive acceleration. It is the consequence of the fact that the derivative we are taking is that of the velocity *magnitude*. That derivative is inherently negative. In short, the N.S.L. expression that will allow us to derive a reasonable velocity function (i.e., one that sees the velocity slowing with time) is:

$$F = - m(dv/dt)$$

c.) Summarizing, in both cases the acceleration is *to the right*, yet depending upon the direction of motion, the sign in front of the *m(dv/dt)* term can either be positive or negative.

5.) So how *do* we remedy the difficulty when dealing with N.S.L. problems in which we are asked to determine a body's velocity as a function of time?

a.) The easiest way to do this is also the most mathematically satisfying. Simply:

i.) Write out the velocity vector, sign unembedded.

ii.) Take the time derivative of that velocity function. That will give you the appropriate $\pm dv/dt$ term.

b.) Example: Consider the block moving *to the left* (i.e., in the $-\boldsymbol{i}$ direction).

i.) The velocity vector in that case can be written as:

$$\begin{aligned} \mathbf{v} &= v\,(-\mathbf{i}) \\ &= -v\,(\mathbf{i}), \end{aligned}$$

where *v* is the magnitude of the velocity vector at any arbitrary point in time and the unit vector $\boldsymbol{i}$ is directed along the line of

motion (note that the negative sign has been detached from the unit vector).

ii.) Taking the time derivative yields:

$$\begin{aligned}\mathbf{a} &= \left(\frac{d\mathbf{v}}{dt}\right) \\ &= \frac{d}{dt}[-v(\mathbf{i})] \\ &= \left(-\frac{dv}{dt}\right)\mathbf{i}.\end{aligned}$$

iii.) N.S.L. becomes:

$$\begin{aligned}F &= ma \\ &= m\left(-\frac{dv}{dt}\right) \\ &= -\,m\left(\frac{dv}{dt}\right).\end{aligned}$$

It works!

c.) Bottom Line: Whenever you find yourself in a situation in which you must use N.S.L. to determine the magnitude of a body's velocity as a function of time, the acceleration term to be substituted into the right side of N.S.L. should be determined using the approach outlined in *Part 5a*.

6.) Parting Shot: Going back to the free fall problem discussed in *Section E*, the positive direction was defined as up, which means the velocity vector should have been written as $\boldsymbol{v} = v(-\boldsymbol{j}) = -\,v(\boldsymbol{j})$. The derivative of this yields an acceleration term of $a = -\,dv/dt$, just as we assumed. The only difference is that now you know *why* the negative sign in front of the acceleration term was appropriate.

QUESTIONS

5.1) Draw a *free body diagram* for the forces acting on each of the bodies in each of the four independent sketches shown in Figure I. Label as completely as you can. Note that in *Part d*, you are being asked to draw f.b.d.s on pulleys. This is for *practice in force recognition*, not because it is something you will ever need to do again. Also, note that in that part, the pin supporting each pulley provides a force.

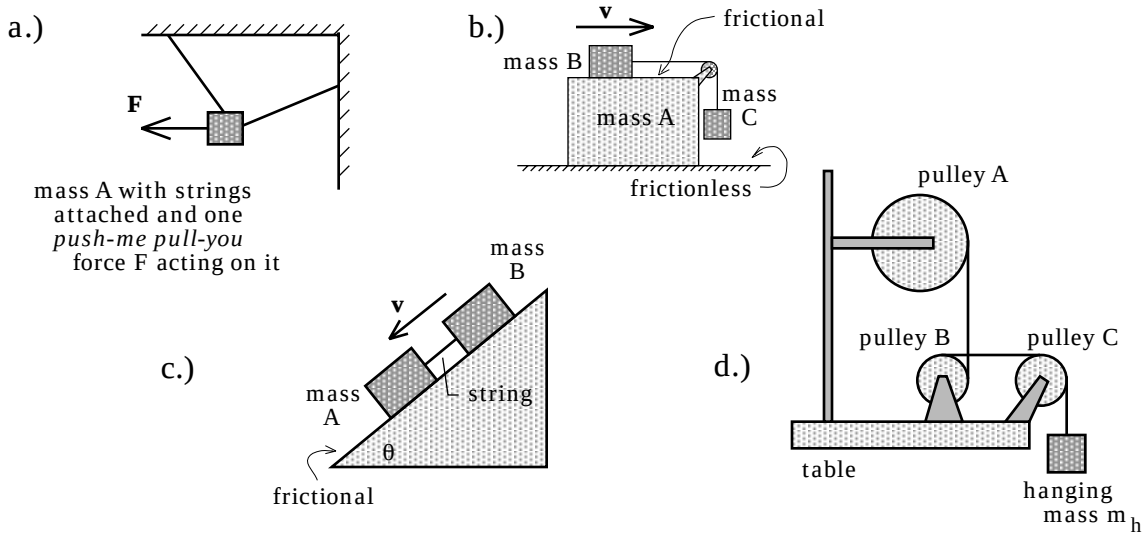


FIGURE I

5.2) In each case below, state the "reaction force" that must exist according to Newton's Third Law.

a.) The floor applies a force to you;

b.) The string applies a force to the weight;

c.) The car applies a force to the tree;

d.) The earth applies a force to the moon.

Important Note: Getting numerical answers for the problems below is not particularly important. You have been given numbers to work with because numbers make students feel secure, but you should initially do each problem algebraically before sticking the numbers in (that is the way you will find the Solutions presented). WHAT IS IMPORTANT is that you can *follow the process* needed to approach and solve these problems, not the bottom line.

BE AWARE: I could very well give you test problems in which there are no numbers, only algebraic symbols.

5.3) A sled whose mass is $m = 30\ kg$ slides horizontally with velocity magnitude $v = 2\ m/s$. It runs into *sticky snow* and experiences a constant *frictional force* of $f_k = 12\ nts$.

a.) What is the sled's acceleration in the sticky snow? (Note that Figure II is for *Part b* only--there is no force F acting in *Part a*).

b.) Your friend decides to help you by pulling the sled with a force $F = 60\ newtons$ at $\theta = 40^o$ with the horizontal (Figure II). For this case, what is the acceleration of the sled? (Note that the vertical component of F will lighten the load, so to speak, lessening N and the *N-related* frictional force--you'll need to determine the coefficient of friction using the information available in *Part a*, then go from there).

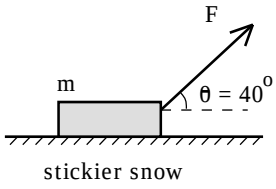


FIGURE II

c.) (**Note:** This will be a difficult question if you are uncomfortable with Calculus; if that be the case, go directly to the Solutions for the approach and execution.) While in the sticky snow, your friend begins to experiment with the *angle of pull* (he has decided that a 40^o pull is not optimal). At what *angle* ϕ will the *minimum force* F_{min} be required to pull the sled *with constant velocity*?

Hint 1: Further explanation: At a large angle, the component of $\boldsymbol{F}$ in the *normal direction* will diminish the normal force $\boldsymbol{N}$ which, in turn, will diminish the frictional force $\mu_k \boldsymbol{N}$. Unfortunately, it will also make the component of $\boldsymbol{F}$ along the line of motion very small--possibly too small to overcome friction. On the other hand, making the angle small will provide more force to counteract friction, but it will do little to diminish the normal force $\boldsymbol{N}$ and, hence, the frictional force. There is an angle that will both diminish $\boldsymbol{N}$ while additionally providing a fairly large component of $\boldsymbol{F}$ in the direction of motion. At that angle, the force required to pull the sled at a constant velocity will be a minimum. Your thrill is to find that angle.

Hint 2: If you can determine the force $\boldsymbol{F}$ as a function of ϕ, you can determine the slope of that function (i.e., $dF(\phi)/d\phi$), then set it equal to zero (see Figure IIa). At the angle at which that is true (i.e., $dF(\phi)/d\phi = 0$), the force will be a minimum.

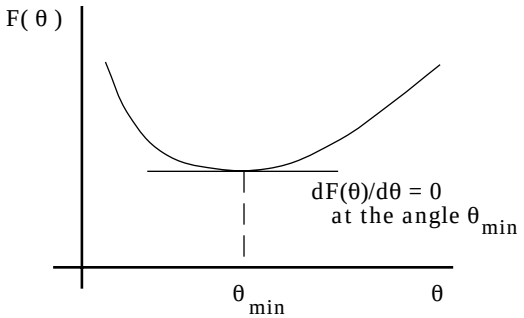


FIGURE IIa

5.4) An elevator whose mass is $m = 400$ *kilograms* is supported by a cable. A frictional force of $f = 80$ *newtons* is applied to the elevator as it moves. Determine the tension T in the cable when the elevator is:

a.) Stationary;

b.) Moving upward while accelerating at 2.8 m/s^2;

c.) Moving upward with a constant velocity-magnitude of 5 m/s;

d.) Accelerating downward at 2.8 m/s^2 with velocity downward;

e.) Moving downward with a constant velocity-magnitude of 5 m/s.

5.5) You are standing on a bathroom scale in an elevator (yes, this is an odd place to find a bathroom scale). Your mass is $m = 60$ *kilograms*; the scale reads $W = 860$ *newtons*. What is the acceleration of the elevator?

5.6) You are sitting in a commercial jetliner. It is on the runway and ready to take off. Before it does, clever soul that you are, you hang a .5 meter long string from the ceiling just overhead and attach to its free end a weight whose mass is $m = .05$ *kg*. The jet begins its acceleration. As it does, the string and mass swing toward you until the string comes into equilibrium at a constant angle of $\theta = 26^o$ with the vertical (see Figure III).

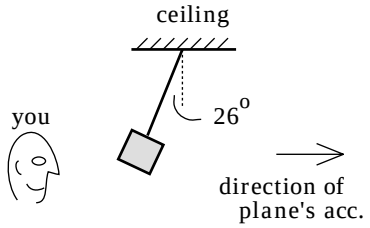


FIGURE III

a.) What is the jet's acceleration?

b.) Once the jet gains altitude and proceeds with a constant velocity, what will the string and mass be doing (that is, will the 26^o angle have changed)? Explain.

5.7) IMPORTANT PROBLEM: Two blocks of mass $m_1 = 2$ *kgs* and $m_2 = 7$ *kgs* are wedged up against one another and against a wall by a horizontal force F (see Figure IV to the right). Doing each section algebraically before putting in the numbers:

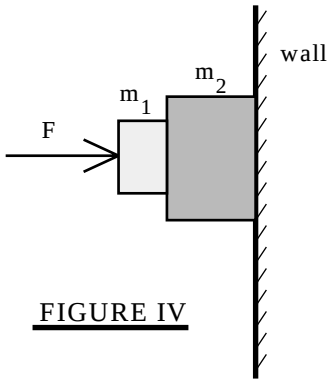


FIGURE IV

a.) What is the *coefficient of static friction* between m_1 and m_2 (call this $\mu_{s,1}$) and between m_2 and *the wall* (call this $\mu_{s,2}$) if the MINIMUM FORCE F_{min} required to keep the blocks from breaking loose and sliding under the influence of gravity if $F_{min} = 25$ *newtons?*

b.) The force F is decreased to 20 newtons. The blocks break loose and begin to fall. If the *coefficients of kinetic friction* between m_1 and m_2 AND between m_2 and *the wall* are $\mu_{k,1} = .15$ and $\mu_{k,2} = .9$ respectively, what are the accelerations of m_1 and m_2? Note that they will NOT be the same.

c.) Why are the accelerations different?

5.8) IMPORTANT PROBLEM: Known masses $m_1 > m_2$ are attached by a string while sitting on an incline of known angle θ (see Figure V). If the coefficient of friction is a known μ_k and if the two are initially sliding UP the incline (no, you don't know *how* they managed it--somebody obviously gave them a push somewhere along the line--what matters is that they are moving UP the incline *when you see them*):

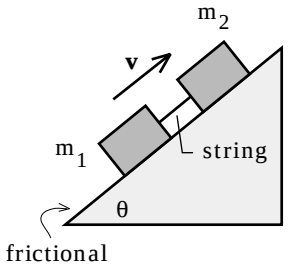


FIGURE V

a.) Draw an f.b.d. for the forces acting on m_1 and m_2.

b.) Derive an expression for the *acceleration* of the system (that is, the acceleration of either m_1 or m_2--both will be the same) as shown.

c.) Derive an expression for the *tension* in the string as shown.

5.9) IMPORTANT PROBLEM: A crate whose mass is m_1 is placed on an incline whose angle is θ. The *coefficient of friction* between the crate and the incline is a known μ_k. A wire attached to the mass proceeds over a frictionless, massless pulley and is attached to a hanging mass m_2. At the instant pictured in Figure VI, m_1 is moving *down* the incline. Additionally, at the instant shown the wire is at an angle ϕ with the line-of-the-incline. *For the moment shown* in the figure, derive general algebraic expressions from which you could determine the crate's acceleration. DO NOT SOLVE THESE EQUATIONS, just generate them!

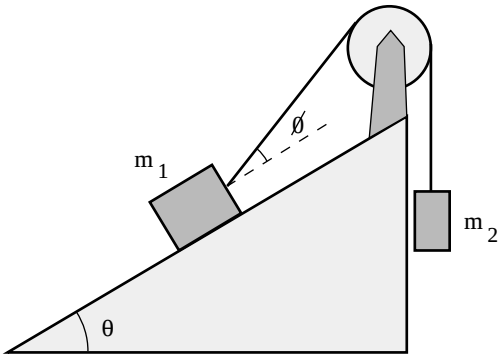


FIGURE VI

Hint: This is not as hard as it looks. Do not look ahead! Content yourself with *following the steps* you have been given in an orderly fashion. This problem will fall out nicely if you do it in pieces. (See additional note below!)

NOTE: This problem is designed to make you think about NSL, but it has one unfortunate characteristic that is not typical of all NSL problems. The acceleration of m_2 is in the same direction and has the same magnitude as the acceleration of the *string* attached to it (this is called a_{string} in the sketch). Unfortunately, the string is attached to m_1 *at an angle*, so m_1*'s* acceleration and the string's acceleration aren't the same. If we call m_1*'s* acceleration a_1 and m_2*'s* acceleration a_2, the relationship between the two parameter's is $a_1 = a_2 \cos \phi$. I'm pointing this out because it is very much a side issue. I don't want you spending a lot of time stewing over it.

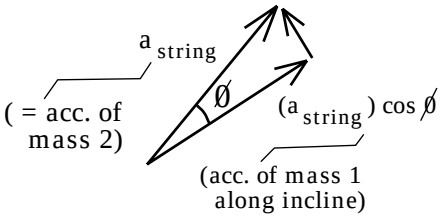


FIGURE VI, supplemental

5.10) A mass m_1 sitting on a frictionless table is attached to a string. The free end of the string is threaded through a hole in the table. Once through, it is attached to a hanging mass m_2 (see Figure VII). If m_1 is then given a shove that moves it into circular motion, what must its velocity be if its radius is to be R?

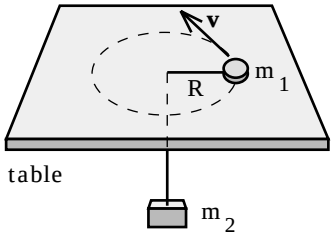


FIGURE VII

5.11) The Loop is a favorite Magic Mountain roller coaster ride that has within it a complete, vertical, 360° loop. The carts that travel through the loop are attached to the track; let's assume one comes loose. What is the *minimum velocity* magnitude required for such a cart to make it through the top of the loop without coming off the track?

Note: At minimum velocity, the cart will just barely free fall through the top of the arc. In that case, the *normal force* applied to the cart by the track will be zero.

5.12) A particular brand of string can withstand a force of $T = 50$ *newtons* of tension before breaking. A rock of mass $m = .2$ *kg* is attached to a string of length $L = 1.2$ *meters*. A kid then takes the *string and mass* and whirls them around her head (see Figure VIII on the next page).

a.) Assuming the radius of motion is, to a good approximation, equal to the string's length (that is, assume the mass is moving very fast and

the *hang-down* angle is zero), what is the *maximum speed* at which the rock can rotate before the string breaks?

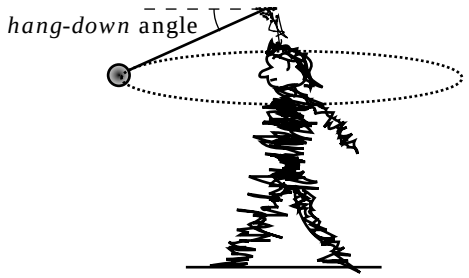


FIGURE VIII

b.) Taking the hang-down into consideration (i.e., assume the string's length is no longer equal to the radius of the rock's motion), what is the angle at which the string breaks?

c.) Use the angle determined in *Part b* to determine the velocity of the mass in that situation.

d.) The rock is observed to be hanging at a 30^o angle. How fast is it moving?

5.13) Until now, we have dealt only with gravitational forces between the earth and small objects close to the earth's surface. Under this circumstance, we have taken the *acceleration of gravity* to be nearly constant and equal to $g = 9.8\ m/s^2$. Newton realized that gravity must exist between *any* two objects no matter how far apart they were. To accommodate the most general situation, he derived a general *gravitational force* expression the magnitude of which is:

$$F_g = G\ m_1 m_2 / r^2.$$

In this equation, G is called the *Universal Gravitational Constant* and is equal to $6.67 \times 10^{-11}\ m^3/kg \cdot s^2$; m_1 and m_2 are the masses generating the force field, and r is the distance between the *center of mass* of the two objects.

a.) The earth has a *mass* $m_e = 5.98 \times 10^{24}\ kg$ and a *radius* $r_e = 6.37 \times 10^6\ meters$. Assuming *your* mass is 70 kg and you are standing on the earth's surface, what is the force of attraction between you and the earth according to Newton's general expression?

After finishing this calculation, determine the gravitational force between you and the earth, given the assumption that your weight equals mg. How do the two numbers compare?

b.) The *acceleration of gravity* on the earth's surface at the poles is $9.83\ m/s^2$. What is the *acceleration of gravity* at the equator? (Think *centripetal acceleration.*)

c.) The moon has a mass $m_m = 7.35 \times 10^{22}\ kg$ and an orbital distance from the earth's center of 3.84×10^8 meters. What must its orbital speed be?

Chapter 6

ENERGY CONSIDERATION

Introduction:

1.) Newton's Second Law is nice because it provides us with a technique for attacking a certain class of problems. "Focus your attention on the *forces* acting on a body," it says, "and you can deduce something about the body's *acceleration*." As useful as this is, there are other ways to approach motion and physical systems. We are about to develop a new perspective that focuses on the *energy content* of a system.

2.) One of the techniques theoretical physicists use to characterize a physical system is to identify all the parameters (i.e., force or displacement or whatever) that govern a phenomenon of interest, then multiply those parameters together. The resulting number or vector then acts as a watermark that allows an individual to predict how pronounced the phenomenon in question will be in a particular instance.

> **a.)** Example: What governs the change of a body's velocity? The *force component along the line of motion* certainly matters; so does the *distance* over which the force is applied. If the product of those two quantities is big, you know the resulting velocity change will be relatively big. If small, the velocity change will be relatively small.

3.) We are about to build a mathematical model that begins with the very product alluded to in *Part 2a*, then look to see where that definition logically takes us. Hold on to your skirts, ladies. This should be fun.

A.) Work:

1.) The beginning definition: As was said above, a change in a body's velocity is governed by the magnitude of the *component of force along the line of the displacement* and the magnitude of the *displacement* itself. The product of those two parameters, $F_{//}$ and d, defines the dot product $\boldsymbol{F}\cdot\boldsymbol{d}$. That quantity is given a special name. It is called *work*.

2.) By definition, the *work* W_F done by a constant force $\boldsymbol{F}$ acting on a body that moves some straight-line distance $\boldsymbol{d}$ (note that $\boldsymbol{d}$ is a vector that de-

fines both the direction and the magnitude of the displacement of the body) is equal to:

$$W = \mathbf{F} \cdot \mathbf{d}$$
$$= |\mathbf{F}||\mathbf{d}| \cos \phi,$$

where ϕ is the angle between *the line of* ***F*** and *the line of* ***d***.

3.) Example: A box of mass $m = 2\ kg$ moving over a *frictional floor* ($\mu_k = .3$) has a force whose magnitude is $F = 25\ newtons$ applied to it at a 30^o angle, as shown in Figure 6.1 (note that ϕ equals the angle θ in the sketch). The crate is observed to move 16 meters in the horizontal before falling off the table (that is, $\boldsymbol{d} = 16\boldsymbol{i}\ meters$). An f.b.d. for the forces acting on the block is shown in Figure 6.2.

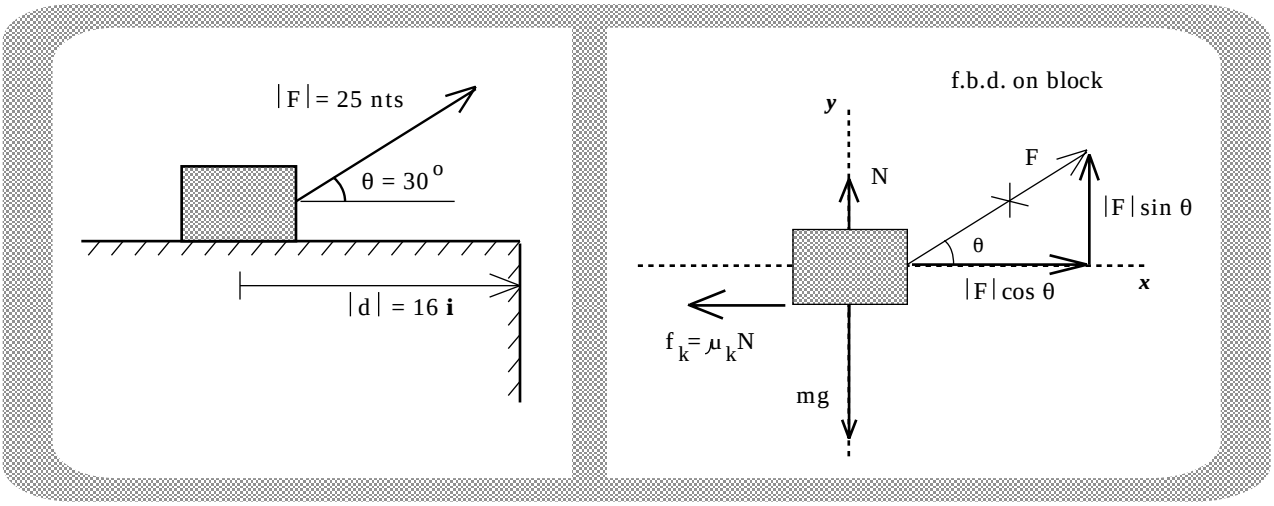


FIGURE 6.1 FIGURE 6.2

a.) How much work does ***F*** do before the crate takes the plunge?

$$W_{\mathbf{F}} = \mathbf{F} \cdot \mathbf{d}$$
$$= \quad |\mathbf{F}| \quad |\mathbf{d}| \quad \cos \theta$$
$$= (25 \text{ newtons})\ (16 \text{ meters}) \cos 30^o,$$
$$= 346.4 \text{ newton-meters}.$$

Note 1: A *newton-meter* (or a $kg{\cdot}m^2/s^2$) is the MKS unit for both work and energy. It has been given a special name--the JOULE. We could, therefore, have written *the work done by F* as "346.4 joules."

Note 2: Work and energy units in the CGS system are *dyne-centimeters* (or gm·cm^2/s^2). That combination has been given the name ERGES. In the English system, work and energy units are in FOOT-POUNDS.

b.) The above *dot product* was done from a polar notation approach (i.e., you multiplied the *magnitude of one vector* times the *magnitude of the second vector* times the *cosine of the angle between the line-of-the-two-vectors*) because the force information was given in polar notation. If the initial information had been given in *unit vector notation*, we would have used the unit vector approach for the dot product.

For the sake of completeness, let us do the problem from that perspective:

i.) The *unit vector* representation of the force vector presented in our problem above is:

$$\mathbf{F} = (21.65\,\mathbf{i} + 12.5\,\mathbf{j})\text{ nts.}$$

ii.) *Dot products* executed in *unit vector notation* are defined as:

$$\begin{aligned}\mathbf{F}\cdot\mathbf{d} &= (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}) \\ &= (F_x\,d_x) + (F_y\,d_y) + (F_z\,d_z).\end{aligned}$$

iii.) As $d_z = 0$, we have:

$$\begin{aligned}W_{\mathbf{F}} &= \mathbf{F}\cdot\mathbf{d} \\ &= (F_x\mathbf{i} + F_y\mathbf{j})\cdot(d_x\mathbf{i} + d_y\mathbf{j}) \\ &= (21.65\,\mathbf{i} + 12.5\,\mathbf{j})\cdot(16\,\mathbf{i}) \\ &= (21.65\text{ newtons})(16\text{ meters}) + (12.5\text{ newtons})(0) \\ &= 346.4\text{ joules.}\end{aligned}$$

This is the same value we determined using the polar approach. As expected, the two approaches yield the same solution.

c.) In our example, how much work does the *normal force* do? The temptation is to assume that we need to determine the magnitude of the normal force before doing this, but a little insight will save us a lot of trouble here. From the definition of *work*:

$$\begin{aligned}W_{\mathbf{N}} &= \mathbf{F}\cdot\mathbf{d} \\ &= |\mathbf{N}||\mathbf{d}|\cos\phi.\end{aligned}$$

The trick is to notice that the angle ϕ between $\boldsymbol{N}$ and $\boldsymbol{d}$ is 90^o (see the *free body diagram* shown in Figure 6.2). As $\cos 90^o = 0$, $W_N = 0$.

In fact, *normal forces* are always *perpendicular* to a body's motion. As such, their work contribution will *always* be ZERO. <u>*Normal forces* do no work on a moving body.</u>

d.) How much work does the *frictional force* do on the body as it moves toward the abyss?

i.) To do this part, we need to determine the *normal force N* so that we can determine the *frictional force* using the relationship $f_k = \mu_k N$. Utilizing both the f.b.d. shown in Figure 6.2 and Newton's Second Law:

$$\underline{\textstyle\sum F_y}:$$
$$N + F(\sin\theta) - mg = ma_y.$$

As $a_y = 0$, rearranging yields:

$$\begin{aligned} N &= -F(\sin\theta) + mg \\ &= -(25 \text{ newtons})(\sin 30^o) + (2 \text{ kg})(9.8 \text{ m/s}^2) \\ &= 7.1 \text{ newtons.} \end{aligned}$$

The frictional force will be:

$$\begin{aligned} f_k &= \mu_k N \\ &= (.3)(7.1 \text{ newtons}) \\ &= 2.13 \text{ newtons.} \end{aligned}$$

ii.) Noticing that the angle between the *line of* $\boldsymbol{f}_k$ and the *line of* $\boldsymbol{d}$ is 180^o, the work done by friction will be:

$$\begin{aligned} W_f &= \mathbf{f}_k \cdot \mathbf{d} \\ &= |\mathbf{f}_k||\mathbf{d}| \cos\phi \\ &= (2.13 \text{ newtons})\ (16 \text{ meters}) \cos 180^o \\ &= -34.1 \text{ joules.} \end{aligned}$$

Note 1: Yes, *work* quantities can be *negative*. In fact, whenever the angle between the *line of* $\boldsymbol{F}$ and the *line of* $\boldsymbol{d}$ is greater than 90^o and less than or equal to 180^o, the cosine of the angle will yield a *negative* number.

Note 2: The *negative sign* is <u>not</u> associated with *direction*. Work is a scalar quantity--IT HAS NO DIRECTION. A negative sign in front of a *work quantity* tells you that the force doing the work is oriented so as to *slow the body down*.

4.) Comments:

a.) Mathematically, the physics concept of *work* is rigidly defined. Hold a 25 pound weight at arm length for fifteen minutes and although there may be sweat pouring off your brow, you will nevertheless be doing no *work*. Why? Because for *work* to occur, a FORCE must be applied to a body *as it moves* over a DISTANCE. If there is no displacement (example: your arm held motionless for fifteen minutes), no *work* is done.

b.) When *work* is done by a single force acting on an object, it *changes the object's motion* (i.e., speeds it up or slows it down). Again, the key is motion. Things become more complicated when many forces act on a body, but in all cases, having some net amount of work being done implies there is motion within the system.

c.) On an intuitive level, forces that do *positive work* are oriented so as to make a body speed up; forces that do *negative work* are oriented so as to make a body slow down. It is as though doing *positive work* puts energy *into* the system while doing *negative work* pulls energy *out of* the system. (We will more fully define the idea of *energy* shortly).

B.) *Work* Due to Variable Forces:

1.) When *work* is done by a force-and-displacement combination that in some way varies as a body moves (i.e., the force changes magnitude along the way or the angle between the force and displacement changes), we can no longer write $W_F = \boldsymbol{F}\cdot\boldsymbol{d}$ and proceed from there. That relationship holds only when all the parameters stay constant throughout the motion.

To deal with the *varying force or angle* situation, we must use Calculus.

a.) If we take a differential displacement $d\boldsymbol{r}$ to be a very small distance traveled along a body's path as it moves from one point to another in a force field $\boldsymbol{F}$, the *differential amount of work* (read this "a very small amount of work in comparison to the whole of the work done") will equal the dot product between $\boldsymbol{F}$ and $d\boldsymbol{r}$. Mathematically, this is:

$$dW = \mathbf{F}\cdot d\mathbf{r}.$$

b.) Summing this *dot product* over the entire path will give us the entire amount of work done by the force over the motion. In short, if we want the work done in such cases, we must determine the integral:

$$W = \int dW = \int \mathbf{F} \bullet d\mathbf{r}.$$

c.) There are two ways to evaluate a *dot product*: using a *unit vector* approach and using a *polar* approach. The easiest way to see how each works is with an example.

2.) Consider a body constrained by a *group of forces* to move along a path described by $y = 1$. If one of those forces (call it ***F***) is defined by the function:

$$\mathbf{F} = k(x\mathbf{i} + y\mathbf{j}),$$

where k is a constant equal to *1 nt/meter* (that is, k is in the problem solely for the sake of units), how much work does the force do as the body moves from $x_1 = 1$ *meter* to $x_2 = 5$ *meters?*

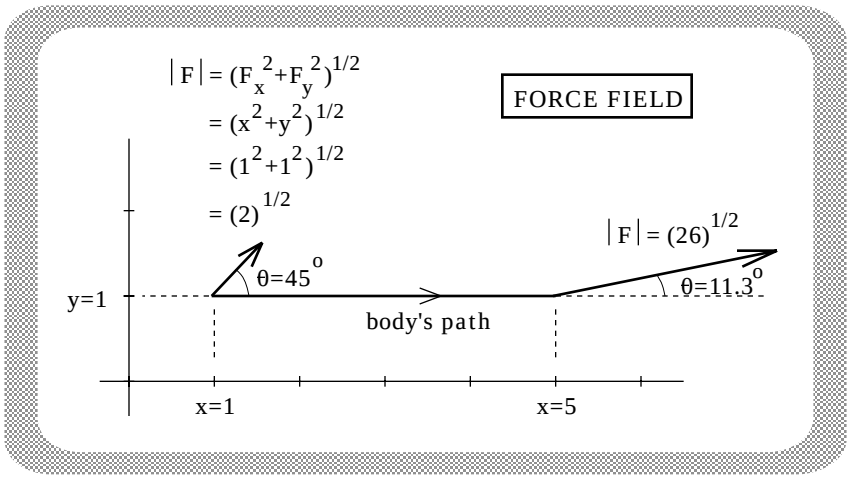


FIGURE 6.3

Note 1: The situation is shown in Figure 6.3, complete with the *force magnitude* and *angle* at the points *(1, 1)* and *(5, 1)*.

Note 2: The other forces acting on the body also do *work*, but we are not interested in them. All we want is the amount of *work* done by force ***F***.

3.) Using a UNIT VECTOR approach:

a.) The most general way to define a *differential displacement* $d\mathbf{r}$ is with the relationship:

$$d\mathbf{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k}.$$

That is, vectorially adding *differential displacements* in the *x, y,* and *z* directions will yield a vector whose net displacement is differential and whose magnitude is:

$$|d\mathbf{r}| = [dx^2 + dy^2 + dz^2]^{1/2}.$$

b.) Looking at Figure 6.4, it is clear that the displacement is in the *x* direction, and the *x* direction only. As such, the *magnitude of differential displacement* in this example will simply be *dx*.

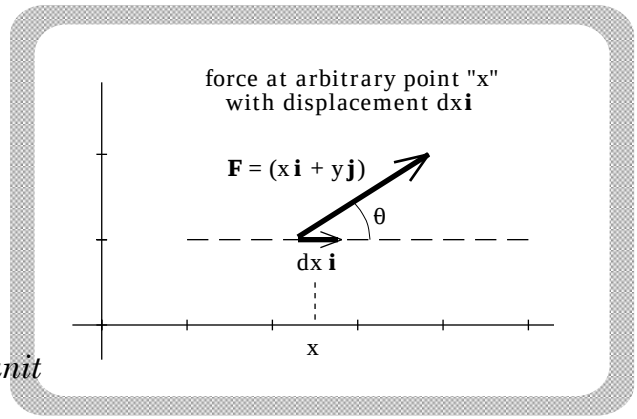


FIGURE 6.4

c.) We know the *force function* (i.e., $\boldsymbol{F} = k(x\boldsymbol{i} + y\boldsymbol{j})$) in *unit vector notation*--that is the way the force was given in the problem. Remembering how to do a *dot product* in *unit vector notation*, and remembering that $k = 1$, we can write our *work expression* as:

$$\begin{aligned} W &= \int dW \\ &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int (kx\mathbf{i} + ky\mathbf{j}) \bullet (dx\mathbf{i}) \\ &= \int_{x=1}^{5} (kx)dx \\ &= \left[k\frac{x^2}{2} \right]_{x=1}^{5} \\ &= \left[(1)\frac{(5)^2}{2} - (1)\frac{(1)^2}{2} \right] \\ &= 12 \text{ joules.} \end{aligned}$$

Note: If you are wondering why the $ky\boldsymbol{j}$ term has not been carried along in the above calculation, think about the definition of *dot product* in *unit vector notation*. The *x components* are multiplied together, as are the *y components* and *z components*, then the products are added together. As there is no *y-direction differential displacement* (i.e., no *dy*), the *y* contribution to the *dot product* is zero leaving only *kx(dx)* under the integral.

4.) Using a POLAR approach:

a.) The magnitude of the *differential displacement* is still:

$$|d\mathbf{r}| = dx.$$

b.) The magnitude of the *force function* (i.e., $\boldsymbol{F} = k(x\boldsymbol{i} + y\boldsymbol{j})$) is:

$$\begin{aligned} |\mathbf{F}| &= [(kx)^2 + (ky)^2]^{1/2} \\ &= k(x^2 + y^2)^{1/2}. \end{aligned}$$

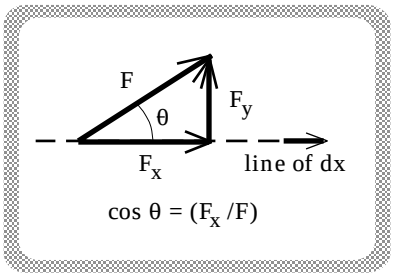


FIGURE 6.5

c.) Looking at Figure 6.5, the *cosine* of the angle between *the line of* $\boldsymbol{F}$ and *the line of* $d\boldsymbol{r} = (dx)\boldsymbol{i}$ is:

$$\begin{aligned} \cos\,\theta &= [F_x]/[F] \\ &= [kx] \,/\, [k(x^2 + y^2)^{1/2}] \\ &= [x] \,/\, (x^2 + y^2)^{1/2}. \end{aligned}$$

d.) Putting it all together:

$$\begin{aligned} W &= \int dW \\ &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int \quad |\mathbf{F}| \quad |d\mathbf{r}| \quad \cos\,\theta \\ &= \int \left[k(x^2 + y^2)^{1/2}\right][dx]\left[\frac{x}{\left[(x^2 + y^2)^{1/2}\right]}\right] \\ &= \int_{x=1}^{5} (kx)dx. \end{aligned}$$

This is the same integral we found using the *unit vector* approach.

C.) Even *More* Fun With Force Functions That Vary:

1.) A ball hangs from a rope attached to a ceiling, as shown in Figure 6.6. A variable, horizontal force $\boldsymbol{F}$ is applied to the ball so that:

a.) ***F*** is ALWAYS horizontal; and

b.) ***F's magnitude*** varies so that the ball moves up the arc with a *constant velocity*; and

c.) The ball's *velocity* is very low.

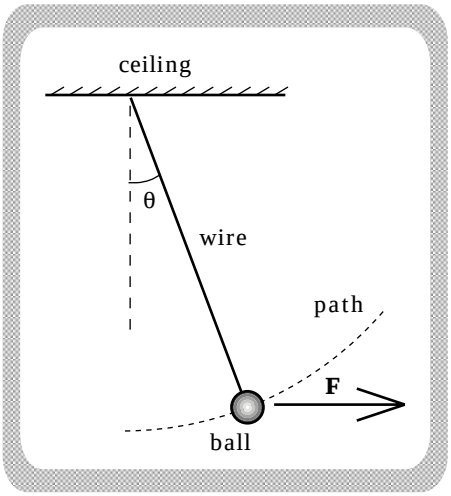


FIGURE 6.6

2.) Assuming the ball's mass is *m*, how much *work* does ***F*** do as the ball moves from $\theta = 0^o$ to $\theta = \theta_1$?

3.) Using a *POLAR* approach:

a.) For a *dot product* using a *polar* approach, we need the *magnitude of the force*, the *magnitude of the displacement*, and the *angle between the line of the force and the line of the displacement*.

b.) Because the angle between the force and the displacement is constantly changing, we must use the *integral form* of the definition of *work*. That means the displacement we want is a *differential displacement d**r*** and the amount of work done by the force ***F(θ)*** will be:

$$W = \int dW = \int \mathbf{F} \cdot d\mathbf{r}.$$

Note: There is a common error people make in doing problems like this. If one determines ***F*** at a known angle (say, $\theta = \theta_1$), the *dot product* under the integral will not be general. As the integral is supposed to sum ***F**·d**r*** terms at all angles between the limits, the *function* that defines ***F*** must be generally good for any arbitrary angle.

c.) We need to use N.S.L. to determine the force as a function of the *angular displacement* of the string (i.e., ***F(θ)***) at any arbitrary angle.

i.) As we have assumed that the *magnitude of the velocity* is constant, the acceleration of the mass along the path will be zero.

ii.) As we have assumed the *magnitude of the velocity* is VERY, VERY SMALL, we can ignore the *centripetal acceleration* (i.e., v^2/R) directed along *the line of tension* as the mass moves up the arc.

iii.) Note that with these assumptions, there is no appreciable acceleration in ANY direction.

d.) We are interested in determining ***F***. As there is no acceleration in any direction, and as ***F*** is in the *horizontal direction*, we can use N.S.L. and the f.b.d. shown in Figure 6.7a to write:

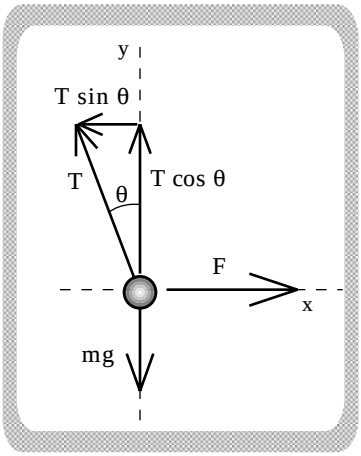


FIGURE 6.7a

$\underline{\sum F_x}$:

$$-T(\sin\theta) + F = ma_x$$
$$= 0 \qquad (\text{as } a_x = 0)$$
$$\Rightarrow \quad F = T(\sin\theta). \qquad \text{(Equation A).}$$

$\underline{\sum F_y}$:

$$T(\cos\theta) - mg = ma_y$$
$$= 0 \qquad (\text{as } a_y = 0)$$
$$\Rightarrow \quad T = mg/(\cos\theta) \qquad \text{(Equation B).}$$

Substituting *Equation B* into *Equation A*, we get:

$$F = T(\sin\theta)$$
$$= [mg/(\cos\theta)](\sin\theta)$$
$$= mg[(\sin\theta)/(\cos\theta)]$$
$$= mg(\tan\theta).$$

e.) The *differential displacement* $d\boldsymbol{r}$ of the mass requires the string to move in such a way as to subtend a *differential angle* $d\theta$ (see Figure 6.7b). If the angle is truly differential, it will be tiny and the *arc length* of the arc upon which the mass moves will approach the net *differential displacement* $d\boldsymbol{r}$. In other words, if we can determine an expression for the arc length associated with $d\theta$, we will have an expression for the magnitude of $d\boldsymbol{r}$.

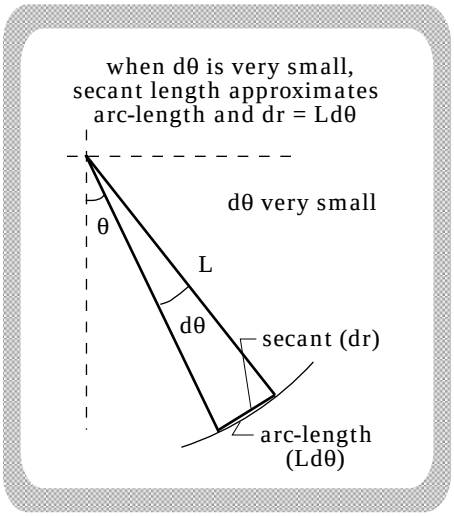


FIGURE 6.7b

If the angle $d\theta$ is measured in *radians*, the arc length will equal $L(d\theta)$. That means:

$$|d\mathbf{r}| = L(d\theta).$$

Note: This is not as obscure as it sounds. A *one-radian angle* is defined as an angle whose *arc length* equals the radius R of the circle upon which it is measured. That means a *one-half radian angle* has an *arc length* of $(1/2)R$, and a θ *radian angle* has an *arc length* of $(\theta)R$. As the angle in our case is $d\theta$, the *arc length* is the radius (i.e., the string's length L) times the angle $d\theta$ or $L\,d\theta$.

f.) The only thing needed is to relate the angle between the *line of* $\boldsymbol{F}$ and the *line of* $d\boldsymbol{r}$ (call this ϕ for the moment). AS CAN BE SEEN in Figure 6.7c, ϕ and θ are the same angle.

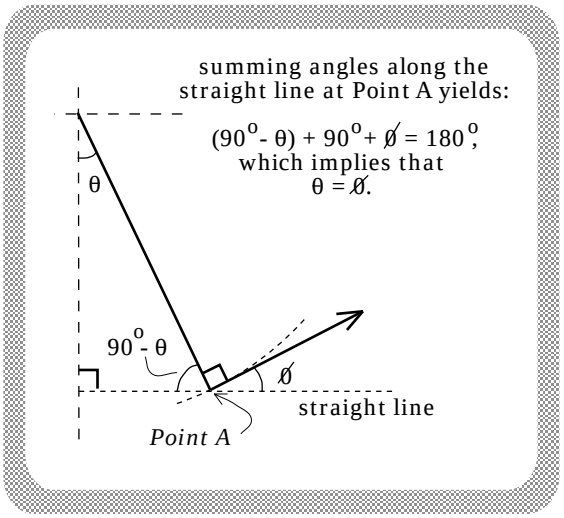


FIGURE 6.7c

g.) We are now ready to use our definition of *work* to determine the amount of work $\boldsymbol{F}$ does as the body moves from $\theta = 0^o$ to $\theta = \theta_1$. Specifically:

$$\begin{aligned}
W &= \int dW \\
&= \int \mathbf{F} \bullet d\mathbf{r} \\
&= \int |\mathbf{F}| \; |d\mathbf{r}| \; \cos\phi \\
&= \int [mg(\tan\theta)][Ld\theta](\cos\theta) \\
&= mgL\int \left[\frac{\sin\theta}{\cos\theta}\right](\cos\theta)d\theta \\
&= mgL\int_{\theta=0}^{\theta_1} \sin\theta d\theta \\
&= mgL[-\cos\theta]_{\theta=0}^{\theta_1} \\
&= mgL\left[(-\cos\theta_1) - (-\cos 0^o)\right] \\
&= mgL[1 - \cos\theta_1]
\end{aligned}$$

4.) Using a *UNIT VECTOR* approach:

a.) For a *dot product* using a *unit vector* approach, we need both the *force* and *displacement* as unit vectors.

b.) We know the force's magnitude is $mg(\tan\theta)$ and its direction is always in the $+\boldsymbol{i}$ direction. That is:

$$\mathbf{F} = [mg\,(\tan\theta)]\mathbf{i}.$$

c.) We also know the displacement is:

$$d\mathbf{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k}.$$

d.) The only problem: The variable $\tan\theta$ is not explicitly a function of x and/or y variables. That is easily remedied by examining the displacement sketched in Figure 6.7d. Using trig on the right triangle, we get:

$$\begin{aligned}\tan\theta &= (\text{opposite})/(\text{adjacent}) \\ &= (dy)/(dx).\end{aligned}$$

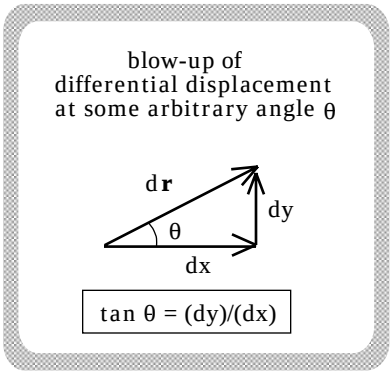


FIGURE 6.7d

e.) Putting it all together, we get:

$$\begin{aligned}W &= \int dW \\ &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int [mg(\tan\theta)\mathbf{i}] \bullet [(dx)\mathbf{i} + (dy)\mathbf{j}] \\ &= \int \left[mg(\frac{dy}{dx})\mathbf{i} \right] \bullet [(dx)\mathbf{i} + (dy)\mathbf{j}] \\ &= \int \left[mg(\frac{dy}{dx})dx \right] \\ &= \int_{y=0}^{y_{\theta_1}} (mg)dy \\ &= mg[y]_0^{y_{\theta_1}} \\ &= mgy_{\theta_1}\end{aligned}$$

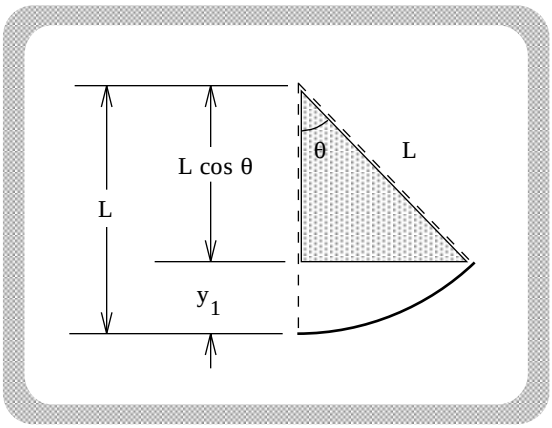


FIGURE 6.8

f.) Determining the *y coordinate* (call this y_1) of the ball when it is located at angle θ_1 is not as difficult as it looks. Consider Figure 6.8. We know the string length is *L*. When in the vertical, the *string length* can also be written $L = L\cos\theta + y_1$. That means:

$$y_1 = L - L(\cos\theta).$$

g.) Using that relationship in our *work* expression, we get:

$$\begin{aligned} W &= mgy_1 \\ &= mg[L - L(\cos\theta)] \\ &= mg[L - L\cos\theta]. \end{aligned}$$

This is exactly the expression we determined using the *polar* approach on the problem.

5.) Don't be put off if a lot of the mathematical maneuvering you've just seen seems a bit mysterious. The reason you are in this class is to see how physicists use math to create theoretical models of the world. If physics was trivially obvious, you'd already know it all and there would be no reason to take the class!

D.) The Work/Energy Theorem:

1.) We would like to relate the *total, net work* done on an object to its resulting change in *velocity*. This next section is the derivation of just such a relationship.

Note 1: You will not be held responsible for duplicating any of the material you are about to read in this part (i.e., *Part D-1*) except the *bottom line*. BUT, if you don't understand how we got there, the *bottom line* won't mean much. Also, the *work/energy theorem* is a half-way point to where we are really going. Understand it, but also understand that there is a more powerful presentation of the same idea coming up soon.

My suggestion is that you read this part, not for memorization purposes but for content. Follow each step as it comes *without* projecting ahead, and when you finally get to the end-result, take the time to reread the section to be sure you know *how* we got from start to finish.

a.) So far, we have been able to calculate the *work* W_F a single force $\boldsymbol{F}$ does on a moving body. It isn't too hard to see that the total, *net work* W_{net} due to all the forces acting on a body will equal the sum of the individual bits of work done by the individual forces.

What might not be so obvious is that there is another way to get that *net work* quantity. How so? We could determine the *net force* $\boldsymbol{F}_{net}$ acting on the body and use *it* in our *work* definition. Doing so yields:

$$W_{net} = \int \mathbf{F}_{net} \cdot d\mathbf{r}.$$

b.) By Newton's Second Law, the *net force* on an object will numerically equal the vector $m\boldsymbol{a} = m(d\boldsymbol{v}/dt)$. If, for ease of calculation, we assume that the *net force* and the *displacement* $d\boldsymbol{r}$ are both in the $\boldsymbol{i}$ *direction*, we can write the *dot product* associated with the *work* definition as:

$$\begin{aligned} W_{net} &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int (m\mathbf{a}) \bullet d\mathbf{r} \\ &= \int \left[m \frac{d(v)}{dt} \mathbf{i} \right] \bullet \left[(dx)\mathbf{i} \right] \\ &= m \int \left(\frac{dv}{dt} \right) dx \qquad \text{(Equation A)} \end{aligned}$$

c.) As the velocity term is *time dependent* (otherwise, we wouldn't be able to determine dv/dt), we would like to write the displacement term dx in terms of time, also. To do so, note that the *rate at which the position changes with time* (i.e., dx/dt) times the *time interval* dt over which the change occurs, yields the *net change in position* dx. Put more succinctly:

$$dx = \left[\left(\frac{dx}{dt} \right) (dt) \right].$$

d.) Substituting this into Equation A and manipulating as only physicists will do (i.e., after canceling out selected dt terms), we get:

$$W_{net} = m\int\left(\frac{dv}{dt}\right)\left[\left(\frac{dx}{dt}\right)dt\right]$$
$$= m\int dv\left[\frac{dx}{dt}\right].$$

e.) Noticing that *dx*/*dt* is the velocity *v* of the body, and taking the limits to be from some velocity v_1 to a second velocity v_2, we can rewrite, then integrate this expression as:

$$W_{net} = m\int_{v_1}^{v_2}(v)dv$$
$$= m\left(\frac{v^2}{2}\right)_{v_1}^{v_2}$$
$$= \left(\frac{1}{2}\right)m(v_2)^2 - \left(\frac{1}{2}\right)m(v_1)^2.$$

f.) This equation, $W_{net} = (1/2)mv_2^2 - (1/2)mv_1^2$, is called the <u>*Work/Energy Theorem*</u>. It is the *bottom line* for this section.

2.) The quantity $(1/2)mv^2$ has been deemed important enough to be given a special name. It is called the *Kinetic Energy* of a body of mass *m* moving with velocity *v*. Its units are (kg)(m/s)^2, or joules--the same units as *work* (as expected).

a.) OBSERVATION: Something is said to have *energy* if it has the ability to do *work* on another "something."

i.) Example--a car traveling at 30 m/s: A car has energy associated with its motion (i.e., kinetic energy). If this is not obvious, imagine stepping in front of one traveling down the road. Any damage done to you by the car will be due to the fact that the car has energy wrapped up in its motion and, as a consequence, has the ability to do work on you.

ii.) Example--a sound wave: If a sound wave didn't carry energy, it wouldn't have *the ability to do work* on the hairs in your ears which, when moved, produce the electrical signals your brain translates into sound.

iii.) In both of the cases cited above, *energy* is associated with *the ability to do work on something else.*

b.) Kinetic Energy--Example #1: What must the *magnitude* of the *velocity* of a 1000 kg car be if it is to have the same *kinetic energy* as a 2 gram bullet traveling at 300 m/s?

Solution:

$$KE_b = (1/2)m_b v_b^2$$
$$= (1/2)\ (.002\ \text{kg})\ (300\ \text{m/s})^2$$
$$= 90\ \text{joules.}$$

If $KE_b = KE_c$:

$$(1/2)\quad m_c\quad v_c^2 = 90\ \text{joules}$$
$$\Rightarrow\quad (1/2)\ (1000\ \text{kg})\ (v_c)^2 = 90\ \text{joules}$$
$$\Rightarrow\quad v_c = .42\ \text{m/s.}$$

c.) Kinetic Energy--Example #2: If one triples a body's velocity, how does the body's *kinetic energy* change?

Solution:

$$KE_1 = (1/2)mv_1^2$$
$$KE_2 = (1/2)m(3v_1)^2$$
$$= 9\ [(1/2)m(v_1)^2].$$

As would be expected when the *kinetic energy* is proportional to the *square of the velocity*, tripling the speed increases the *kinetic energy* by a factor of three-squared, or nine.

3.) Back to the *Work/Energy Theorem*: whenever a *net amount of work* is done on a body, the body will either acquire or lose energy. That change will ALWAYS show itself as a change in the *kinetic energy* of the body.

More succinctly: the net work done on a body will always equal the *change of the body's kinetic energy*.

4.) Example: At a given instant, a 2 kg mass moving to the right over a frictional surface has a force $F = 5\ nts$ applied to the left at an angle $\theta = 30^o$ below the horizontal (see Figure 6.9). The

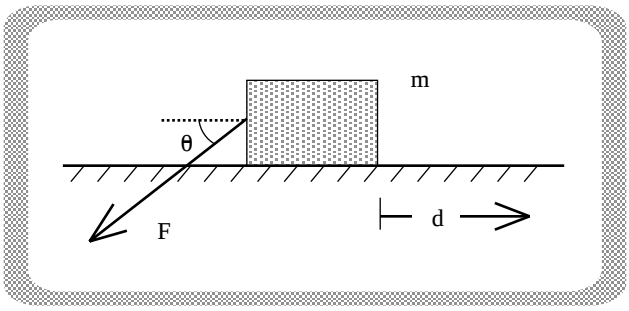


FIGURE 6.9

average frictional force acting on the box is $f_k = 1.5$ *nts*. If the block is initially moving with velocity 9 m/s, how fast will it be moving after traveling a distance 4 meters?

Note: You could have been given μ_k and been expected to use N.S.L. to determine the normal force N required to use $f_k = \mu_k N$. That twist hasn't been included here for the sake of simplicity, but it is a perfectly legitimate problem for your next test.

a.) Someone well-familiar with the *work/energy theorem* would do the problem as shown below (if the pieces making up the expressions aren't self explanatory, a derivation of each follows in *Part b*):

$$W_{net} = \Delta KE$$

$$W_F + W_f = \Delta KE$$

$$(-Fd\cos\theta) + (-f_k d) = (1/2)mv_2^2 - (1/2)mv_1^2$$

$$-(5 \text{ nts})(4 \text{ m})(.866) - (1.5 \text{ nts})(4 \text{ m}) = (1/2)(2 \text{ kg})(v_2)^2 - (1/2)(2 \text{ kg})(9 \text{ m/s})^2$$

$$\Rightarrow \quad v_2 = 7.59 \text{ m/s}.$$

b.) The following shows how each quantity used in the above equations was derived:

i.) The *Work/Energy Theorem* states that:

$$W_{net} = \Delta KE.$$

ii.) The left-hand side of the equation is equal to the sum of all the work done by all the forces acting on the block. The f.b.d. shown in Figure 6.10 identifies those forces. Note that the work done by the *normal force* will always equal zero (the line of motion and the line of the *normal* are perpendicular to one another). The work due to *gravity* will, in this case, also equal zero for the same reason.

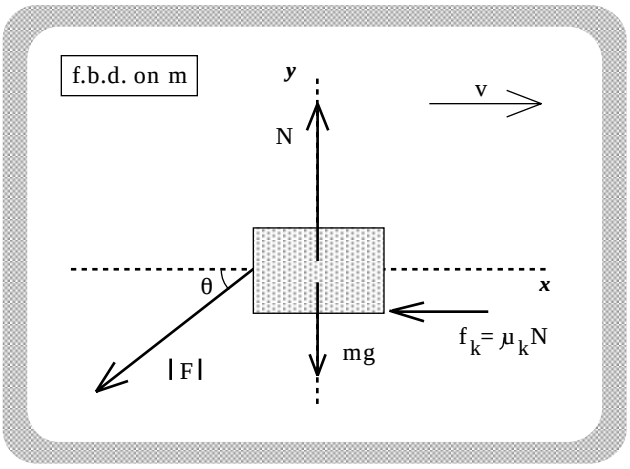


FIGURE 6.10

iii.) That leaves $W_{net} = W_F + W_{f_k}$.

iv.) Using the definition of work, we get:

$$\begin{aligned} W_F &= \mathbf{F} \cdot \mathbf{d} \\ &= |\mathbf{F}|\ |\mathbf{d}| \cos \phi \\ &= (F)\ (d) \cos (180^o - \theta) \\ &= -\ (F)\ (d) \cos \theta. \end{aligned}$$

and

$$\begin{aligned} W_{f_k} &= \mathbf{f}_k \cdot \mathbf{d} \\ &= |\mathbf{F}|\ |\mathbf{d}| \cos \phi \\ &= (f_k)\ (d) \cos 180^o \\ &= -\ f_k\ d. \end{aligned}$$

Note 1: The angle between *the line of motion* and the force ***F*** is not so obvious--we really did need to write out the *work* derivation for that force.

Note 2: Friction pulls energy out of the system, hence the negative work quantity. That energy is usually dissipated as *heat*.

v.) Putting it all together, we get:

$$\begin{aligned} W_{net} &= W_F + W_{f_k} \\ &= (-Fd \cos \theta) + (-f_k d). \end{aligned}$$

vi.) Returning to the Work/Energy theorem:

$$W_{net} = \Delta KE \qquad \text{(Equation A)}$$

$$\Rightarrow \quad (-Fd\cos \theta) + (-f_k d) = (1/2)mv_2^{\,2} - (1/2)mv_1^{\,2} \qquad \text{(Equation B).}$$

vii.) We know everything except v_2. Solving for that variable, assuming $F = 5$ *newtons,* $f_k = 1.5$ *newtons,* $\theta = 30^o$, $d = 4$ *meters,* $m = 2$ *kg,* and $v_1 = 9$ *m/s*, we get:

$$(-Fd\cos\theta) + (-f_k d) = (1/2)\,mv_2^2 - (1/2)mv_1^2$$
$$-(5\text{ nts})(4\text{ m})(.866) - (1.5\text{ nts})(4\text{ m}) = (1/2)\,(2\text{ kg})\,(v_2)^2 - (1/2)\,(2\text{ kg})\,(9\text{ m/s})^2$$
$$\Rightarrow \quad v_2 = 7.59\text{ m/s}.$$

Note 1: Do not memorize the final form of the above equation. The key is to understand *how we got it*. It is the approach that is important here, not the final result!

Note 2: Going back for another look at the original formulation of the problem (i.e., the way *you* ought to present a test problem should you be asked to use the *work/energy theorem* to solve a problem):

$$W_{net} = \Delta KE$$
$$W_F + W_f = \Delta KE$$
$$(-Fd\cos\theta) + (-f_k d) = (1/2)mv_2^2 - (1/2)mv_1^2$$
$$-(5\text{ nts})(4\text{ m})(.866) - (1.5\text{ nts})(4\text{ m}) = (1/2)\,(2\text{ kg})\,(v_2)^2 - (1/2)\,(2\text{ kg})\,(9\text{ m/s})^2$$
$$\Rightarrow \quad v_2 = 7.59\text{ m/s}.$$

Note 3: For a moment, think about *the approach*. It allows you to relate the total amount of *energy-changing work* W_{net} done on the body to the way the body's energy-of-motion (its *kinetic energy*) changes. Forces come into play in calculating the "work" part of the relationship. That means N.S.L. is still important (you could need it to determine an expression for the magnitude of an unknown force), but the main thrust is wrapped up in the question, "How does the system's ENERGY change?"

Note 4: Although the *Work/Energy Theorem* is important, we will shortly be using it to derive an even more important relationship. We haven't yet gotten to the "bottom line" of this approach.

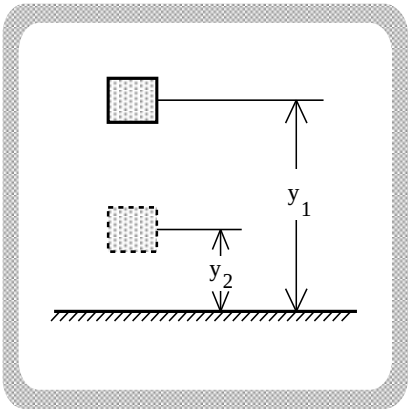


FIGURE 6.11

E.) Conservative Forces:

1.) Background: A body of mass *m* moves from y_1 (call this Position 1) to y_2 (call this Position 2) with a constant velocity (see Figure 6.11). How much work does *gravity* do on the body as it executes the motion?

Note: There are at least two forces acting on the body in this case, one provided by gravity and one

provided by an outside agent like yourself. Our only interest in this problem is in the work *gravity* does.

a.) Noting that the angle between the *line of the gravitational force* and the *line of the displacement vector* is 0^o, we can use our definition of *work* to write:

$$W_{gr} = \mathbf{F}_g \cdot \mathbf{d}$$
$$= |\mathbf{F}|\,|\mathbf{d}| \cos 0^o$$
$$= (mg)(y_1 - y_2)(1) \qquad \text{(Equation A)},$$

which could be written:

$$= -(mg)(y_2 - y_1)$$
$$= -(mg)(\Delta y).$$

Note: By definition, Δy is the *final height* y_2 minus the *initial height* y_1. The last two steps of the above derivation were included to make use of this fact (this particular notation will come in handy later).

2.) Let us now replay the situation with a small alteration. Assume now that the block moves from *Point 1* to *Point 2* following the path outlined in Figure 6.12. How much work does *gravity* do on the block in this situation?

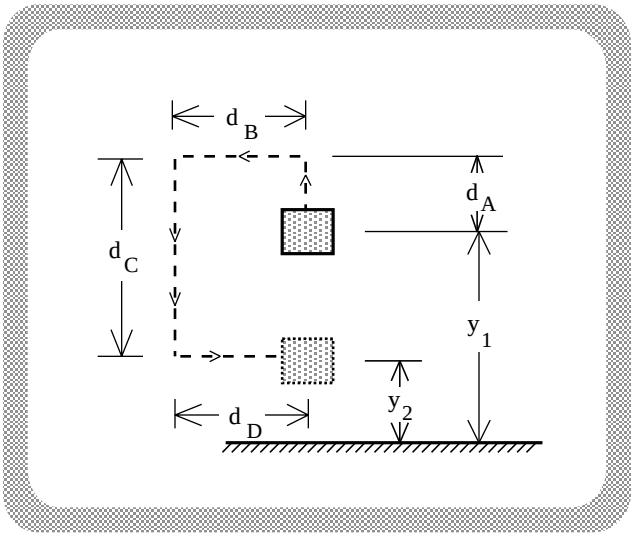


FIGURE 6.12

a.) Noting that the *total work* gravity does will equal the work done by gravity through each section of the displacement, we get:

$$W_{gr} = W_{d_A} + W_{d_B} + W_{d_C} + W_{d_D}.$$

b.) We know that the distance $d_C = d_A + (y_1 - y_2)$. Using that and the definition of *work*, we can write:

$W_{gr} = (mg)(d_A)\cos 180^o + (mg)(d_B)\cos 90^o + (mg)(d_A + y_1 - y_2)\cos 0^o + (mg)(d_D)\cos 90^o.$

c.) Setting *cos 180^o= -1*, *cos 90^o= 0*, and *cos 0^o= 1,* this becomes:

$$\begin{aligned} W_{gr} &= -(mg)(d_A) + (mg)(d_A + y_1 - y_2) \\ &= -(mg)(d_A) + (mg)(d_A) + (mg)(y_1 - y_2) \\ &= +(mg)(y_1 - y_2) \\ &= -(mg)(y_2 - y_1). \end{aligned}$$

d.) Notice that this is the same amount of work *gravity* did when the body followed the first path. In fact, no matter what path the body takes in moving from *Point 1* to *Point 2*, the amount of work gravity does on the body will *always be the same.*

Put another way, the *amount of work* gravity does on a body as the body moves from one point to another in the gravitational field is PATH INDEPENDENT. *FORCE FIELDS THAT ACT THIS WAY ARE CALLED CONSERVATIVE FORCE FIELDS.*

e.) A corollary to this *path independence* observation is the fact that the amount of work a *conservative force field* does on a body that moves around a *closed path* in the field will always be ZERO!

Note: "Moving around a closed path" means the body ends up back where it started.

i.) Reasoning? Consider a body that moves upward a vertical distance *d*. The work *gravity* does on the body will be *-mgd* (negative because the angle between the *displacement vector* and the *gravitational force* is 180^o). When the body is brought back down to its original position, the work *gravity* does is *+mgd*. The *total work* gravity does on the body as it moves through the round trip is (*-mgd* + *mgd*), or ZERO.

Gravity is a *conservative force field*.

f.) An example of a force field that is not *conservative* is *friction*. Common sense dictates that the further a body moves under the influence of friction, the more work friction will do on the body. As an example, anyone who has ever dragged a fingernail across a chalkboard knows that the further one drags, the more *work* friction does on his fingernails (and the more his listening friends will want to murder him).

From another perspective, frictional forces always *oppose* the direction of relative motion between two bodies. This means that a frictional force will either do all *negative work* or all *positive work* (99% of the time it's negative), depending upon the situation. That, in turn, means that the work due to friction on a body moving around a closed path can never equal zero.

Friction is a *non-conservative force*.

Note: For those of you who are wondering if there are other kinds of non-conservative force fields, all *time-varying* force fields qualify. You will not be asked to deal with time-varying fields until much later; the only *non-conservative* force you will have to worry about for now is friction.

F.) Preamble to the *Gravitational Potential Energy* Function:

Note 1: We are about to consider a concept you have heard about in past science classes but that was most probably never addressed in a truly rigorous way. To eliminate as much intellectual stress as possible, my suggestion is that you forget everything you have ever been told about *potential energy* and start from scratch with the presentation that follows.

Note 2: You will not be held responsible for duplicating any of the material you are about to read except *the bottom line*. BUT, if you don't understand the following material you *won't* understand the *bottom line*, and if you don't understand the "bottom line" you will undoubtedly find yourself totally lost later. Therefore, read the next section, not for memorization purposes but for content. Follow each step as it comes without projecting ahead. When you finally get to the end-result, read back over the material to be sure you know what assumptions were made in proceeding to the endpoint.

1.) Consider a conservative force field--gravity, for instance. A body of mass m moves from y_1 (call this Position 1) to y_2 (call this Position 2) with a constant velocity. How much work does the *gravitational force field* do on the body as the body so moves?

a.) This was the question posed at the beginning of the "Conservative Forces" section. The solution was found to be:

$$W_{gr} = - (mg) (y_2 - y_1) \qquad \text{(Equation A).}$$

b.) One of the important conclusions drawn from that section was the observation that as a body moves from *Point 1* to *Point 2* in a gravitational field, the work done by the field is not dependent upon the path taken. Gravity is a *conservative force*.

c.) With that in mind, let's consider a novel idea. If the path counts for nothing--if the endpoints are all that are important when determining the work gravity does--might it not be possible to somehow define a number N_1 that can be attached to *Point 1*, and a number N_2 that can be attached to *Point 2*, and cleverly make them such that the *difference* between them would yield the amount of *work done by gravity* as the body proceeds from *Point 1* to *Point 2*?

d.) This surely is a strange idea, but whether you see the usefulness of it or not, could it be done?

The answer is "yes."

e.) Example: Figure 6.13 shows just such a possible situation. Assuming the numbers have been chosen appropriately, the work done on the body due to gravity as the body goes from *Points 1* to *2* should be:

$$W_{gr} = (N_2 - N_1)$$
$$= [(12 \text{ joules}) - (25 \text{ joules})]$$
$$= -13 \text{ joules.}$$

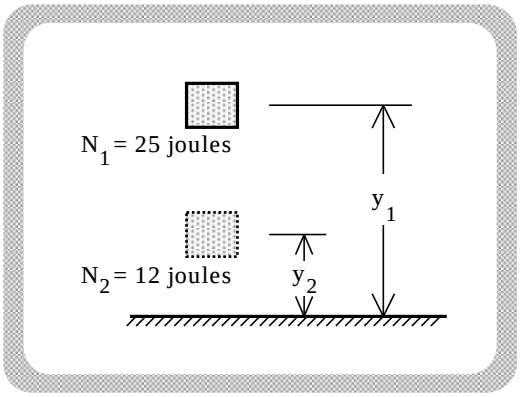


FIGURE 6.13

f.) There is only one difficulty with this.
We have assigned *zero* to ground level making all numbers above ground level *increase* with elevation. That means that when a body moves from a *higher* (big number) position to a *lower* (small number) position, the difference between the second number and first number ($N_{low} - N_{hi}$) will be negative (just as we found in our example). The problem here is that if we proceed from high to low (i.e., move in the direction of *mg*), the work *gravity* does should be *positive*!

To make our scheme work, we need to modify our original model by re-defining the "numbers expression." We will do so by putting a *negative sign* in front of the relationship. This yields:

$$W_{gr} = - (N_2 - N_1) \qquad \text{(Equation B)}$$
$$= - [(12 \text{ joules}) - (25 \text{ joules})]$$
$$= + 13 \text{ joules.}$$

g.) With our modification, we now have numbers attached to our initial and final points that, when correctly manipulated, give us the *work done by gravity* as the body moves from *Point 1* to *Point 2*.

Note: Kindly notice that we can do this only because the gravitational force is *conservative* and, hence, the work done due to gravity is *path independent*. If the work done depended upon the path taken, none of this would make any sense at all.

h.) It would be nice to have some handy mathematical function that would allow us to define our *N* numbers. Fortunately, we already have such a function for gravity. Using the definition of work, the *work done by gravity on a body moving from Point 1 to Point 2 in a gravitational field is*:

$$W_{gr} = - (mgy_2 - mgy_1).$$

We determined this expression earlier.

i.) By comparing this equation with our "number expression":

$$W_{gr} = - (N_2 - N_1),$$

we find by inspection that:

$$N_2 = mgy_2 \quad \text{and} \quad N_1 = mgy_1.$$

j.) Written in general (i.e., written as *mgy* where *y* is the <u>*vertical distance*</u> above some arbitrarily chosen *zero-height level*--the ground in our example), this function is important enough to be given a special name. It is called the "*gravitational potential energy*" function, normally characterized as U_g.

k.) <u>Bottom Line</u>: Although we have done this analysis using a gravitational force field, EVERY *conservative force field* has a *potential energy* function associated with it. Furthermore, there is a formal, Calculus-driven approach for deriving *potential energy* functions which will be presented shortly.

Whether you are given a potential energy function or have to derive it, understand that when a body moves through a *conservative force field* the amount of *work done by the field* as the body moves from *Point 1* to *Point 2* will always be:

$$W_{field} = - (U_{pt.2} - U_{pt.1})$$
$$= - \Delta U.$$

This is the bottom line on *potential energy*.

G.) Comments and Problems--Potential Energy Functions in General:

1.) Gravitational potential energy is not an absolute quantity.

a.) Consider the table and chalk shown in Figure 6.14. If we take y to be measured from the table's top (i.e., y_1 in the sketch), we are safe in saying that the amount of *potential energy* the chalk has is equal to mgy_1. If we want to determine the amount of work gravity does on the chalk as it rises to a second point at y_2, we can use the above-derived expression relating *gravitational potential energy* to *the work gravity does*, and get:

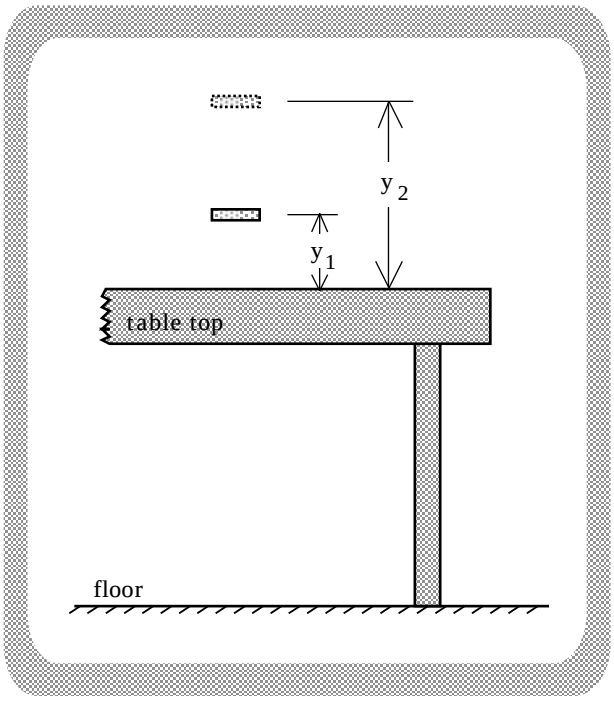


FIGURE 6.14

$$W_{gr} = - \Delta U_{gr}$$
$$= - (U_2 - U_1)$$
$$= - (mgy_2 - mgy_1).$$

b.) Could we have used *the floor* as the *zero potential energy level*, making all y measurements from there?

ABSOLUTELY! The chalk would be assigned an initial *potential energy* value of mgy_3 (see Figure 6.15 on the next page), etc., and the work calculation would proceed as before:

$$W_{gr} = - \Delta U_{gr}$$
$$= - (U_4 - U_3)$$
$$= - (mgy_4 - mgy_3).$$

c.) The amount of work gravity does as the chalk rises to its new position can be determined correctly using *either* approach (notice that $y_2 - y_1$ is numerically equal to $y_4 - y_3$).

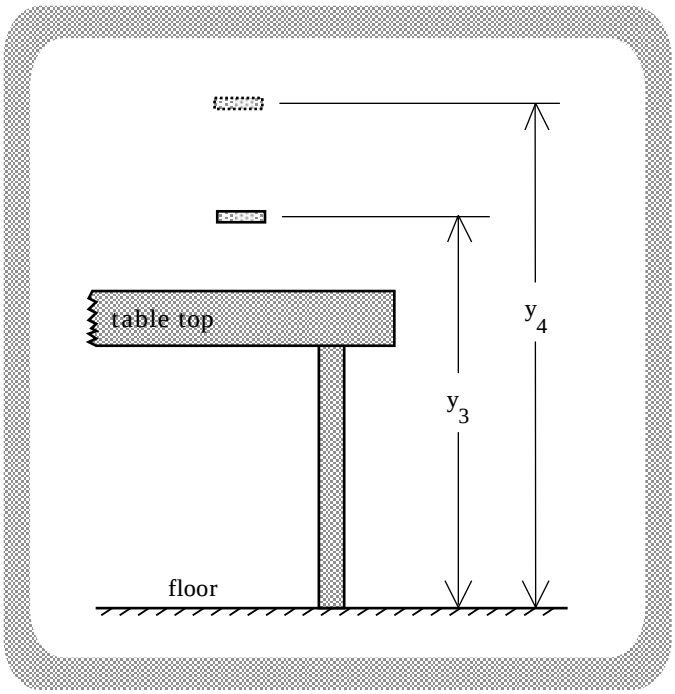


FIGURE 6.15

Why does this seemingly nonsensical situation exist? Because what is important is not the *amount* of gravitational potential energy an object has when at a particular point. What is important is the *change* of the gravitational potential energy of a body *as it moves from one point to another*. That is what allows us to determine the *amount of work* done on the body as it moves through the gravitational field. Thus, *work determination* is the *ONLY USE* you will ever have for *potential energy functions* . . . ever.

2.) Although most students associate *potential energy* with <u>*gravitational*</u> *potential energy*, there are actually many other *conservative force fields*. For instance, an ideal spring produces a force that is, at least theoretically, conservative. *All conservative forces have potential energy functions associated with them.*

Their use?

If you want to know *how much work* a conservative force field does on a body moving from one point to another within the field, and if you know the field's *potential energy function*, the work done by the field will always equal *minus the change of the potential energy function between the start and end points*, or:

$$W_{\text{cons.force}} = -\,\Delta U.$$

3.) <u>A Work/Energy-Theorem, Potential-Energy Example Problem:</u> A plane oriented at 30° above the horizontal moves at 300 m/s. It is 1200 meters above the ground when a coke bottle becomes free and sails out of the window

a' la the movie The Gods Must Be Crazy (see Figure 6.16). Neglecting air friction, how fast will the bottle be moving just before it hits the ground?

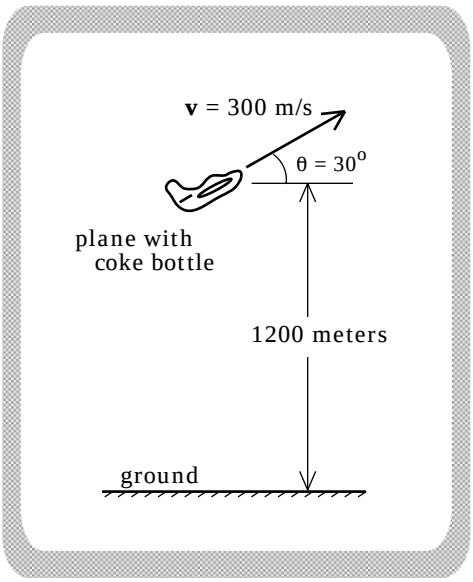


FIGURE 6.16

a.) The work/energy theorem states that the *net work* done on a body must equal the body's change in *kinetic energy* (ΔKE). Mathematically, this is stated as:

$$W_{net} = \Delta KE.$$

b.) In this case, the W_{net} consists solely of the *work done by gravity* W_g. Coupling this with the fact that there is a change in the *kinetic energy* $\Delta KE = (1/2)mv_2^2 - (1/2)mv_1^2$, we can write:

$$W_g = (1/2)mv_2^2 - (1/2)mv_1^2.$$

c.) If we had to calculate the work due to gravity using only the definition, the task would require Calculus (the bottle's *direction of motion* is constantly changing, which means the *angle* between the gravitational force and the displacement is constantly changing--see Figure 6.17) which would be nasty. Fortunately for us, we can easily determine the work gravity does in this situation because:

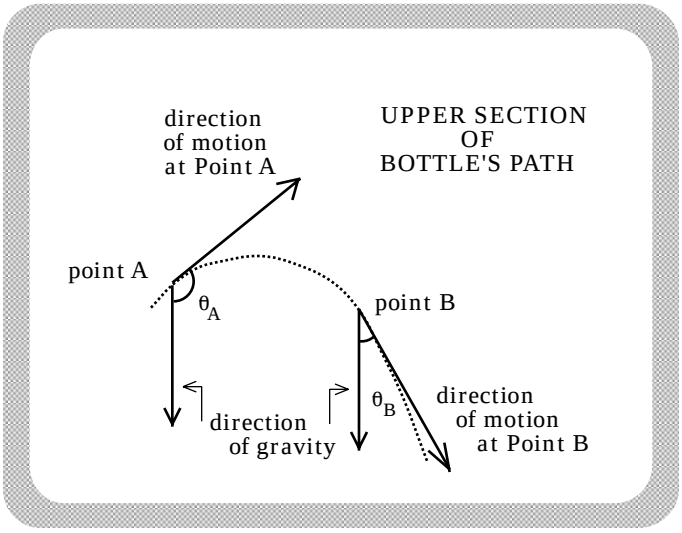


FIGURE 6.17

i.) We know the *potential energy function* for gravity is *mgy*; and

ii.) We know that:

$$W_g = -\Delta U_g = -(U_{2,g} - U_{1,g}).$$

d.) Utilizing these facts, we find:

$$W_g = (1/2)mv_2^2 - (1/2)mv_1^2$$
$$-(U_2 - U_1) = (1/2)mv_2^2 - (1/2)mv_1^2$$
$$-(mgh_2 - mgh_1) = (1/2)mv_2^2 - (1/2)mv_1^2.$$

e.) Solving for v_2 yields:

$$v_2 = [v_1^2 - 2\,(gh_2 - gh_1)]^{1/2}.$$

f.) Putting in the numbers yields:

$$v_2 = [\,(300\text{ m/s})^2 - 2\,[(9.8\text{ m/s}^2)(0\text{ m}) - (9.8\text{ m/s}^2)(1200\text{ m})]\,]^{1/2}$$
$$= 336.9\text{ m/s}.$$

Note: As usual, memorizing this result is a waste of time. What is important is the technique involved. Whenever you need to know *how much work a conservative force does* on a body moving through its force field, that quantity will always equal $-\Delta U$, where U is the *potential energy function* associated with the field.

H.) Deriving the *Potential Energy Function* for a Known Force Field:

1.) We created the idea of a *potential energy function* out of the need to easily determine the amount of work gravity does as a body moves from one point to another in a gravitational field. We then concluded that *any* conservative force can have a *potential energy function* associated with it. The only requirement? That:

$$W_{cons.fld.} = -\Delta U \qquad \text{(Equation A)},$$

where the symbol U was used to denote the *potential energy function* associated with the conservative force field with which we happen to be dealing.

2.) It is possible to use the above expression to design an approach by which the *potential energy function* of a known, conservative force field can be derived. Specifically:

a.) In general, the work done by any force is:

$$W = \int dW = \int \mathbf{F} \cdot d\mathbf{r} \qquad \text{(Equation B).}$$

b.) Putting our two *work* expressions together (i.e., Equation A and Equation B), we get:

$$W_{cons.fld.} = -\Delta U$$
$$= \int \mathbf{F}_{cons.force} \cdot d\mathbf{r}.$$

More succinctly, if a body is moving in the *x direction* from coordinate x_1 to coordinate x_2, we can write:

$$[U(x_2) - U(x_1)] = -\int_{x_1}^{x_2} \mathbf{F}_{cons.force} \bullet d\mathbf{r},$$

where the *differential displacement* is $d\boldsymbol{r} = dx\boldsymbol{i}$ and the integral must be evaluated (as shown) between x_1 and x_2.

Note: In advanced physics books, this relationship is expressed in a more general expression:

$$[U(\mathbf{r}_2) - U(\mathbf{r}_1)] = -\int \mathbf{F}_{cons.fld} \cdot d\mathbf{r},$$

where the *differential displacement* is $d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$ and the integral must be evaluated between the generalized, two or three dimensional coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Also, the symbol used for *potential energy* in some books is *PE* instead of *U*.

c.) How does this help us define a *potential energy function*? Being clever, we can set x_1 equal to the coordinate at which the *potential energy* is to equal zero, then set x_2 to any arbitrary coordinate *x*. In that way, we end up with an integral we can do (we know ***F***) that has limits we can evaluate, and that is equal to *U(x) - 0* or, simply, *U(x)*.

That is exactly what we want--a function that is general and that reflects the *potential energy of the system* at any coordinate *x*.

3.) An example: The gravitational force field close to the earth's surface applies a force equal to $-mg\boldsymbol{j}$ *newtons* to any body placed in the field. What is the *potential energy function* for this *force field*?

a.) The first thing to decide is where the *potential energy function* is to equal zero. IN MOST CASES, THE *POTENTIAL ENERGY FUNCTION* IS DEFINED AS ZERO WHERE THE *FORCE FUNCTION,* ITSELF, EQUALS ZERO. The only cases in which this isn't true are cases in which the force is a constant (hence, the force *has* no place where it is zero). Gravity is one such force. That means there is no preferred *zero potential energy level* for gravity close to the earth's surface. As such, we will arbitrarily choose *ground level* (i.e., $y = 0$) to be the *zero level* and work from there.

b.) Defining $U(y = 0) = 0$, we can use our definition of *potential energy* to write:

$$\left[U(y) - U(y=0)\right] = -\int \mathbf{F} \bullet d\mathbf{r}$$

$$\Rightarrow \quad U(y) - 0 = -\int_{y=0}^{y} \left[-mg\mathbf{j}\right] \bullet \left[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right]$$

$$\Rightarrow \quad U(y) = \int_{y=0}^{y} (mg)dy$$

$$= mgy\Big|_{y=0}^{y}$$

$$= mgy$$

Note: $U(y = 0)$ is ZERO, not because $y = 0$ but because we have chosen to start our *work* calculation at the place where the *potential energy* has been defined as ZERO. In this case, that just happens to be at the coordinate $y = 0$.

c.) This is exactly the *potential energy function* we determined using the hand-waving arguments employed in the previous section.

4.) A more challenging example: The gravitational force field between *any two bodies* varies depending upon how massive the bodies are and how far their *centers of mass* are from one another. Newton deduced this general force function for gravity as:

$$\mathbf{F} = \left[-G(m_1\, m_2)/r^2\right]\mathbf{r},$$

where G is called the *universal gravitational constant*, m_1 and m_2 are the masses in question, r is the distance between their mass centers, and $\boldsymbol{r}$ is a unit vector in the *radial* direction (gravitational forces are always directed along a line between the two bodies--i.e., in a *radial* direction). The question? What is the *potential energy function* for this *force field?*

a.) For simplicity, assume the gravitational force is pointed in the *x direction*, making

$$\mathbf{F} = [-G(m_1\, m_2)/x^2]\mathbf{i}.$$

b.) Having made that simplification, the first thing to decide is where the *potential energy function* is equal to zero. Using the criterion suggested previously, the *potential energy* should be zero where the *force function* is equal to zero. That occurs at $x = \infty$.

Note: Remember what we are essentially doing when we use this approach? We are calculating the amount of work our force field does (i.e., $\int \boldsymbol{F}\cdot d\boldsymbol{r}$) as we move from the *zero potential energy point* to an arbitrary point within the field. In this case, we are moving from $x = \infty$ to some point at coordinate x.

c.) Defining $U(x = \infty) = 0$, we can use our definition of *potential energy* to write:

$$\begin{aligned}
\left[U(x) - U(x=\infty)\right] &= -\int \mathbf{F} \bullet d\mathbf{r} \\
\Rightarrow \quad \left[U(x) - 0\right] &= -\int \left[-G\frac{m_1 m_2}{x^2}\mathbf{i}\right] \bullet \left[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right] \\
&= -\int_{x=\infty}^{x} \left[-G\frac{m_1 m_2}{x^2}\right] dx \\
&= \left(G m_1 m_2\right)\left[\frac{-1}{x}\right]_{x=\infty}^{x} \\
&= \left(G m_1 m_2\right)\left[\left(\frac{-1}{x}\right) - \left(\frac{-1}{\infty}\right)\right] \\
&= -G\frac{m_1 m_2}{x}
\end{aligned}$$

d.) This is the *potential energy function* for gravitational fields anywhere. Does it work? Let's see. According to the theory, we should be able to calculate the amount of work gravity does as a body moves from one point to another in a conservative force field using:

$$W = -\Delta U.$$

Assume your mass is 85 kg. You're in an elevator moving upward from ground level to a position 200 meters above the ground. How much work does *gravity* do as you so move?

i.) Using the *gravitational potential energy function* we derived for situations near the surface of the earth (i.e., $U_{mg,near} = mgy$, where we can assume ground level is the *zero potential energy level*), the amount of work done by gravity is found to be:

$$\begin{aligned} W_{grav} &= -[\quad U(y_2 = 200) \quad - U(y_1 = 0)] \\ &= -[\quad mgy_2 \quad - \quad mgy_1 \quad] \\ &= -[(85\text{ kg})(9.8\text{ m/s}^2)(200\text{ m}) - \quad 0 \quad] \\ &= -166{,}600\text{ joules.} \end{aligned}$$

ii.) We would like to do the same problem using the <u>*general*</u> *potential energy function* for gravity (i.e., $-Gm_1m_2/r^2$, where r is the distance between the center of masses of the interacting objects--in this case, you and the earth). To do so, note that:

--the universal gravitational constant $G = 6.67x10^{-11}\ m^3/kg{\cdot}s^2$;

--the mass of the earth is $m_e = 5.98x10^{24}\ kg$;

--the radius of the earth is $r_e = 6.37x10^6\ m$.

iii.) Remember how *potential energy functions* are used. If you want the amount of work done by a conservative field as a body moves from one point to another in the field, evaluate the *potential energy function* for the field AT THE START POINT and AT THE END POINT, then take *minus* the difference of that amount. The value you end up with will be the work done by the field during the motion. Up until now, all you have worked with has been the simple version of *gravitational potential energy*--a function with an adjustable *zero level*. You are about to use a *potential energy function* with a fixed *zero point* (remember, $U = 0$ at <u>*infinity*</u> for this function). Even though you may be in the habit of treating ground level as the *zero point*, that isn't true of this function! With that in mind:

iv.) If we let r_e be the distance between the earth's center and your center of mass when standing on the earth's surface (this is essentially the radius of the earth), then $r_e + 200$ will be the distance between your *center of mass* when at 200 meters above the earth's surface. We can write:

$$W_{grav} = -\,[\;U(r_e + 200) - U(r_e)\;]$$
$$= -\,\big[[-Gm_e m_{you}/(r_e + 200)] - [-Gm_e m_{you}/(r_e)]\big].$$

v.) Pulling out the constants, eliminating the units for the sake of space, and wholly ignoring significant figures, this becomes:

$$W_{grav} = G\; m_e\; m_{you}[1/(r_e + 200) - 1/(r_e)]$$
$$= (6.67\text{x}10^{-11})\,(5.98\text{x}10^{24})\,(85)\,[1/(6{,}370{,}200) - 1/(6{,}370{,}000)]$$
$$= 5322220652 - 5322387755$$
$$= -167{,}103 \text{ joules.}$$

vi.) Using the *near Earth potential energy function* in *Part d-i* above, we found that gravity did -166,600 joules of work. If we had not used rounded values for G, r_e, and m_e, these two numbers would have been the same.

vii.) Bottom Line: Our approach for determining *potential energy functions* generates functions that work as expected. LEARN THE APPROACH!

I.) The Potential Energy Function for an Ideal Spring:

1.) An ideal spring loses no energy as it oscillates back and forth. The amount of work such springs do through one full cycle is zero, which is to say that the force they provide is a conservative one. As such, we can derive a *potential energy function* for an ideal spring using the approach outlined above.

2.) The position of a body attached to an ideal spring is measured from the system's equilibrium position (i.e., the position at which the force on the body *due to the spring* is zero). It has been experimentally observed that when a mass is attached to a spring and the spring is elongated or compressed:

a.) The magnitude of the *spring force* exerted on the body is proportional to the spring's displacement from the equilibrium, and

b.) The direction of the force always points toward the equilibrium position.

c.) Assuming the force is in the *x direction,* these observations can be mathematically expressed as:

$$\mathbf{F} = -kx\mathbf{i},$$

where *k* is a constant that defines the amount of force required to compress the spring *one meter*, and *x* is the distance the spring is displaced from its equilibrium position (see Figure 6.18 to the right).

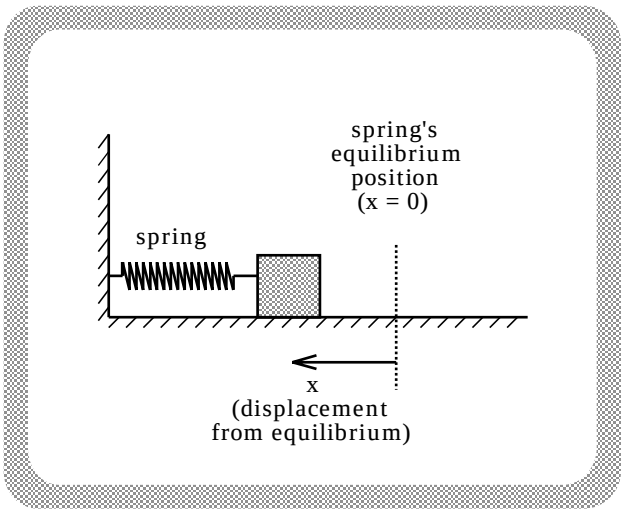


FIGURE 6.18

Note: The displacement term *x* is really a Δx, but as usual the convention is to assume that the initial position is at $x = 0$. This leaves the displacement term as $\Delta x = x - 0 = x$.

3.) Noting that the force function is ZERO at $x = 0$ (hence, the *potential energy function* must be defined as *ZERO at* $x = 0$), we can write:

$$\begin{aligned}
\left[U(x) - U(x=0)\right] &= -\int_{x=0}^{x} \mathbf{F}_{spr} \bullet d\mathbf{r} \\
\Rightarrow \qquad U(x) &= -\int_{x=0}^{x} \left[-kx\mathbf{i}\right] \bullet \left[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right] \\
&= \int_{x=0}^{x} (kx)dx \\
&= k\left[\frac{x^2}{2}\right]_{x=0}^{x} \\
&= \left(\frac{1}{2}\right)kx^2
\end{aligned}$$

4.) Having derived the *potential energy function* for an ideal spring as

$$U_{sp} = (1/2)k(x)^2,$$

we can now use that function in any problem in which an ideal spring does work.

5.) A Problem Involving a *U Function* Other Than Gravity (i.e., that of a Spring): A 2 kilogram block on a horizontal surface is placed without attachment against a spring whose spring constant is $k = 12\ nt/m$. The block is made to compress the spring .5 meters (see Figure 6.19 below). Once done, the block is released and accelerates out away from the spring. If it slides over 2 meters of *frictionless surface* before sliding onto a *frictional* surface, and if it then proceeds to travel an *additional* 13 meters on the *frictional surface* before coming to rest, how large was the *frictional force* that brought it to rest?

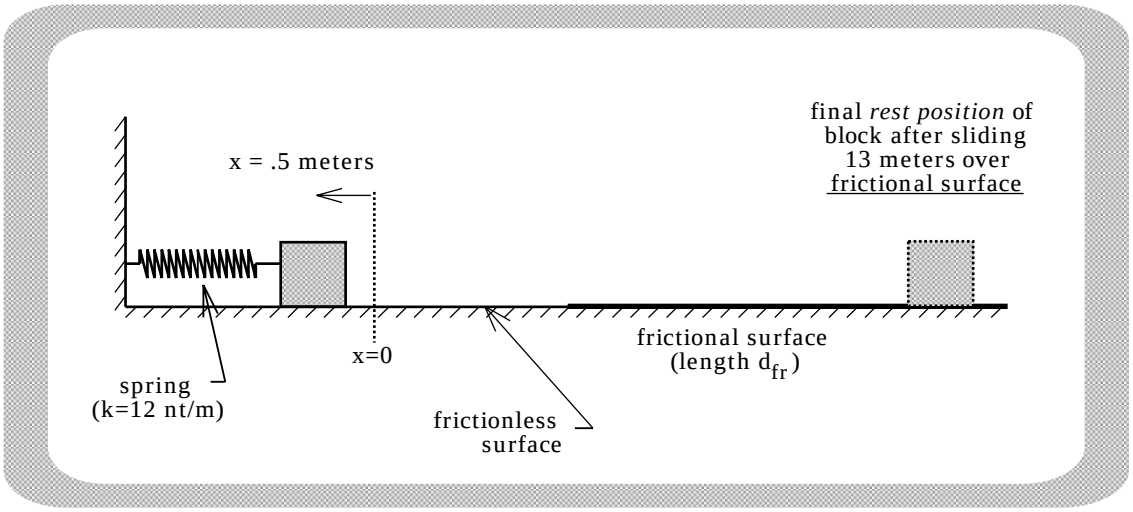

final *rest position* of block after sliding 13 meters over frictional surface
x = .5 meters
x=0
spring (k=12 nt/m)
frictionless surface
frictional surface (length d_{fr})

FIGURE 6.19

Note: Do not get too comfortable with using the *work/energy theorem*. It is an OK approach in some cases, but there is a much easier way to deal with the kind of information given in this problem using the concept of *energy conservation*. That alternative approach will be presented shortly. This example is given SOLELY to allow you to see a *potential energy function* other than *gravity* in a problem.

a.) Looking at this problem from a work/energy perspective, we need to determine two different quantities: *the net change of the body's kinetic energy* (i.e., its final *kinetic energy* minus its initial *kinetic energy*), and *the amount of work done by all forces acting on the body* between the beginning and end of its motion. In short, we need to determine:

$$W_{net} = \Delta KE.$$

b.) As the mass does not *rise* or *fall* in this problem, gravity does no work and there is no reason to include the *potential energy function* for gravity in the *work/energy* expression.

c.) Writing this out as you would on a test (should you be asked to use the *work/energy theorem* on a test), we get:

$$W_{net} = \Delta KE$$

$$\Rightarrow \quad W_{sp} + W_{fr} = KE_2 - KE_1$$

$$-\Delta U_{sp} + (-f_k d_{fr}) = (1/2)\, m\, v_2^2 - (1/2)\, m\, v_1^2$$

$$-[0 - (1/2)kx^2] + (-f_k d_{fr}) = (1/2)\, m\, v_2^2 - (1/2)\, m\, v_1^2$$

$$.5(12 \text{ nt/m})(.5 \text{ m})^2 + (-f_k)(13 \text{ m}) = .5(2 \text{ kg})(0)^2 - .5(2 \text{ kg})(0)^2$$

$$\Rightarrow \quad f_k = .115 \text{ nts.}$$

Note: Once again, THE WORK DONE BY A *CONSERVATIVE FORCE FIELD* ON A BODY MOVING THROUGH THE FIELD WILL ALWAYS EQUAL *-(U_2 - U_1),* ASSUMING THE *POTENTIAL ENERGY FUNCTION* USED IS THE PROPER FUNCTION FOR THE FORCE FIELD.

J.) Force Derived from Known Potential Energy Function:

1.) We have already established an approach for deriving the *potential energy function* associated with a given *force function*. How might we go the other way (i.e., derive a *force function* for a given *potential energy function*)?

2.) In one dimension:

a.) We can write:

$$\int_{x_1}^{x_2} d(U) = \left[U(x_2) - U(x_1)\right] = -\int_{x_1}^{x_2} \mathbf{F} \bullet d\mathbf{x},$$

where the first integral is equal to the middle expression by definition, and the second expression equals the third due to the derivation of the potential energy function.

b.) This implies that:

$$d(U) = -\mathbf{F} \bullet d\mathbf{x},$$

which, in turn, implies that:

$$F_x = -\frac{d(U)}{dx}.$$

3.) It would be useful to have a way to denote this operation in its most general sense so that we can determine the magnitude of components of the force function along with appropriate unit vectors. That possibility exists using partial derivatives and the *del operator*. Specifically:

a.) As:

$$\nabla(U) = \left[\frac{\partial(U)}{\partial x}\mathbf{i} + \frac{\partial(U)}{\partial y}\mathbf{j} + \frac{\partial(U)}{\partial z}\mathbf{k} \right]$$
$$= -\left[F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \right].$$

b.) We can write:

$$\mathbf{F} = -\nabla(U).$$

Note: Interesting observation: This means that the *rate of change* of a *potential energy function* in a particular direction numerically equals the *force component in that direction* (that is, the force component of the force function that is associated with the potential energy function).

K.) MODIFIED CONSERVATION OF ENERGY Theorem: Or, Getting to the Bottom of the Bottom Line:

Note: We are about to put the work/energy theorem into a considerably more useful form. To do so, we will spend some time with the derivation behind "the bottom line." You will not be asked to duplicate this derivation, but if you do not understand it, you will most probably not be able to use the end result to its full extent. Read the following section; think about it; then read it again. It is important that you know what is being done here.

1.) Consider an object with numerous forces acting on it as it moves from *Point A* to *Point B*. The *work/energy theorem* relates the *amount of work done* on the body to the body's *change of kinetic energy*. Writing this out, we get:

$$W_{net} = \Delta KE.$$

The left-hand side of this equation is simply the sum of the work done by all the forces acting on the body. This equation could be written as:

$$W_A + W_B + W_C + W_D + \ldots = \Delta KE,$$

where W_A is the work done by force $\boldsymbol{F}_A$, W_B is the work done by force $\boldsymbol{F}_B$, etc. For the sake of argument:

a.) Assume forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are *conservative forces* with *known potential energy* functions U_A and U_B. If we define the body's *potential energy* when at *Point 1* due to force field $\boldsymbol{F}_A$ as $U_{A,1}$, and the *potential energy* when at *Point 2* due to force field $\boldsymbol{F}_A$ as $U_{A,2}$, then the *work done* by $\boldsymbol{F}_A$ as the body moves from *Point 1* to *Point 2* in the force field will be:

$$\begin{aligned} W_A &= -\ \Delta U_A \\ &= -\ (U_{A,2} - U_{A,1}). \end{aligned}$$

Likewise, the *work done* on the body due to $\boldsymbol{F}_B$ will be:

$$\begin{aligned} W_B &= -\ \Delta U_B \\ &= -\ (U_{B,2} - U_{B,1}). \end{aligned}$$

b.) Assume the forces associated with W_C and W_D are either *non-conservative forces* that have no *potential energy function* or *conservative forces* for which we don't know the *potential energy function*. If that be the case, we will have to determine those *work* quantities using:

$$W_C = \mathbf{F}_C \cdot \mathbf{d}$$

and

$$W_D = \mathbf{F}_D \cdot \mathbf{d}.$$

c.) Having made these assumptions, we can write the *work/energy theorem* as:

$$W_A + W_B + W_C + W_D + \ldots = \Delta KE,$$

or

$$[-(U_{A,2} - U_{A,1})] + [-(U_{B,2} - U_{B,1})] + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^2 - (1/2)mv_1^2.$$

d.) Multiplying the *potential energy quantities* by the *-1* outside their parentheses, we get:

$$(-U_{A,2} + U_{A,1}) + (-U_{B,2} + U_{B,1}) + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^2 - (1/2)mv_1^2.$$

e.) The expression we end up with has:

i.) A number of *potential energy functions* evaluated at t_1 (i.e., when the body is at *Point 1);*

ii.) A number of *potential energy functions* evaluated at t_2 (i.e., when the body is at *Point 2*);

iii.) The *kinetic energy function* evaluated at t_1;

iv.) The *kinetic energy function* evaluated at t_2;

v.) And all the other work done on the body that we haven't been able to keep track of using *potential energy functions*, but that has been done on the body as it moved from *Point 1* to *Point 2*.

f.) If we move all the *time 1* terms to the left-hand side of the equation and all the *time 2* terms to the right-hand side, our equation will look like:

$$(1/2)mv_1^2 + U_{A,1} + U_{B,1} + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^2 + U_{B,2} + U_{A,2}.$$

g.) What we have now is the *kinetic energy* of the body at *Point 1* added to the sum of the *potential energies* attributed to the body while at *Point 1* added to all the extraneous work done on the body (extraneous in the sense that we haven't kept track of it with potential energy functions) between *Points 1* and *2* equaling the *kinetic energy* of the body when at *Point 2* added to the sum of the *potential energies* of the body while at *Point 2*.

Written in shorthand, this is:

$$KE_1 + \Sigma U_1 + \Sigma W_{extraneous} = KE_2 + \Sigma U_2.$$

h.) This is called the *modified conservation of energy* equation. If we identify the sum of the kinetic and potential energies of a body while at a particular point (that is, $KE_1 + \Sigma U_1$) as "the total mechanical energy E_1" of the body at that point in time, the *modified conservation of energy* equation can be written in an even more compact way:

$$E_1 + \Sigma W_{extraneous} = E_2.$$

In this form, the equation states that the *total energy* of the body when at *Point 1* will equal the *total energy* of the body when at *Point 2*, modified only by the "extraneous work" done to the body as it moves *from Points 1 to 2*. In other words, this equation keeps track of the ENERGY the body either *has* or *has-the-potential-of-picking-up* as it moves from one point to another.

Note: The word "conserved" here means "not changing with time." If we have no extraneous bits of work being done as the body moves from *Point 1* to *Point 2*, which is to say we know the *potential energy functions* for all the forces doing work on the body as it moves and there are no non-conservative forces acting on the system, we can write $E_1 = E_2$. This is the true "conservation of energy" equation. That equation is the mathematical way of saying, "The *total energy* of the system will always be the same--the body's *kinetic energy* may change and its *potential energy* may change, but the *sum* of the *kinetic* and *potential energies* will be a constant throughout time."

By adding the possibility of dealing with non-conservative or oddball conservative forces (one for which we haven't a potential energy function), the "modified" *conservation of energy* equation is extremely powerful. It allows for the analysis of situations in which E_1 and E_2 are *not* equal but are related in a deducible way.

2.) Bottom Line: When approaching a problem from the standpoint of energy considerations:

a.) Determine the amount of *kinetic energy* the body has to start with (this may be nothing more than writing down $(1/2)mv_1^2$) and place that information on your sketch next to the body's position at *Point 1*. Do the same for *Point 2*.

b.) Identify any conservative forces for which you know *potential energy functions*. Once identified, determine the *amount of potential energy* the body has when at *Point 1* and put that information on your sketch. Do the same for *Point 2*.

Note: If *gravity* is the only force with *potential energy function* in the problem, this last step may amount to nothing more than writing $U_1 = mgh_1$ next to *Position 1* on your sketch with a similar notation at *Position 2*.

c.) Identify any forces that do work on the body as it moves from *Point 1* to *Point 2*, but for which you don't have *potential energy functions*. Determine the *amount of work* they do over the motion and place that information in a convenient spot on your sketch.

d.) Take the information gleaned from *Parts a, b,* and *c,* and after writing out $KE_1 + \Sigma\ U_1 + \Sigma\ W_{extraneous} = KE_2 + \Sigma\ U_2$, plug the information in where appropriate. Solve for the unknown(s) in which you are interested.

3.) A Simple Example: Consider a ball of mass .25 kilograms positioned at $y_1 = +4$ *meters* above the ground. It is given an initial upward velocity of 6 m/s at a 60^o angle with the horizontal. The ball freefalls, finally reaching $y_2 = 1$ *meter* above the ground. If friction does 7 joules of work on the ball during the trip, how fast is the ball moving when it gets to $y_2 = 1$ *meter*?

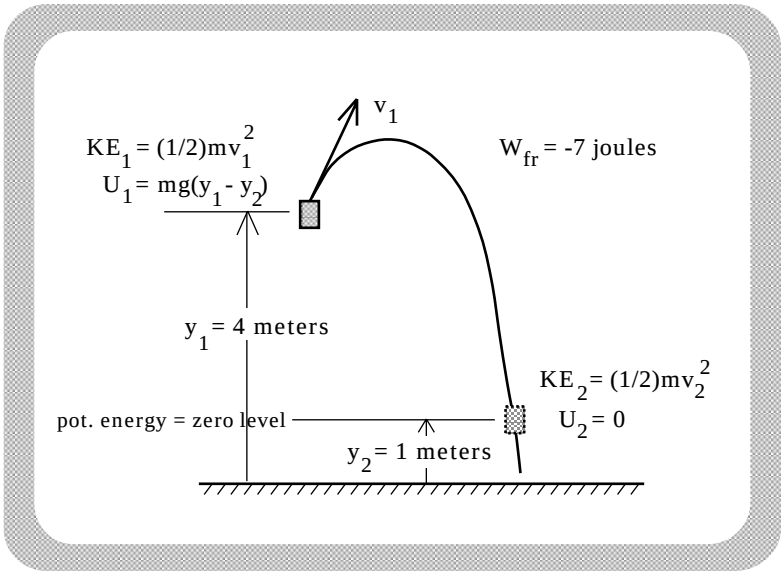


FIGURE 6.20

a.) Consider the sketch in Figure 6.20. In it is placed all the information needed to solve this problem. WE WILL ASSUME THE *ZERO POTENTIAL ENERGY* LEVEL is AT THE "FINAL POSITION" (i.e., y_2).

b.) Remembering that the *work due to friction* is negative and that the *zero potential energy level* is at y_2, we can begin with the *modified conservation of energy equation* and write:

$$KE_1 + \Sigma\ U_1 + \Sigma\ W_{extraneous} = KE_2 + \Sigma\ U_2.$$

c.) Spreading out that equation to see what goes where, then solving, we get:

$$\begin{array}{ccccccc} KE_1 & + & \Sigma\ U_1 & + \Sigma\ W_{ext} = & KE_2 & + \Sigma\ U_2 \\ (1/2)mv_1^2 & + & mg(y_1 - y_2) & + & W_{fr} = (1/2)mv_2^2 & + & 0 \end{array}$$

$$.5(.25\text{ kg})(6\text{ m/s})^2 + (.25\text{ kg})(9.8\text{ m/s}^2)[(4\text{ m}) - (1\text{ m})] + (-7\text{ j}) = .5(.25\text{ kg})v_2^2 + 0$$

$$\Rightarrow \quad v_2 = 6.23\text{ m/s}.$$

Note 1: Important point: Notice the angle had nothing to do with this problem. As far as the concept of energy is concerned, it does not matter whether the body is moving downward or upward or sideways. The amount of energy the body has at a given instant is solely related to the body's mass and velocity, NEVER ITS DIRECTION. As such, do not waste time breaking *velocity vectors* into their component parts. All you need is the velocity's *magnitude*.

Note 2: You could just as easily have taken ground level to be the *zero potential energy level*. If you had, the initial potential energy would have been mgy_1 instead of $mg(y_1 - y_2)$ and the final potential energy would have been mgy_2 instead of zero. Both ways work (if you don't believe me, try it); there is no preferred way to attack the problem.

4.) Example You've Already Seen, Done the Easy Way: A 2 kilogram block on a horizontal surface is placed without attachment against a spring whose spring constant is $k = 12\ nt/m$. The block is made to compress the spring .5 meters (see Figure 6.21 on the next page). Once done, the block is released and accelerates out away from the spring. If it slides over 2 meters of *frictionless surface* before sliding onto a *frictional* surface, and if it then proceeds to travel an *additional* 13 meters on the *frictional surface* before coming to rest, how large is the *frictional force* that brought it to rest?

Note: All the information concerning the *energy state of the system* when the block is at *Point 1* is shown on the sketch. The same is true for *Point 2*. Even the *work done by forces not accommodated by potential energy functions* is written onto the sketch. All the information you need to use the *modified conservation of energy* expression is laid out in its entirety. All that has to be done from there is to put the information into the *c. of e.* equation.

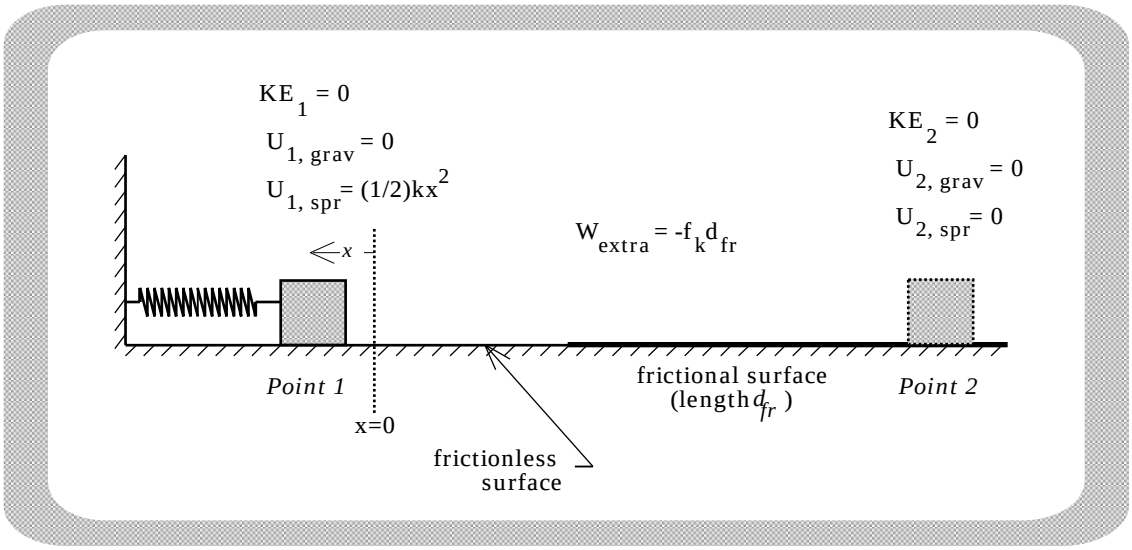


FIGURE 6.21

a.) According to the *modified conservation of energy* expression:

$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$

$$(1/2)mv_1^2 + [U_{1,gr} + U_{1,sp}] + [W_{f_k}] = (1/2)mv_2^2 + [U_{2,gr} + U_{2,sp}]$$

$$0 + [0 + (1/2)kx^2] + [-f_k d_{fr}] = 0 + [0 + 0]$$

$$\Rightarrow f_k = [k x^2] / [2 d_{fr}]$$

$$= [(12 \text{ nt/m})(.5 \text{ m})^2] / [2(13 \text{ m})]$$

$$= .115 \text{ nts.}$$

b.) When this example was done in the *work/energy* section, you were told not to get too attached to the *work/energy* approach. Why? Because another approach was coming that was purported to be easier to use.

You have now seen the other technique--the *modified conservation of energy* approach. What makes it so easy? It is primarily end-point dependent. Indeed, you have to manually determine the amount of work done on the body in-between the end-points if you have forces for which you haven't *potential energy functions*, but that is considerably easier than hassling with *work* calculations for *each* force on an individual basis.

Bottom line: In short, the *modified conservation of energy* approach is easier to execute. Get to know it, understand it, practice it, and you'll learn to love it!

5.) <u>A More Complex Example</u>: A block of mass m is pressed against an unattached spring whose equilibrium position is $d_1 = 3$ *meters* above ground and whose spring constant is $k = 25.6mg/d_1$ (see Figure 6.22). The block is made to compress the spring a distance $d_1/8$ *meters*. The block is additionally forced against the side-wall by your little sister. The force she applies ($\boldsymbol{F}_{sis}$) has a magnitude of $mg/4$ (no, mg does *not* stand for milligrams; it is the *weight of the block--mass* times *gravity*) at an angle of 60^o with the vertical. The wall is *frictional* with a coefficient of friction of $\mu_k = .4$ (see Figure 6.23 below). Once released by *you* (your sister is still pushing), the block falls. How fast will it be traveling when it reaches *Position 2* a distance $d_1/4$ from the ground?

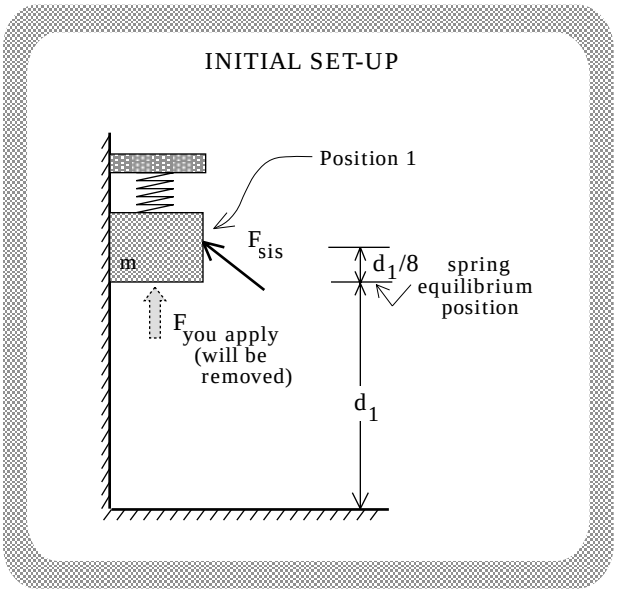


FIGURE 6.22

a.) We need an equation that will allow us to determine the velocity of the block after it has moved to $y = d_1/4$. As the *conservation of energy* approach is related to <u>distances traveled</u> (these are wrapped up in the *work* calculations and *potential energy* functions) and velocities (these are wrapped up in the *kinetic energy* calculations), we will try to use that approach here.

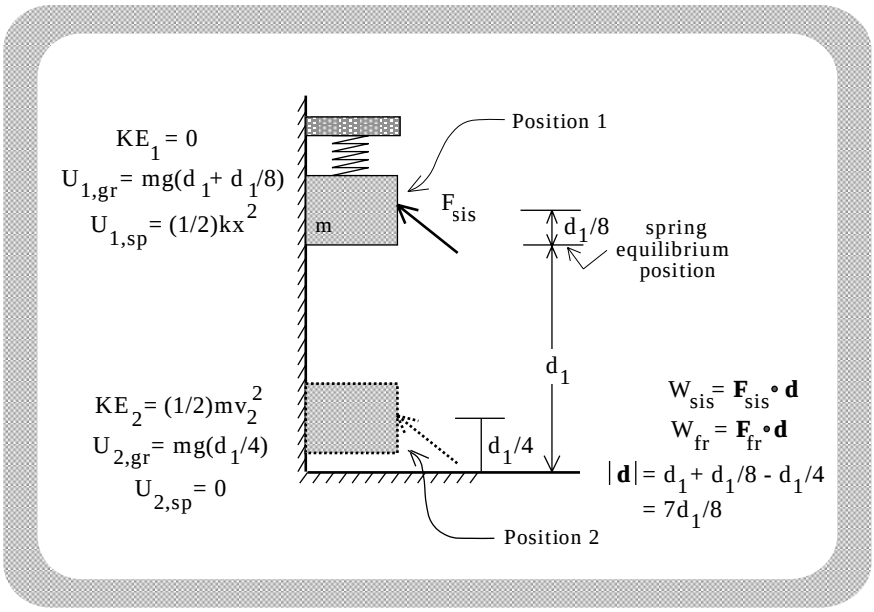


FIGURE 6.23

Note 1: As all our *distance measurements* are relative to the ground, we might as well take the *zero potential energy level* for gravity to be at ground-level.

Note 2: When the block is released, it is accelerated downward by *gravity* and the *spring* but is also retarded in its acceleration by *friction* and *your little sister*. We know *potential energy* functions for *gravity* and the *spring*, but we have no function for *your sister's force* or *friction*.

b.) In its bare bones form, the *modified conservation of energy* equation yields (justification for each part is given below in *Section 5c*):

$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$

$$0 + [\,U_{1,gr} + U_{1,sp}\,] + [\,W_{sis} + W_{f_k}\,] = (1/2)mv_2^2 + [U_{2,gr} + 0]$$

$$0 + [mg(d_1 + d_1/8) + (1/2)kx^2\,] + [\,\mathbf{F}_{sis}\cdot\mathbf{d}_{sis} + (-f_k)(d_{fr})\,] = (1/2)mv_2^2 + [mg(d_1/4) + 0]$$

$$0 + [\,mg(9d_1/8) + .5(25.6mg/d_1)(d_1/8)^2] + [(mg/4)(7d_1/8)\cos 120^\circ + (-\mu_k N)(7d_1/8)] = (1/2)mv_2^2 + [.25mgd_1 + 0]$$

$$0 + [(1.125mgd_1) + (.2mgd_1)\,] + [\,(-.11mgd_1) + (-.074mgd_1)\,] = (1/2)mv_2^2 + [.25mgd_1]$$

$$\Rightarrow \quad v_2 = [1.78\ gd_1]^{1/2}$$
$$= [1.78\ (9.8\ m/s^2)\ (3\ m)]^{1/2}$$
$$= 7.23\ m/s.$$

c.) If the pieces used in the above expression are obvious, skip this section and continue onward. If they are not obvious, the following should help:

i.) At *Point 1*, as the block is not initially moving:

$$KE_1 = 0.$$

ii.) At *Point 1*, the block has *gravitational potential energy*

$$U_{1,gr} = mg(d_1 + d_1/8) = 1.125mgd_1$$

and *spring potential energy*

$$U_{1,sp} = (1/2)kx^2 = (1/2)(25.6mg/d_1)(d_1/8)^2 = .2mgd_1.$$

iii.) At *Point 2*, the block has gravitational potential energy

$$U_{2,gr} = mg(d_1/4) = .25mgd_1.$$

iv.) At *Point 2*, the block has no *spring potential energy* (as the spring *exerts no force on the block* when the block is at *Point 2*, the spring provides *no potential energy* to the block when at that point):

$$U_{2,sp} = 0.$$

v.) At *Point 2*, the block will have *kinetic energy*

$$KE_2 = (1/2)mv_2^2.$$

vi.) In between *Points 1* and *2*, "extraneous" work is done by *little sister* in the amount of:

$$\begin{aligned} W_{sis} &= \mathbf{F}_{sis} \cdot \mathbf{d} \\ &= |\mathbf{F}|\,|\mathbf{d}| \cos \phi \\ &= (F_{sis})\,(d) \cos 120^o \\ &= (mg/4)(7d_1/8)(-.5) \\ &= -.11\ mgd_1. \end{aligned}$$

vii.) In between *Points 1* and *2*, "extraneous" work is done by friction in the amount of:

$$\begin{aligned} W_{f_k} &= \mathbf{f}_k \cdot \mathbf{d} \\ &= |\mathbf{F}|\,|\mathbf{d}| \cos \phi \\ &= (f_k)\,(d) \cos 180^o \\ &= f_k(7d_1/8)(-1) \\ &= -.875 f_k d_1. \end{aligned}$$

viii.) To solve this, we need f_k. The easiest way to determine f_k is with Newton's Second Law (the *free body diagram* shown in Figure 6.24 is for the body in mid-flight--it looks a bit different from the fbd for the section of flight during which the spring is still engaged, but the horizontal components are identical in both cases). Doing so yields:

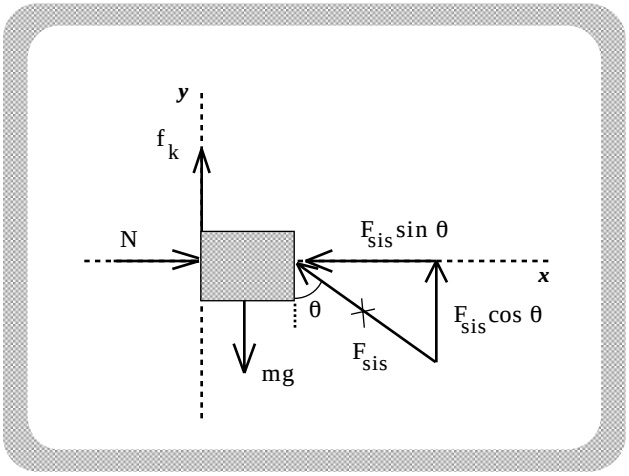


FIGURE 6.24

$\underline{\sum F_x}$:

$$N - F_{sis} \sin\theta = ma_x = 0$$
$$\Rightarrow \quad N = F_{sis} \sin\theta$$
$$= (mg/4) \sin 60$$
$$= .215\ mg.$$

The frictional force is, therefore:

$$f_k = \mu_k N$$
$$= (.4)\ (.215\ mg)$$
$$= .085\ mg.$$

With that, we can determine the work friction does:

$$W_{f_k} = -.875 f_k d_1$$
$$= -.875\ (.085\ mg)\ d_1$$
$$= -.074\ mgd_1.$$

d.) As we did in the beginning, putting it all together yields:

$$KE_1 + \sum U_1 + \sum W_{ext} = KE_2 + \sum U_2$$
$$0 + [\ U_{1,gr} + U_{1,sp}\ ] + [\ W_{sis} + W_{f_k}\ ] = (1/2)mv_2^2 + [U_{2,gr} + 0]$$
$$0 + [mg(d_1 + d_1/8) + (1/2)kx^2\ ] + [\ \mathbf{F}_{sis}\cdot\mathbf{d}_{sis} + (-f_k)(d_{fr})\ ] = (1/2)mv_2^2 + [mg(d_1/4) + 0]$$
$$0 + [\ mg(9d_1/8) + .5(25.6mg/d_1)(d_1/8)^2] + [(mg/4)(7d_1/8)\cos 120^\circ + (-\mu_k N)(7d_1/8)] = (1/2)mv_2^2 + [.25mgd_1 + 0]$$
$$0 + [(1.125mgd_1) + (.2mgd_1)\ ] + [\ (-.11mgd_1) + (-.074mgd_1)\ ] = (1/2)mv_2^2 + [.25mgd_1]$$

$$\Rightarrow \quad v_2 = [1.78\ gd_1]^{1/2}$$
$$= [1.78\ (9.8\ m/s^2)\ (3\ m)]^{1/2}$$
$$= 7.23\ m/s.$$

L. One More Twist–Energy Considerations with Multiple-Body Systems:

1.) The idea behind the *modified conservation of energy* equation is that it is possible to keep track of not only the amount of energy in a system, but also how the energy is distributed throughout the system.

2.) Up until now, all we have examined have been single-body systems. It is possible to extend the *energy considerations* approach to take into account the energy of a whole group of objects.

3.) Executing this expanded version of the *modified conservation of energy* approach:

a.) Calculate the *total kinetic energy* (i.e., the kinetic energy of each body in the system added together) at time t_1.

b.) To that, add the *total potential energy* (i.e., all potential energy of all sorts acting on each body in the system, all added together) at time t_1.

c.) To that, add the *total extra work* done on all the bodies in the system between times t_1 and t_2.

d.) Put the above sum equal to the *total kinetic energy* plus the *total potential energy* in the system at time t_2 .

4.) The modified *modified conservation of energy equation* thus becomes:

$$\sum KE_{1,tot} + \sum U_{1,\,tot} + \sum W_{extra,tot} = \sum KE_{2,tot} + \sum U_{2,tot}.$$

5.) Example: An Atwood Machine is simply a string threaded over a pulley with a mass attached to each end (see Figure 6.25). Assuming the pulley is ideal (i.e., massless and frictionless) and that $m_1 < m_2$, how fast will m_1 be moving if the system begins from rest and freefalls a distance h meters?

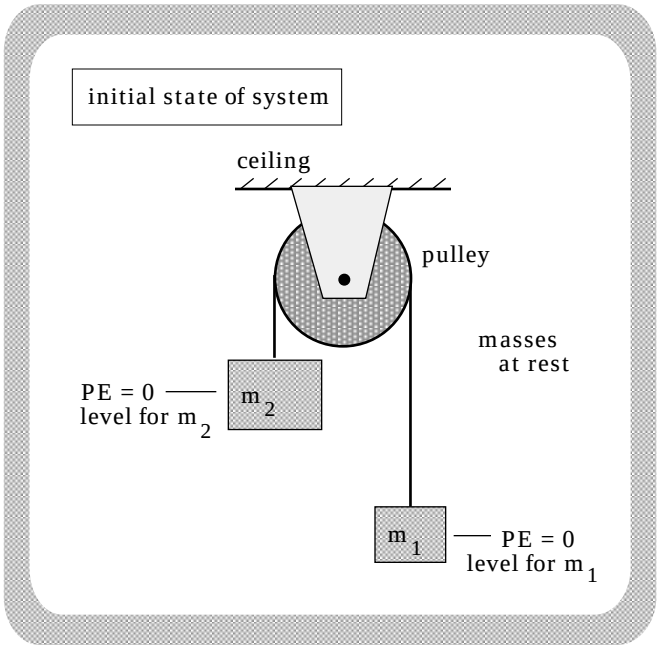


FIGURE 6.25

a.) The system in its initial state is shown in Figure 6.25. Notice that each body is assigned a *zero gravitational-potential-energy level* of its own.

Note: We could have assigned a common level, but it is easier the other way (remember, *where* the zero is for a given body doesn't matter--it is *changes* in *potential energy* that count).

b.) Figure 6.26 shows the system after the freefall. Notice that mass m_2 has moved *below* its zero-potential-energy-level, making the *potential energy* at that point *negative.*

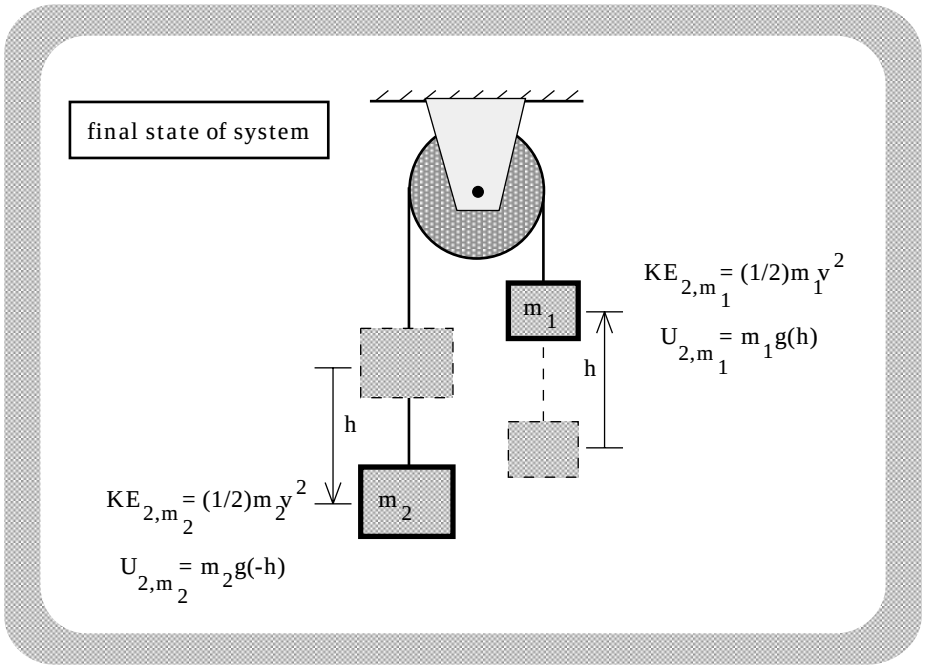


FIGURE 6.26

Note: The amount of *work* tension does on m_1 is *+T(h)*, whereas the amount of *work* tension does on m_2 is *-T(h).* As such, the two *work quantities* associated with the *tension in the line* add to zero.

c.) Putting everything together and executing the *modified conservation of energy* approach, we get:

$$\sum KE_{1,tot} + \sum U_{1,tot} + \sum W_{extra,tot} = \sum KE_{2,tot} + \sum U_{2,tot}.$$

$$[KE_{1,m_1} + KE_{1,m_2}] + [U_{1,m_1} + U_{1,m_2}] + [T(h) + T(-h)] = [KE_{2,m_1} + KE_{2,m_2}] + [U_{2,m_1} + U_{2,m_2}]$$

$$[\,0 + 0\,] + [\,0 + 0\,] + [\,0\,] = [.5m_1v^2 + .5m_2v^2] + [m_1gh + m_2g(-h)]$$

$$\Rightarrow \quad v = [[-m_1gh + m_2gh] / [.5(m_1 + m_2)]]^{1/2}.$$

M.) Power:

1.) There are instances when knowing how much work is done by a force is not enough. As an example, it may seem impressive to know that a particular motor can do 120,000 joules of work, but not if it takes ten years for

it to do so. The amount of *work per unit time* being done is often more important than *how much* work can be done.

2.) The physics-related quantity that measures "work per unit time" is called *power*. It is defined as:

$$P = W/t,$$

where t is the time interval over which the work W is done.

3.) The units for power in the MKS system are $kg \cdot m^2/s^3$. This is the same as a *joules/second,* which in turn is given the special name *watts*. Although the watt is a unit most people associate with electrical devices (the light bulb you are using to read this passage is probably between 60 watts and 150 watts), the quantity is also used in mechanical systems. Automobiles are rated by their *horsepower*. One horsepower is supposedly the amount of *work* a "standard" horse can do *per unit time*. As *formally* defined, one horsepower equals 746 watts.

4.) A special relationship is often derived in physics books that relates the amount of power provided by a force $\boldsymbol{F}$ as it is applied to a body that moves a distance $\boldsymbol{d}$ with constant velocity $\boldsymbol{v}$. Simply presented:

$$\begin{aligned} P_F &= W/t \\ &= (\mathbf{F} \cdot \mathbf{d})/t \\ &= \mathbf{F} \cdot (\mathbf{d}/t) \\ &= \mathbf{F} \cdot \mathbf{v}. \end{aligned}$$

This manipulation has been included for the sake of completeness.

QUESTIONS

6.1) A 3 kilogram mass moving at 2 m/s is pulled 35 meters up a 25^o incline by a force F (see Figure I). If the coefficient of friction between the mass and the incline is .3:

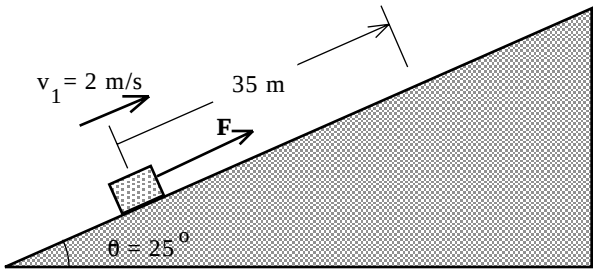


FIGURE I

a.) How much work does *gravity* do as the mass moves up the incline to the 35 meter mark?

b.) How much work does *friction* do as the mass moves up the incline to the 35 meter mark?

c.) How much work does the *normal force* do as the mass moves up the incline to the 35 meter mark?

d.) How much *kinetic energy* does the mass initially have?

e.) WORK/ENERGY PROBLEM: Assuming the mass's *velocity* at the 35 meter mark is 7 m/s, use the *work/energy theorem* to determine the force F. *Do this as you would on a test*. That is, forget for the moment that you have done any work above and lay this problem out completely in algebraic form before putting in numbers.

6.2) A force $\boldsymbol{F}$ is applied to a mass $m = .5\ kg$ as it proceeds up a frictional, hemispherical dome ($\mu_k = .2$) of radius $R = .3\ meters$. The force is ALWAYS at an angle of 12^o, relative to the mass's motion (see Figure II). $\boldsymbol{F}$ also varies so as to keep the mass's very slow velocity constant throughout the motion. Assuming the velocity is additionally very, very small (i.e., so that the centripetal acceleration v^2/R is minuscule), how much work does $\boldsymbol{F}$ do on the mass as it moves from an angle of 20^o to an angle of 60^o up the dome? Put the numbers in last.

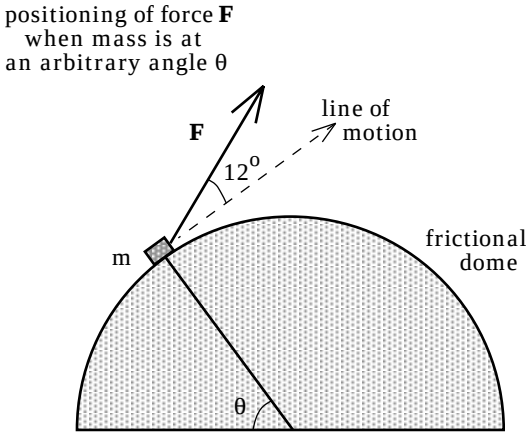


FIGURE II

6.3) How much energy is stored in a spring compressed 20 centimeters (.2 meters) if the spring's *spring constant* is *k=120 nts/m*?

6.4) Derive an expression for the force function associated with the *potential energy* function shown below.

$$U(x,y) = -(k_1/x)e^{-ky}.$$

You may assume the *k* terms have magnitudes of *one* and have the appropriate units.

6.5) Determine the *potential energy function* for the force function:

$$\mathbf{F} = [(k_1 \ln x) - 3]\mathbf{i} - [k_2 y^2]\mathbf{j}.$$

Assume the magnitudes k_1 and k_2 are both *one* and that each has the appropriate units. Also, assume that $x > 1$.

6.6) Tarzan ($m_T = 80\ kg$) stands on a 12 meter high knoll (see Figure III). He grabs a taut, 15 meter long vine attached to a branch located 17 meters above the ground and swings down from rest to Jane perched on a 5 meter high mole hill (they breed particularly big moles in Africa).

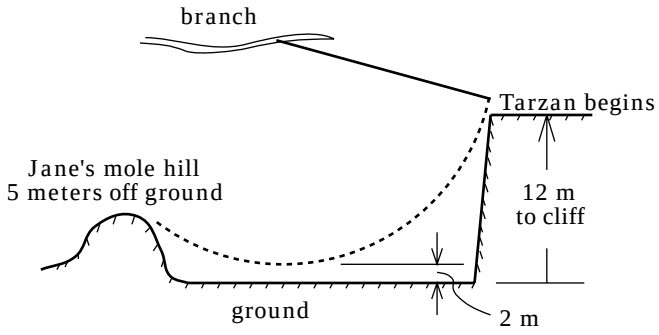


FIGURE III

a.) What is Tarzan's velocity when he reaches Jane?

b.) What is the tension in the vine when Tarzan is at the bottom of the arc? (Note: Tarzan is moving through a CIRCULAR path).

c.) What is the tension in the vine just before Tarzan lets go upon reaching Jane?

6.7) A 12 kilogram crate starts from rest at the top of a curved incline whose radius is 2 meters (see Figure IV). It slides down the incline, then proceeds 18 additional meters before coming to rest. What is the *frictional force* between the crate and the supporting floor (both curved and horizontal)?

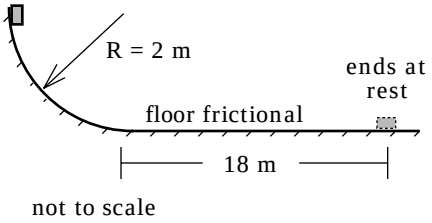


FIGURE IV

Assume this *frictional force* is constant throughout the entire motion.

6.8) Pygmies use blow-guns and 15 gram (.015 kg) darts dipped in the poison curare to immobilize and kill monkeys that live in the tree-top canopy of their forest home. Assume a pygmy at ground level blows a dart at 85^{o} (relative to the horizontal) at a monkey that is 35 vertical meters up (over 100 feet). Assuming a dart must be moving at 4 m/s to effectively pierce monkey skin, what is the *minimum velocity* the dart must be moving as it leaves the blow-gun if it is to pierce the monkey?

6.9) A freewheeling 1800 kilogram roller coaster cart is found to be moving 38 m/s at *Point A* (see Figure V). The actual distance between:

Point A and *Point B* is 70 meters;
Point B and *Point C* is 60 meters;
Point C and *Point D* is 40 meters.

If the average frictional force acting throughout the motion is 27 newtons, the radius of the loop is 20 meters, the first hill's height 25 meters, the first dip 15 meters, and the incline just after the loop coming directly off the loop's bottom at an angle of 30^{o}:

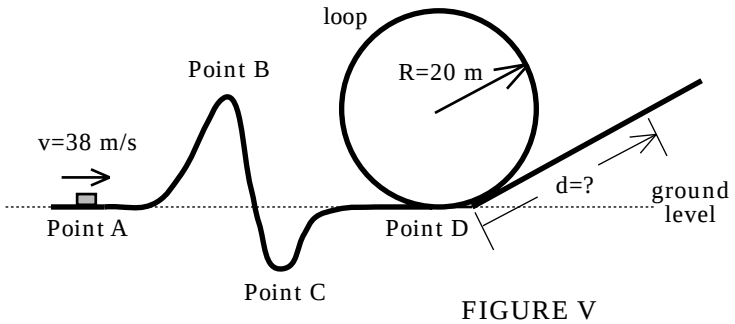


FIGURE V

a.) How fast is the cart moving at *Point C*?

b.) How far up the incline *d* will the cart travel before coming to rest?

c.) What must the cart's *minimum velocity* be at *Point A* if it is to just make it through the top of the loop without falling out of its CIRCULAR MOTION (hint, hint).

Note: The phrase "just making it through the top" means that for all intents and purposes, the *normal force* applied to the cart by the track goes to zero leaving gravity the only force available to affect the cart's motion at the top.

6.10) A spring-loaded bumper is placed on a 55^{o} frictional incline plane. A 60 kilogram crate breaks loose a distance 3 meters up the incline above the bumper and accelerates down the incline (see Figure VI). If the average

frictional force applied to the crate by the incline is 100 newtons and the spring constant is 20,000 newtons/meter:

a.) How much will the bumper spring compress in bringing the crate to rest? (Assume there is friction even after the crate comes in contact with the bumper).

b.) The crate compresses the bumper's spring which then pushes the crate back up the incline (the crate effectively bounces off the bumper). If a total of three-quarters of the crate's *kinetic energy* is lost during the collision, how far back up the incline will the crate go before coming to rest?

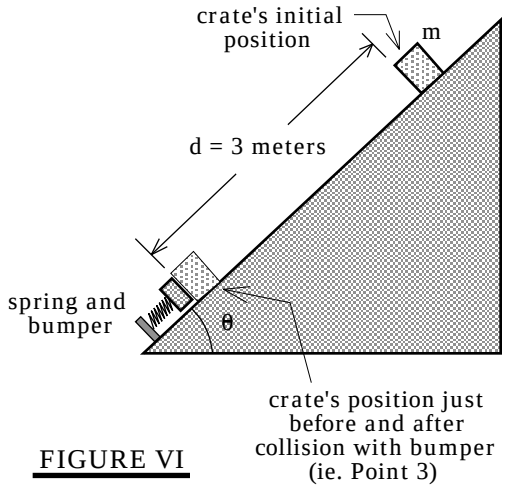

crate's initial position
m
d = 3 meters
spring and bumper
θ
crate's position just before and after collision with bumper (ie. Point 3)

FIGURE VI

6.11) Because *gravitational attraction* between you and the earth becomes less and less as you get higher and higher above the earth's surface, the *gravitational potential energy function* for a body of mass m_1 that is a substantial distance *d units* away from the earth's surface is not *mgh*; it is:

$$U = - Gm_1m_e/(r_e+d),$$

where *G* is called *the universal gravitational constant* (6.67×10^{-11} $m^3/kg \cdot s^2$), m_e is the mass of the earth (5.98×10^{24} kg), and r_e is the radius of the earth (6.37×10^6 m).

A satellite is observed moving at 1500 m/s when 120,000 meters above the earth's surface. It moves in an elliptical path which means its height and velocity are not constants. After a time, the satellite is observed at 90,000 meters. Ignoring frictional effects, how fast is the satellite traveling at this second point?

6.12) A string of length *L* is pinned to the ceiling at one end and has a mass *m* attached to its other end. If the mass is held in the horizontal and released from rest, it freefalls down through an arc of radius *L* until the string collides with a peg located a distance *L/3* from the bottom of the arc (see Figure VIIa). From there it proceeds along an arc of lesser radius (i.e., a radius of *L/3*).

Assuming one-tenth of the energy in the system is lost during this collision, what will the tension T in the string be as the body moves through the *top* of its final arc (see Figure VIIb)?

INITIAL POSITION

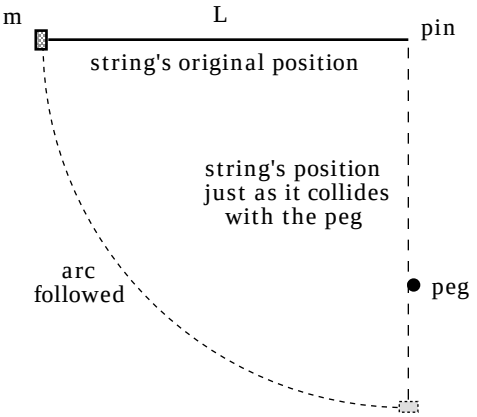


FIGURE VIIa

AT TOP OF SWING

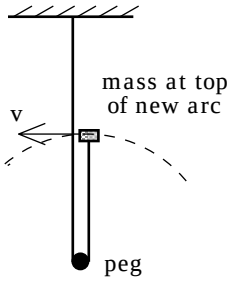


FIGURE VIIb

Chapter 7

M O M E N T U M

A.) Momentum and Impulse:

1.) What parameters are important if you want to determine how relatively large an applied force must be to stop a moving body *in a given amount of time*?

Answer? The body's *resistance to motion-change* (i.e., its inertia . . . or mass) and *how fast* it's moving (i.e., its velocity).

When Newton developed his Second Law, these insights motivated him to multiply mass m and velocity $\boldsymbol{v}$ together to create a new vector. He called it *momentum*, he labeled it $\boldsymbol{p}$ (because he couldn't spell--he thought it was *pmomentum* . . . just kidding), and he defined it as:

$$\mathbf{p} = m\mathbf{v}.$$

He then observed that the *net force* $\boldsymbol{F}_{net}$ applied to a *mass* vectorially equals the *rate at which the momentum changes with time*, or:

$$\begin{aligned} \mathbf{F}_{\text{net}} &= d\mathbf{p}/dt \qquad\qquad \text{(Equation A)} \\ &= d(m\mathbf{v})/dt \\ &= m[d\mathbf{v}/dt] + [dm/dt]\mathbf{v}. \end{aligned}$$

Setting $d\boldsymbol{v}/dt = \boldsymbol{a}$, and assuming the mass does not change in the system (i.e., $dm/dt = 0$), this equation becomes $\boldsymbol{F}_{net} = m\boldsymbol{a}$--the form of *Newton's Second Law* you have come to know and love.

2.) Taking *Equation A* and multiplying both sides by dt yields:

$$\mathbf{F}_{\text{net}}dt = d\mathbf{p}.$$

The quantity $\boldsymbol{F}_{net}dt$ is called the *impulse* and is equal to the differential change of a body's momentum $d\boldsymbol{p}$ during the differential time period dt. Over a *long* period of time, assuming $\boldsymbol{F}_{net}$ is constant, this equation becomes $\boldsymbol{F}_{net}\Delta t = \Delta \boldsymbol{p}$.

3.) Example: A 2 kg sliding puck whose initial velocity-magnitude is $v_1 = 10\ m/s$ strikes a wall at a 30^{o} angle and bounces off (see Figure 7.1a). If it

leaves the wall with velocity-magnitude $v_2 = 10$ *m/s* (that's right, we are assuming no mechanical energy is lost during the collision), and if the collision takes a total of .02 seconds to complete, what is the *average force* applied to the puck by the wall?

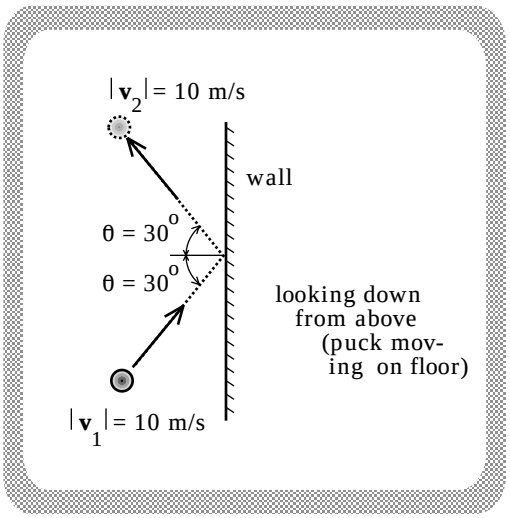


FIGURE 7.1a

Note: This is clearly an *impulse* problem. The puck's *change of momentum* can be determined, and the *time* over which the change occurred is known. All that isn't known is the variable we are interested in--$\boldsymbol{F}_{net}$.

a.) *Momentum* is a VECTOR. Whenever it is used in a problem, you have to treat it like a vector. Figure 7.1b shows the components of the *before-bounce* momentum $\boldsymbol{p}_1$ and the *after-bounce* momentum $\boldsymbol{p}_2$.

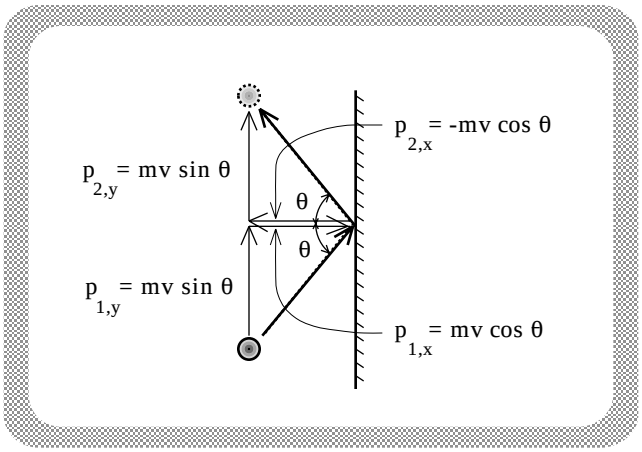


FIGURE 7.1b

b.) In the *x direction*, the incoming momentum $p_{1,x}$ is:

$$p_1 = mv_1 \cos \theta.$$

Remembering that *positive x velocities* are directed toward the *right* while *negative velocities* are directed toward the *left*, the *x component* of the outgoing momentum $p_{2,x}$ is:

$$p_2 = - mv_2 \cos \theta.$$

c.) The velocity-magnitudes v_1 and v_2 are equal in this problem. Calling both v for simplicity, the change of momentum in the *x direction* (hence, the impulse in the *x direction*) is:

$$\Delta(p_x) = p_{2,x} - p_{1,x}$$
$$= (-mv\cos\theta) - (mv\cos\theta)$$
$$= -2\,(mv\cos\theta)$$
$$= -2\,(2\text{ kg})\,(10\text{ m/s})\cos 30^{o}$$
$$= (-34.6\text{ kg m/s})$$

$$\Rightarrow \quad F_{net,x} = \Delta p_x / \Delta t$$
$$= (-34.6\text{ kg m/s}) / (.02\text{ sec})$$
$$= (-1730\text{ nts}).$$

d.) Following a similar process in the *y direction* yields $F_{net,y} = 0$ (try it).

Note: The temptation for many students when first confronted with a problem like this is to reason as follows: "The *initial momentum* is $mv_1 = 20\ kg{\cdot}m/s$. The *final momentum* is $mv_2 = 20\ kg{\cdot}m/s$. The difference between the two must, therefore, be zero. If that is true, the *net force* must be zero!"

This obviously doesn't make much sense--there has to be a force if the ball is to change its direction. The problem is that beginners often treat *momentum* quantities as though they are scalars. Don't be led astray. Momentum must be treated *as a vector*. SIGNS COUNT!

B.) Calculating the *Center of Mass* for a System of Discrete Masses:

1.) Until now, we have analyzed situations in which moving objects have been approximated as point masses (i.e., as structures that neither spin nor have pieces moving internally with different speeds). Even big objects like cars have been treated like point masses (if you know the speed of the car's radio, you know the speed of the car's transmission). The question arises, "What about *systems of objects* in which each part has its own velocity--two billiard balls colliding with one another, or a rocket that blows into a known number of pieces while in flight?"

In such cases, we need an approach that is not tied to the motion of a *specific* body but that deals with the *entire system*. What follows is the theoretical basis for doing just that.

2.) A qualitative description of the *center of mass* of a *single object* is "the *average position* of the mass in the body."

a.) A basketball, for instance, has its *center of mass* at the geometric center of the ball. No physical mass exists at that point--the basketball is, after all, hollow--but the *average* position of the mass making up the

ball is at the ball's geometric center (we are assuming the ball is homogeneous--that it doesn't have a valve or any anomalies).

b.) The center of mass of an object does *not* have to be at the geometric center of the object. Anyone who has ever tried *lawn bowling* knows that when a lawn bowling ball rolls, it does not go straight.

The reason? The *center of mass* of the ball is off-set from the geometric center of the ball.

c.) Examples of a few other *center of mass* situations are shown in Figure 7.2.

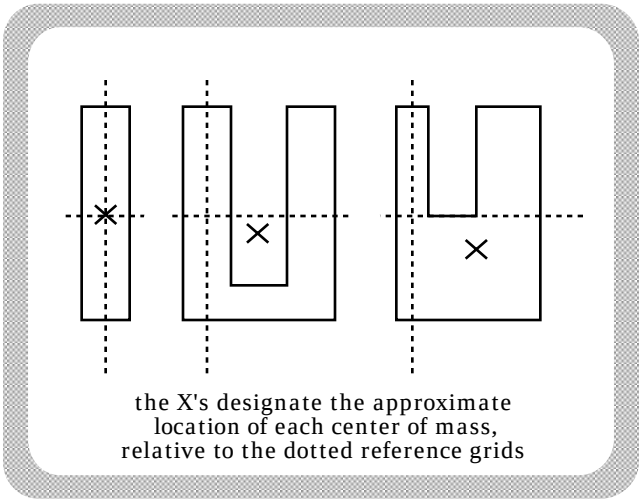


FIGURE 7.2

d.) An additional characteristic of a body's *center of mass*: when an object is thrown, its *center of mass* will follow the parabolic arc normally associated with a free-falling object. Other points on the object will not follow such an arc.

i.) Example: Throw a tomahawk end-over-end. The tip of the handle will follow a convoluted path; the *center of mass* will follow a parabolic arc. Both paths are simulated in Figure 7.3.

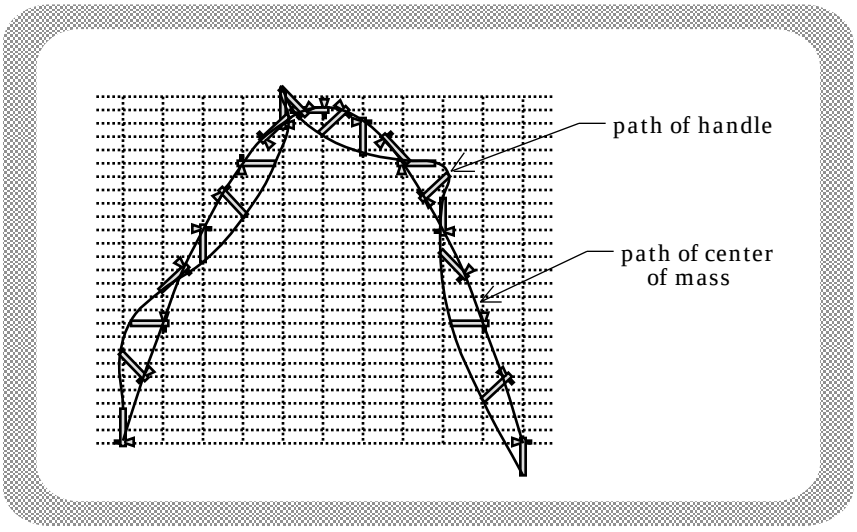


FIGURE 7.3

3.) A *group* of objects also has a *center of mass*. The *x coordinate* of a group's *center of mass* is determined by using the following equation:

$$x_{cm} = \Sigma\ (m_i x_i) / \Sigma\ (m_i)$$
$$= \Sigma\ (m_i x_i) / M,$$

where $\Sigma\ (m_i x_i)$ is the sum of the product of each mass and its *x coordinate* (i.e., $m_1 x_1 + m_2 x_2 + \ldots$ etc.) and $\Sigma\ (m_i)$ is the sum of all the mass in the system (i.e., the total mass *M*).

Similar equations define the *y* and *z coordinates* of a body's *center of mass*, relative to the coordinate axis used to define the body's position.

4.) The best way to see how this works is to do an example:

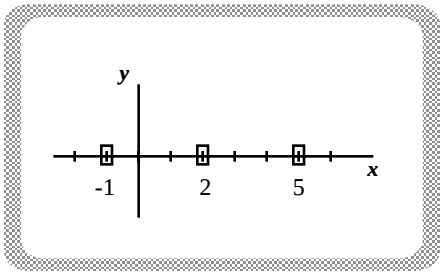


FIGURE 7.4a

a.) Consider the equal masses shown in Figure 7.4a. From the symmetry of the situation, it should be obvious that the *x coordinate* of the system's center of mass is at *x = 2 meters*. Checking this with our math, we get:

$$x_{cm} = \Sigma\ (m_i x_i) / \Sigma\ (m_i)$$
$$= (m_1 x_1 + m_2 x_2 + m_3 x_3)/(m_1 + m_2 + m_3)$$
$$= [(-1)(m) + (2)(m) + (5)(m)]/3m$$
$$= 2 \text{ meters.}$$

b.) If the bodies had differing *y coordinates*, a similar operation would have been used for the *y direction* and our final center of mass would have been some ordered pair (x_{cm}, y_{cm}).

C.) Calculating the *Center of Mass* for a Continuous Mass:

1.) The *center of mass* is defined by a set of coordinates that identifies the location of the *average position of all the mass* in a structure.

2.) For a discrete set of masses, the *x coordinate* of the *center of mass* is $x_{cm} = \Sigma\ (m_i x_i)/M$, where *M* is the total mass in the system.

3.) How do we deal with a situation in which the mass is not in lumps but, rather, is continuously distributed throughout the structure? To determine the *x coordinate* of the *center of mass* for such a case, we must identify all of the

differential mass (*dm*) having the same *x coordinate*, multiply that *x* by *dm*, sum over all possible *x's* through integration, then divide by *M*. In other words, when dealing with an extended object:

$$x_{cm} = (1/M)\int x\,dm.$$

A similar expression holds for the *y* center of mass coordinate.

4.) An example will help. Determine the *x coordinate* of the center of mass for a rectangular solid of width *a*, height *b*, thickness *t*, and whose mass is *M* (see Figure 7.4b).

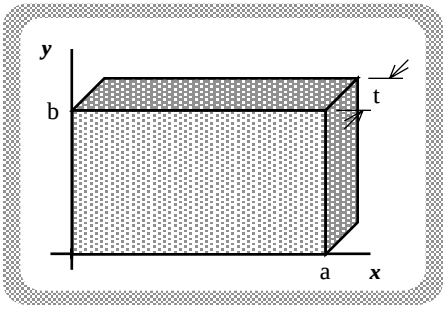


FIGURE 7.4b

a.) Consider Figure 7.4c. In that sketch, a slab of differential width *dx* and differential mass *dm* is drawn a distance *x* units from the *y axis* (note that *dm* is all the mass *x* units out!).

b.) To determine the *x* center of mass, we must multiply *x* by *dm*, then integrate. The problem? We need some way of relating *dm* to its *x position*.

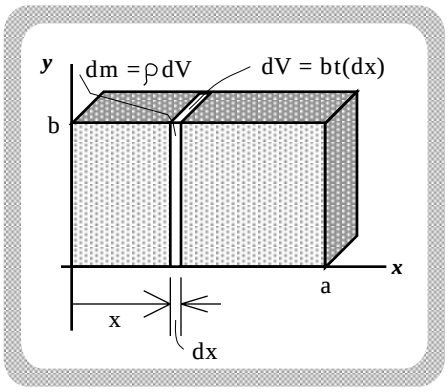


FIGURE 7.4c

c.) The trick here is to define a *volume-density function* ρ. Because the mass of the structure is homogenous (i.e., uniformly distributed throughout the structure), the definition of ρ allows us to write:

$$\rho = (\text{mass})/(\text{volume}).$$

This density function can be expressed in two ways:

i.) Looking at the entire structure macroscopically (i.e., as a whole), the *volume density* can be written as:

$$\rho = M/abt,$$

where *M* is the solid's total mass and *abt* is its total volume.

ii.) Looking microscopically, the *volume density* can be written as:

$$\rho = dm/dV,$$

where *dm* is the differential mass wrapped up in the differential slab and *dV* is the differential volume of the slab.

iii.) The differential volume is:

$$\begin{aligned} dV &= (\text{height})(\text{thickness})(\text{differential width}) \\ &= (b)\quad(t)\quad(dx). \end{aligned}$$

This means that:

$$\begin{aligned} \rho &= (dm)/(dV) \\ &= (dm)/[bt(dx)]\ . \end{aligned}$$

iv.) Taking this last expression and solving for *dm*, we get:

$$\begin{aligned} dm &= \rho\, dV \qquad\qquad \text{(Equation A)} \\ &= \rho\,[bt(dx)]. \end{aligned}$$

This is the expression we will use to replace *dm* in the integral.

Note: The relationship presented in *Equation A* is ALWAYS true.

d.) We are now in a position to do the problem:

$$\begin{aligned} x_{cm} &= \left[\frac{1}{M}\right]\int (x)\,dm \\ &= \left[\frac{1}{M}\right]\int_{x=0}^{a} (x)\left[(\rho bt)\,dx\right] \\ &= \frac{\rho bt}{M}\int_{x=0}^{a} (x)\,dx. \end{aligned}$$

Substituting in $\rho = M/abt$, doing the integral, then canceling, we get:

$$\begin{aligned} x_{cm} &= \frac{\left[\frac{M}{abt}\right]bt}{M}\left[\frac{x^2}{2}\right]_{x=0}^{a} \\ &= \frac{1}{a}\left[\frac{a^2}{2}-\frac{0^2}{2}\right] \\ &= \frac{a}{2}. \end{aligned}$$

e.) Huzzah! This is exactly what we would have expected for the *x coordinate* of the *center of mass* of a rectangular solid.

Note: As the thickness variable *t* canceled out in the problem, we did not necessarily need to use a *volume* density function. Noting that the differential volume *dV* of our strip was equal to the differential surface-area *dA* (this equaled *bdx*) times the body's thickness *t* (i.e., $dV = b(dx)t$), we could have ignored the thickness term and instead defined an *area* density function σ (i.e., the *mass per unit area*). Using this approach:

$$dm = \sigma dA,$$

where *dA* is the differential face-area associated with the mass *dm*. This approach will be used in the next example.

5.) Next example: Determine the *y coordinate* of the center of mass of a solid, flat, thin, semi-circular plate of mass *M* and radius *R*.

a.) How much mass is *y* units up? Figure 7.4d shows a strip of differential mass *dm* with differential thickness *dy* drawn a distance *y* units from the *x axis*.

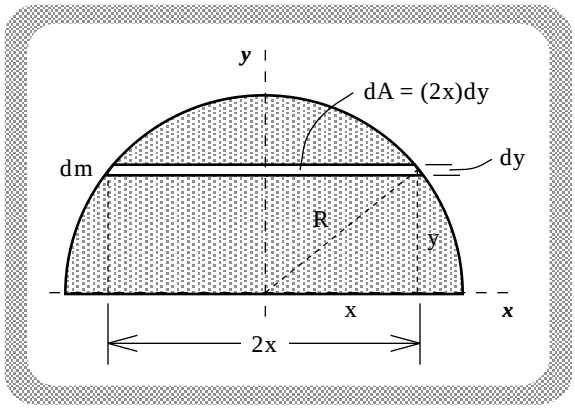


FIGURE 7.4d

b.) We can define σ in two ways: macroscopically and microscopically.

i.) Macroscopically:

$$\sigma = M/[(\pi R^2/2)]$$
$$= 2M/(\pi R^2),$$

where $\pi R^2/2$ is the area of the half-disk and *M* is its mass.

ii.) Microscopically:

$$\sigma = dm/dA,$$

where *dm* is the differential mass associated with the differential surface-area *dA*. This implies that:

$$dm = \sigma dA.$$

iii.) Using the variables provided in the sketch, we can write:

$$\begin{aligned} dA &= (\text{width})(\text{height}) \\ &= (2x)\quad dy. \end{aligned}$$

We need to relate x to y. To do so, notice that $x^2 + y^2 = R^2$. Manipulating, we can write:

$$dA = [2(R^2 - y^2)^{1/2}dy]$$

and

$$\begin{aligned} dm &= \sigma dA \\ &= \sigma[2(R^2 - y^2)^{1/2}dy]. \end{aligned}$$

c.) We are now ready to integrate:

$$\begin{aligned} y_{cm} &= \frac{1}{M}\int y\,dm \\ &= \frac{1}{M}\int_{y=0}^{R} y\left[\sigma\left[2\left(R^2 - y^2\right)^{1/2}\right]\right]dy \\ &= \frac{2\sigma}{M}\int_{y=0}^{R} y\left(R^2 - y^2\right)^{1/2}dy \\ &= -\frac{2\sigma}{3M}\left[\left(R^2 - y^2\right)^{3/2}\right]_{y=0}^{y=R} \\ &= -\frac{2\sigma}{3M}\left[\left(R^2 - R^2\right)^{3/2} - \left(R^2 - 0^2\right)^{3/2}\right] \\ &= \frac{2\sigma}{3M}R^3. \end{aligned}$$

Plugging in $\sigma = 2M/(\pi R^2)$, we get:

$$\begin{aligned} y_{cm} &= \frac{2\left[\frac{2M}{\pi R^2}\right]R^3}{3M} \\ &= \frac{4}{3\pi}R \\ &= .424R. \end{aligned}$$

D.) Systems of Bodies and Their Collective Motion:

1.) Although you won't be required to use the following material to solve problems, there are some interesting consequences that come from the definition of the *center of mass*. Specifically, if discrete bodies in a system are moving, the *center of mass* will also move. For a one-dimensional situation:

$$x_{cm} = \sum (m_i x_i) / M$$
$$\Rightarrow \quad M x_{cm} = \sum (m_i x_i)$$
$$\Rightarrow \quad M (\Delta x_{cm}/\Delta t) = \sum [m_i (\Delta x_i/\Delta t)]$$
$$\Rightarrow \quad M (v_{cm}) = \sum [m_i (v_i)]$$
$$\Rightarrow \quad M v_{cm} = m_1 v_1 + m_2 v_2 + m_3 v_3 + \ldots$$

In other words, the *sum of the momenta* of all the pieces within the system will be equal to the *momentum* of a single particle whose mass *M* is the *total mass* of the system and whose *velocity* is the velocity of the system's *center of mass*. Similar expressions can be determined for *y* and *z momenta*.

2.) Taking this one step further, if the *velocities* of each of the various pieces of a system change due to forces acting on them, the *rate of change of the velocities* will be:

$$M (\Delta v_{cm}/\Delta t) = \sum [m_i (\Delta v_i/\Delta t)]$$
$$\Rightarrow \quad M (a_{cm}) = \sum [m_i (a_i)]$$
$$\Rightarrow \quad M a_{cm} = m_1 a_1 + m_2 a_2 + m_3 a_3 + \ldots$$

The *net force* F_1 on particle #1 (i.e., $m_1 a_1$) added to the *net force* F_2 on particle #2, etc., will vectorially equal the total mass *M* of the system times the *acceleration of the system's center of mass*.

3.) The conclusion and importance of this section is simple: the motion of any rigid, "extended" object (a car, a cannonball, whatever) can be legitimately analyzed by treating it as though *all of its mass* were lumped together and positioned at the body's *center of mass*. This is the theoretical basis from which we have been allowed to treat, say, a car as though it were a point mass.

E.) The Modified *Conservation of Momentum* Equation:

Note: With the exception of the "bottom line," you will *not* be held responsible for duplicating any of the material you are about to read. BUT, if you don't understand what has been done in getting to the "bottom line," the end result won't mean much and your ability to use it won't be worth a tinker's dam.

My suggestion: read the next section, not for memorization but for content. Follow each step as it comes *without* projecting ahead. When you finally get to the end result, take the time to reread the section to be sure you know the general trend and principles involved in the derivation.

1.) Begin by considering a collision between two free (unconstrained) masses. Mass m_1 moves in the *x direction* on a frictionless table. It strikes a second mass m_2 that is moving in the *-x direction* and that additionally has a rocket on its back that is constantly applying a force F_{ext} to it. Figures 7.5a and 7.5b directly below show the *before and after* set-up.

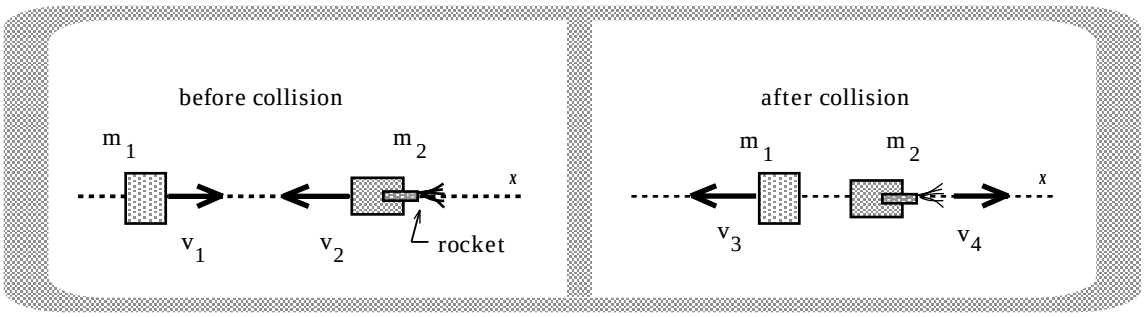


FIGURE 7.5a FIGURE 7.5b

All the motion is in the *x direction* so we need not worry about subscripting the *momentum magnitudes* with an *x* (we don't have to worry about confusing the momentum in the *x direction* for that in the *y direction*--there *is* none in the *y direction*). With that simplification, we can write the momenta of masses m_1 and m_2 <u>*just before*</u> and <u>*just after*</u> the collision as:

<u>before</u>:

$$\mathbf{p}_{1,\text{before}} = p_{1,\text{bef}}\,\mathbf{i}$$
$$\mathbf{p}_{2,\text{before}} = p_{2,\text{bef}}\,\mathbf{i}$$

<u>after</u>:

$$\mathbf{p}_{1,\text{after}} = p_{1,\text{aft}}\,\mathbf{i}$$
$$\mathbf{p}_{2,\text{after}} = P_{2,\text{aft}}\,\mathbf{i}.$$

where $p_{1,bef}$ stands for "the magnitude of the *momentum* of mass m_1 *before* the collision," etc.

2.) Consider just the motion of m_1 through the collision:

a.) The <u>*net force*</u> acting on m_1 in the *x direction* comes solely from the strike it takes from m_2 as the two collide (an f.b.d. on m_1 is shown in Figure 7.6). As the magnitude of the force varies (it starts out very small at the beginning of the hit, gets larger as the collision proceeds, then diminishes at the end), we will assume an *average* collision force F_i and go from there.

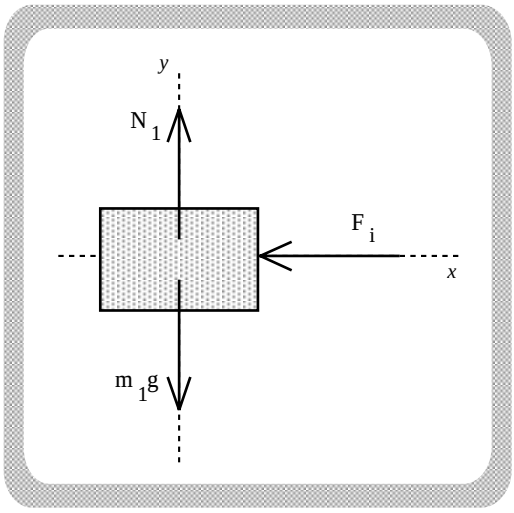


FIGURE 7.6

Note: Because the force F_i is generated from the interaction of m_1 with "other members of the system" (in this case, the *only* other member in the system--m_2), this kind of force is called an "*internal force*." To highlight this, the subscript "i" has been used to denote *internal*.

b.) Assuming the collision takes time Δt to be complete, the <u>*impulse equation*</u> ($F\Delta t = \Delta p$) written for m_1 during the time Δt yields:

$$-F_i \Delta t = (p_{1,aft} - p_{1,bef}) \qquad \text{(Equation A)}.$$

Note: The force on m_1 due to its collision with m_2 is in the *negative* direction, hence the unembedded *negative* sign in front of F_i in the impulse equation. (Notice I haven't unembedded the momenta signs.)

3.) Consider just the motion of m_2 through the collision:

a.) The <u>*net force*</u> acting on m_2 in the *x direction* comes from two sources (see the f.b.d. shown in Figure 7.7):

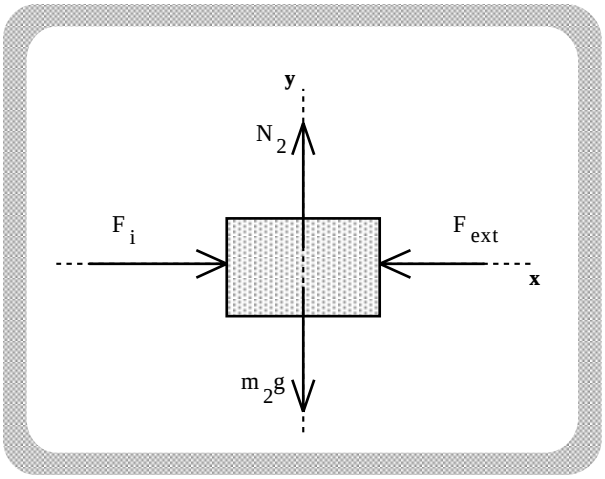


FIGURE 7.7

i.) The force $\boldsymbol{F}_{ext,2}$ generated by the jet attached to m_2 (the force is subscripted "external" because it is provided by a source that is not *internal* to the two-body system); and

ii.) The internal force magnitude F_i generated by m_2*'s* collision with m_1.

Note: The *internal force* on m_2 is the Newton's Third Law counterpart to the *internal force* acting on m_1 (for every force there is an *equal* and *opposite* reaction force, etc.). That means we can use the same symbol for that force as was used in Equation A, with the exception that the unembedded sign must be *positive*.

b.) As was the case with mass m_1, the collision takes time Δt to complete. The impulse equation $F\Delta t = \Delta p$ written for m_2 during the time Δt yields:

$$F_i \Delta t - F_{ext,2} \Delta t = (p_{2,aft} - p_{2,bef}) \qquad \text{(Equation B).}$$

4.) We would now like to vectorially add up all the *impulse* quantities acting on all the bodies within the system to get the system's *net impulse*. In this case, that is tantamount to adding the left-hand side of *Equation A* to the left-hand side of *Equation B*, etc.

a.) Doing the addition yields:

$$(- F_i \Delta t) + (F_i \Delta t - F_{ext,2} \Delta t) = (p_{1,aft} - p_{1,bef}) + (p_{2,aft} - p_{2,bef}).$$

b.) Noting that all the impulses provided by internal forces add to zero, the final equation can be rearranged to look like:

$$- F_{ext,2} \Delta t = (p_{1,aft} - p_{1,bef}) + (p_{2,aft} - p_{2,bef}).$$

c.) Putting all the *before* momenta on the left-hand side of the equation while leaving all the *after* momenta on the right-hand side, we get:

$$p_{1,bef} + p_{2,bef} - F_{ext,2} \Delta t = p_{1,aft} + p_{2,aft}.$$

d.) Written in a more general way:

$$\Sigma\ p_{bef} + \Sigma\ (F_{ext}\,\Delta t) = \Sigma\ p_{aft}.$$

e.) The general equation shown above is called the *MODIFIED CONSERVATION OF MOMENTUM* equation. It maintains that the *sum of the momentum* of all the parts of a system *before* a collision will equal the *sum of the momentum* of all the parts *after* a collision UNLESS there are *impulses* ($\Delta F_{ext}\,\Delta t$) due to *external forces* acting on the system during the time interval Δt.

Note 1: Everything done above has been for motion in the *x direction*. A similar equation holds true for both *y motion* and *z motion*. In other words, the most general form of this equation is:

$$\Sigma\ \mathbf{p}_{bef} + \Sigma\ (\mathbf{F}_{ext}\ \Delta t) = \Sigma\ \mathbf{p}_{aft}.$$

which, in turn, can be expressed as:

$$\Sigma\ p_{bef,x} + \Sigma\ (F_{ext,x}\,\Delta t) = \Sigma\ p_{aft,x},$$
$$\Sigma\ p_{bef,y} + \Sigma\ (F_{ext,y}\,\Delta t) = \Sigma\ p_{aft,y},$$
$$\Sigma\ p_{bef,z} + \Sigma\ (F_{ext,z}\,\Delta t) = \Sigma\ p_{aft,z}.$$

Note 2: Even though these equations were derived from a special situation--a collision--they are true in general. The *net momentum* of a system <u>in a particular direction</u> will remain the same throughout time--it will be CONSERVED--if there are no *external forces* acting in that direction.

That is not to say there cannot be change *within* the system. In fact, there may be *changes in the momentum* of *individual parts* of the system due to interaction-forces between those members, but the *net momentum* (as a vector) of the system will *not change with time* if there are no external forces acting on the system in the appointed direction.

5.) The <u>BOTTOM LINE</u> and a few conclusions:

a.) If there are no *external forces* acting in a particular direction, $F_{ext}\,\Delta t = 0$ and $\Sigma p_{bef} = \Sigma p_{aft}$ in that direction.

b.) Even if there are relatively small external impulses (the impulse gravity exerts, for instance), collisions and explosions characteristically

take so little time that Δt is minuscule and $F\Delta t$ is tiny and the *total momentum* of the system in a particular direction *just before the collision* will equal the *total momentum* of the system in that direction *just after the collision*. In such cases, momentum is said to be conserved *through the collision* (even though this is technically a misuse of the word *conserved*).

Note that THIS IS TRUE ONLY if all pieces of the system are allowed to respond to collisions freely. If you take a ball and a wall as *the system*, the ball hitting the wall will not see momentum conserved because the wall is constrained--it is not free to respond to the impulse applied to it (on the other hand, if the system is just the ball, it will not see momentum conserved either--the wall will exert a very large external force to change the ball's momentum over a very small period of time).

Bottom line: Assuming external forces are small in a *free mass* system, you *can* assume that the *net momentum* of a system is conserved *through the collision.*

c.) Because the *x, y,* and *z directions* are independent, it is possible for momenta to be conserved in one direction and not in another. An example: a projectile in air. The momentum in the *y direction* will change over time because there is an *external force* acting in that direction (the force of gravity). If we ignore air friction, there are *no external forces* acting in the *x direction* so momentum in that direction is conserved.

F.) Collision Examples:

1.) A one dimensional situation: A puck of mass m_1 moving with velocity magnitude v_1 strikes a second puck of mass m_2 moving in the opposite direction with velocity $2v_1$ head on (see Figure 7.8).

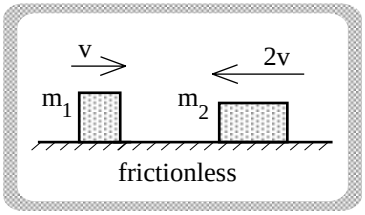


FIGURE 7.8

a.) What determines the direction each mass goes after the collision?

b.) What must the puck's mass ratio m_1/m_2 be if the two pucks are to leave the collision in opposite directions *with the same velocity magnitude* v?

c.) And, if a hard shelled bug happens to get squashed between the blocks during the collision, how will this change the final velocities relative to what would have happened if no bug were present during the collision (obscure? Yes . . . but informative . . .).

2.) And the answers are . . .

a.) The first query is more of a conceptual question than anything else. It asks, *what determines what will happen after the collision*? The answer to that depends upon the size of m_1 and m_2.

i.) If m_1 is a lot larger than m_2, its momentum will carry through the collision and the two will both proceed to the right with different velocities. In fact, if m_1 is a whole lot larger than m_2, its velocity will change very little. A determination of m_2*'s* velocity in that case is a little tricky. That kind of calculation will be treated at the end of the chapter.

ii.) If m_2 is a lot larger than m_1, its momentum will carry through the collision and the two will both proceed to the left with different velocities. Again, if m_2 is a whole lot larger than m_1, it's velocity will change very little as it proceeds toward the left.

iii.) If the two masses are comparable in size, they will reverse themselves at collision and leave with velocities *that conserved the total momentum* of the two-body system.

b.) This second query assumes that the collision occurs and the two masses respond by moving off in opposite directions but with the same velocity magnitude v. To determine the mass ratio that will motivate this to happen, we need equations.

i.) As both masses are free to respond to the impulse supplied by the other during the collision, and as there are no enormous external forces acting in the *x direction* during the collision, momentum will be conserved through the collision. Unembedding negative signs and writing out the *conservation of momentum* expression, we get:

$$\Sigma p_{before} = \Sigma p_{after}$$

$$m_1v_1 - m_2(2v_1) = -m_1v + m_2v \qquad \text{(Equation A).}$$

ii.) This gives us one relationship with four unknowns: m_1, m_2, v_1, and v. We clearly need more equations. At this juncture, all you can do is hope that some information is given about the *energy content* of the system.

iii.) Although it is highly unlikely that such a collision will ever occur, let's assume that the collision is elastic (mechanical energy conserved) and see where that leads. Observing that there are no potential energy changes anywhere in the system, we can write the *conservation of energy* expression as:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2(2v_1)^2 = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 \qquad \text{(Equation B).}$$

iv.) After canceling out the *1/2* terms in the energy expression and expanding, it is tempting to rewrite Equations A and B as:

$$(m_1 - 2m_2)v_1 = (-m_1 + m_2)v \qquad \text{(Equation A modified)}$$

and $\qquad (m_1 + 4m_2)v_1^2 = (m_1 + m_2)v^2 \qquad$ (Equation B modified),

then try to solve by writing v_1 in terms of v from the momentum equation and substituting that back into the energy equation. The problem is that this leaves us with the very unpleasant expression:

$$(m_1 + 4m_2)\left(\frac{-m_1 + m_2}{m_1 - 2m_2}\right)^2 v^2 = (m_1 + m_2)v^2.$$

v.) Although this expression will allow us to solve for the mass ratio, it is kinda ungodly. Even more to the point, if we had been interested in solving for, say, unequal final velocities (should that have been the case), using that approach would have been even uglier. Fortunately, there is a clever way to shuffle the variables so as to create a third independent equation that is considerably easier to

solve in conjunction with the momentum expression than was the original pair. That technique follows.

vi.) ESPECIALLY TRICKY APPROACH (ETA): For both conservation equations, group the m_1 terms on one side of the *equal sign* and the m_2 terms on the other side of the equal sign so that:

$$m_1v_1 - m_2(2v_1) = -m_1v + m_2v \qquad \text{(Equation A)}$$

becomes

$$\mathbf{m_1(v_1 + v) = m_2(v + 2v_1)} \qquad \text{(Equation C)}$$

and

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2(2v_1)^2 = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 \qquad \text{(Equation B)}$$

becomes

$$\mathbf{(1/2)m_1(v_1^2 - v^2) = (1/2)m_2(v^2 - 4v_1^2)} \quad \text{(Equation D).}$$

vii.) The *1/2's* cancel. Notice that $(v_1^2 - v^2)$ in the energy equation can be factored and equals $(v_1 + v)(v_1 - v)$, and that we can factor both sides of Equation D. We can also *divide* the left side of the factored version of Equation D by the left side of Equation C, and the right side by the right side, yielding:

$$\frac{m_1(v_1 + v)(v_1 - v)}{m_1(v_1 + v)} = \frac{m_2(v + 2v_1)(v - 2v_1)}{m_2(v + 2v_1)}.$$

Both mass terms and one of the factors on each side will cancel leaving:

$$v_1 - v = v - 2v_1,$$
$$\Rightarrow \quad v = 3v_1/2.$$

Putting this into the original momentum equation (Equ. A) yields:

$$m_1v_1 - m_2(2v_1) = -m_1(3v_1/2) + m_2(3v_1/2)$$
$$\Rightarrow \quad m_1/m_2 = 7/5.$$

viii.) In other words, this technique requires you to group velocities associated with each mass, factor the energy equation, divide one equation into the other, then use the resulting equation in conjunction with the *conservation of momentum* expression to solve for the required unknowns. It sounds complicated, but it's a lot easier than trying to deal with the squares in the original *conservation of energy* equation.

c.) So what's the deal with the squashed bug?

i.) To begin with, a problem similar to this was originally presented as a conceptual question on a quiz given to the freshman physics class at Caltech in 1999. I've included it because students who are learning physics for the first time don't generally have the time, experience, or in many instances, inclination, to think creatively and deeply about exotic problems. This problem could be done mathematically by power-grunting through it, or it could be taken apart conceptually by simply using your head. What you are about to read approaches the problem from that latter perspective. That is, the analysis you are about to read will not highlight an approach that will always work for you whenever you run into something exotic. What you are about to read is more *seat of the pants* thinking (versus *algorithmic* thinking). In short, you aren't going to be tested on this stuff, but do read it. If I put you on a desert island, gave you this problem, told you I'd be back in 24 hours with a machine gun . . . and that you'd better have the correct answer for me, I can guarantee you would sooner or later get past your belief that you couldn't possibly figure out what was going on, would get into the problem, and would begin to follow lines of thought that would be similar to the ones you are about to read. For you, for now, I'm doing the think-work gratis. Enjoy the luxury.

ii.) The first thing to notice is that work has to be done to squash the bug, so mechanical energy is no longer conserved (i.e., we are now dealing with an inelastic collision). That being the case, what do the velocities do? Well, at least one velocity must go down so that the total final mechanical energy will be less than it was for the conserved situation. But if one must go down, the other must go down also. Why? Because momentum must still be conserved.

Think about it. Let's assume that the total momentum of the system before the energy-conserved collision occurs is *2 kg·m/s*, and let's say m_1*'s* after-collision momentum in that case is $p_1 = 6$ *kg·m/s* and m_2*'s* is $p_2 = -4$ *kg·m/s* (note that the total after-collision momentum is still *2*

kg·m/s . . . $p_1 + p_2$ = (6 kg·m/s) + (-4 kg·m/s) = 2 kg·m/s). Now let's assume that the collision is not conserved so that at least one of the after-collision velocities, say that of m_1, decreases. That means the magnitude of its after-collision momentum will also decrease to, say, *5.8 kg·m/s*. The only way the sum of the momentum after the collision will equal the total momentum before the collision (remember, it was *2 kg·m/s*) is if m_2*'s* momentum magnitude also drops (in this case, to *-3.8 kg·m/s . . .* now $p_{1,new} + p_{2,new}$ = *5.8 kg·m/s + (-3.8) kg·m/s = 2 kg·m/s*). In other words, if one momentum magnitude decreases, so must the other. That means that both velocities must decrease.

iii.) But by how much? Though you probably wouldn't think to do so on your own, at least not without a fair amount of cogitating, the easiest way to determine this is to assign new final velocities (v_2 for m_2 and v_3 for m_1) and play with the momentum expression. Noting that the initial momentum is a constant (we'll call it *c*), and remembering that the mass ratio implies that $m_1 = (7/5)m_2$, we can write the *conservation of momentum* as:

$$\text{initial total momentum} = -\, m_1 v_3 + m_2 v_2$$
$$\Rightarrow \quad c = -\,(7m_2/5)\, v_3 + m_2 v_2.$$

Dividing by v_3 yields:

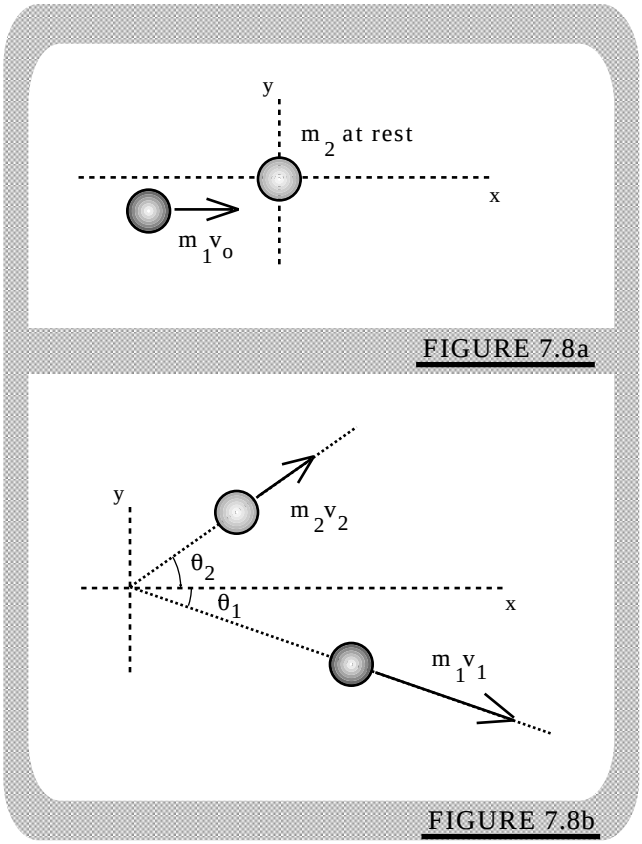
$$c/v_3 = -\,(7m_2/5) \; + m_2(v_2/v_3).$$

iv.) Look at this expression and think. If v_3 decreases (relative to its earlier value), as we've decided it must, then the left side of the equation increases. The first term on the right is a constant, so that means that the second term on the right must go UP if the equal sign is to continue to hold. To do that, the denominator of that piece must be smaller than the numerator, which means that v_2 (the velocity associated with m_2) must be larger than v_3. In other words, both velocities will change, and the velocity associated with m_2 will change the least (if it's bigger than v_3, it's closer to the original v which means it hasn't changed as much as has v_3).

UNDERSTAND, this is all a consequence of the need for the system's momentum to be conserved coupled with the relationship the

velocities have to satisfy with respect to energy. For a given situation, there is only one set of velocities that will do the trick. That's just the way nature works.

3.) Two dimensional situation: Looking down on a frictionless table, a puck of mass $m_1 = 4\ kg$ moves in the *x direction* with a velocity-magnitude of $v_o = 12\ m/s$. It strikes a second mass $m_2 = 6\ kg$ initially at rest (see Figure 7.8a). After the collision, the two move off at angles 25^o and 40^o respectively (see Figure 7.8b). What are the magnitudes of the after-collision velocities v_1 and v_2?

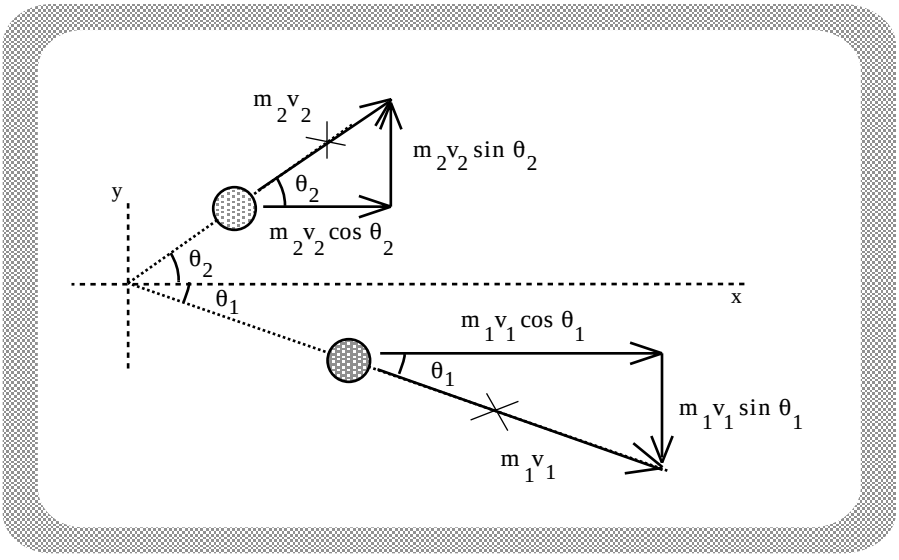


Note: This problem will be solved in *Parts a* through *e*, complete with generalized algebra for ease of reading and explanation. In *Part f*, the problem is done again as it would be done on a test--that is, with math but *without* mountains of explanatory verbiage.

a.) The only forces acting in either the *x* or *y directions* during the collision are *internal forces*: the force m_2 applies to m_1 and the force m_1 applies to m_2. As such, momentum is conserved in both the *x* and *y directions* through the collision.

b.) The momenta are broken into their component parts in Figure 7.8c.

c.) Using the *conservation of momentum*



equation for motion in the *x direction* yields:

$$\underline{\sum p_{bef,x} = \sum p_{aft,x}:}$$

$$\Rightarrow \quad m_1 v_o + m_2(0) = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2$$

$$\Rightarrow \quad v_1 = (m_1 v_o - m_2 v_2 \cos\theta_2)/(m_1 \cos\theta_1) \qquad \text{(Equ. A).}$$

d.) For the motion in the *y direction*, we get:

$$\sum p_{bef,y} = \sum p_{aft,y}:$$

$$\Rightarrow \quad m_1(0) + m_2(0) = -m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 \qquad \text{(Equ. B).}$$

Note: I've unembedded the sign of the after-collision velocity of m_1, leaving the v_1 term a *magnitude*.

e.) Plugging *Equation A* into *Equation B* yields:

$$0 = [-m_1 [(m_1 v_o - m_2 v_2 \cos\theta_2)/m_1 \cos\theta_1] \sin\theta_1] + m_2 v_2 \sin\theta_2$$

$$\Rightarrow \quad v_2 = [m_1 v_o \sin\theta_1/\cos\theta_1]/[m_2 \sin\theta_2 + (m_2 \cos\theta_2/\cos\theta_1)\sin\theta_1].$$

With some manipulation, this becomes:

$$v_2 = [(m_1 v_o / m_2)] (\tan\theta_1)/(\sin\theta_2 + \cos\theta_2 \tan\theta_1)$$
$$= [(4 \text{ kg})(12 \text{ m/s})/(6 \text{ kg})] (\tan 25^o)/(\sin 40^o + \cos 40^o \tan 25^o)$$
$$= [(4 \text{ kg})(12 \text{ m/s})/(6 \text{ kg})] (.47)/(.64 + .36)$$
$$= 3.76 \text{ m/s}.$$

Plugging back into *Equation A*:

$$v_1 = (m_1 v_o - m_2 v_2 \cos\theta_2)/(m_1 \cos\theta_1)$$
$$= [(4 \text{ kg})(12 \text{ m/s}) - (6 \text{ kg})(3.76 \text{ m/s})(\cos 40^o)]/(4 \text{ kg})(\cos 25^o)$$
$$= 8.47 \text{ m/s}.$$

f.) The problem solved *without verbiage*:

Write:

$\sum p_{bef,x} = \sum p_{aft,x}$:

$$\Rightarrow \quad m_1 v_o = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2$$

$$\Rightarrow \quad (4\text{ kg})(12\text{ m/s}) = (4\text{ kg})v_1 \cos(25^o) + (6\text{ kg})v_2 \cos(40^o)$$

$$\Rightarrow \quad v_1 = [48 - (4.6)\, v_2\,]/3.63$$

$$\Rightarrow \quad v_1 = 13.24 - 1.27 v_2 \qquad \text{(Equ. A).}$$

$\sum p_{bef,y} = \sum p_{aft,y}$:

$$\Rightarrow \quad 0 = -m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2$$

$$\Rightarrow \quad 0 = -(4\text{ kg})v_1 \sin(25^o) + (6\text{ kg})v_2 \sin(40^o)$$

$$\Rightarrow \quad 0 = -1.69\, v_1 + 3.86\, v_2 \qquad \text{(Equ. B).}$$

Substituting v_1 from *Equation A* into *Equation B* yields:

$$0 = -1.69\,(13.24 - 1.27 v_2) + 3.86\, v_2$$

$$\Rightarrow \quad v_2 = 3.73\text{ m/s}.$$

Plugging back into *Equation A* will yield:

$$v_1 = 13.24 - 1.27\,(3.73\text{ m/s})$$

$$= 8.5\text{ m/s}.$$

Note: These answers are not exactly as above (off by a few hundredths)--the discrepancy is due to round-off error.

4.) Example 2: A string of length L is attached at one end to the ceiling and at the other end to a ball of mass m_1. The ball is positioned as shown in Figure 7.9 and, from that position, allowed to freefall from rest. At the bottom of its arc, the ball strikes a second mass m_2 sitting at rest on a frictionless table. Assuming the ball loses three-quarters of its mechanical energy as it *inelastically* bounces off m_2, what will both the ball

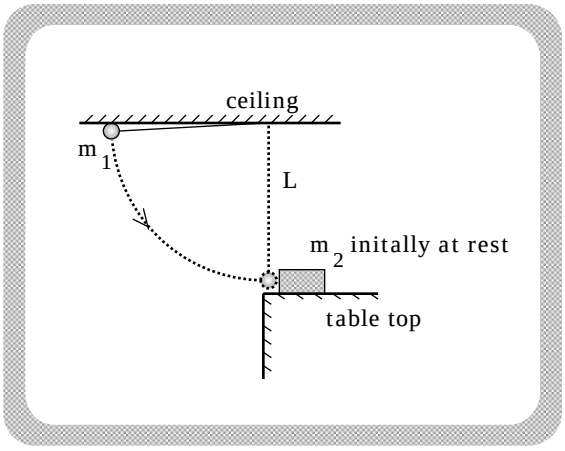


FIGURE 7.9

and block's after-collision velocities be (call them v_1 and v_2 respectively)?

Note 1: It is customary to term a collision *elastic*, *inelastic*, or *perfectly inelastic*. The definition of each follows:

Note 2a: An *elastic collision* is one in which *mechanical energy is conserved.* This situation never really occurs in nature (though the repulsion between like-charged electric particles comes very close), but is often assumed for the sake of mathematical simplification.

Minor Point: It should be noted that because collisions usually happen very quickly, the *potential energy* of the particles involved does not change through the collision (or if it does, it changes so slightly that it can be ignored). In other words, when *mechanical energy* is said to be *conserved through a collision*, we are really saying that the *sum of the kinetic energies of all the particles before the collision will equal the sum of the kinetic energies after the collision.* Due to this, many books simply define an *elastic collision* as one in which the *total kinetic energy* of the system is conserved *through the collision.*

Note 2b: An *inelastic collision* is one in which the kinetic energy of the system (i.e., the *kinetic energy* of all the pieces within the system added together) is *not* conserved. In most cases, energy is "lost" to heat, sound, or, in some cases, light. There can also be chemical changes internal to some part of the system (example: a piece of exploding dynamite).

Note 2c: A *perfectly inelastic collision* is one in which energy is not only *not* conserved, it's a collision in which the bodies additionally *stick together* upon contact. One obvious consequence of this situation: after the collision, everything is moving with the same *velocity*.

a.) Figures 7.10a and 7.10b show the ball at the bottom of the arc. It is moving solely in the *x direction* at that point. As momentum is always assumed to be

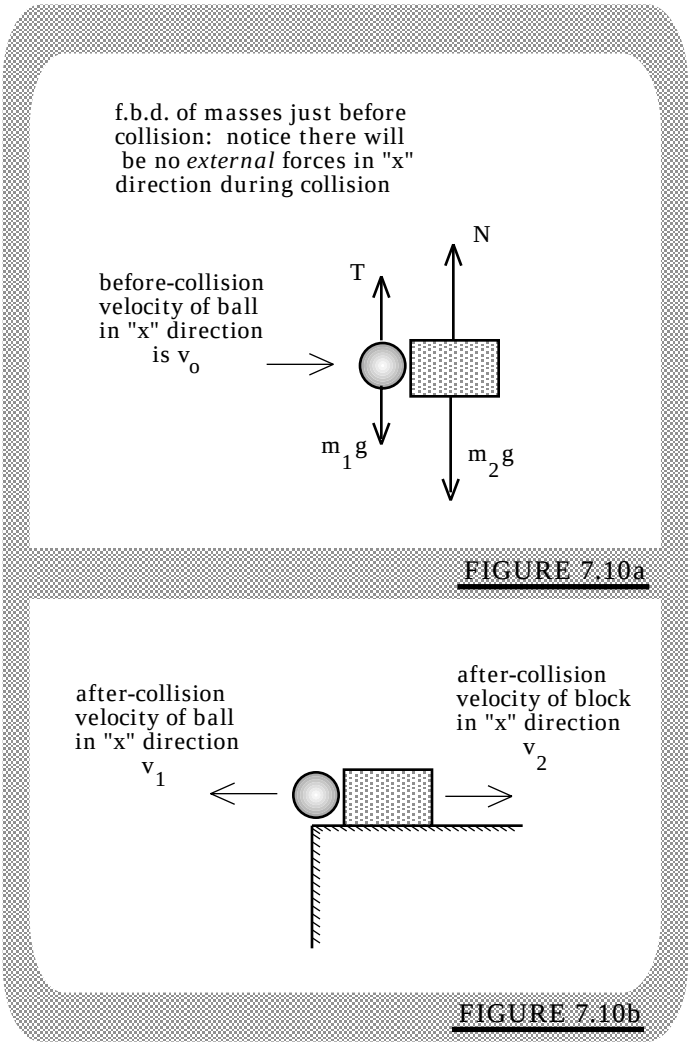

f.b.d. of masses just before collision: notice there will be no *external* forces in "x" direction during collision
N
T
before-collision velocity of ball in "x" direction is v_o
$m_1 g$
$m_2 g$
after-collision velocity of ball in "x" direction v_1
after-collision velocity of block in "x" direction v_2

FIGURE 7.10a

FIGURE 7.10b

conserved *through a collision*, we can equate the *x direction* momentum of the ball/block system before and after the strike. With m_1's velocity at the bottom of the arc equal to v_o and m_2's velocity equal to zero, we get:

$\underline{\sum p_{bef,x} = \sum p_{aft,x}:}$

$$\Rightarrow \quad m_1 v_o + m_2 (0) = - m_1 v_1 + m_2 v_2$$
$$\Rightarrow \quad m_1 v_o = - m_1 v_1 + m_2 v_2.$$

b.) There are *three* unknowns in this equation: v_0, v_1, and v_2. We know that mechanical energy is not conserved through the collision, but if we ignore air friction we know that mechanical energy will be conserved from the time the ball begins to fall until *just before* it strikes the block. Using *conservation of energy* for that part of the motion yields:

$$m_1 gh = (1/2) m_1 v_o^2$$
$$\Rightarrow \quad v_o = (2gh)^{1/2}.$$

c.) We now have two equations; we need a third. At this point, the only other help we can hope for is information about *energy loss* during the collision. If, for instance, we are told that *three-quarters* of the *before-collision* energy is lost, we know there will be *one-quarter* of the original kinetic energy left. In other words, *one-quarter* of the *before-collision* kinetic energy E_{bef} will equal the *after-collision* kinetic energy E_{aft} in the system. Mathematically, this gives us:

$$(1/4)\ E_{bef} = E_{aft}$$
$$\Rightarrow \quad (1/4)\ [(1/2) m_1 v_o^2 + (1/2) m_2 (0)^2] = (1/2)\ m_1 v_1^2 + (1/2)\ m_2 v_2^2.$$

This gives us our third equation which, though it may be a pain to solve, allows us to complete the problem.

5.) Parting shot: *Momentum* is a *vector*. Its change ($\Delta \mathbf{p}$) is equal to the impulse ($\mathbf{F} \Delta t$) received by the body in question. In cases where a number of bodies interact with each other providing "internal" impulses to one another, the momentum of individual pieces might change but the *total momentum* of all the pieces in a given direction will always sum to the same amount. In short, the *total momentum* in that direction will be *conserved*.

Under some circumstances, *energy consideration* will come in handy in providing one-more-solvable-equation, but one needs to be careful that *energy conservation* is truly justified before using the approach (Translation: know the *criteria* for the use of both the *conservation of energy* and the *conservation of momentum*--they are two very different conservation theorems).

G.) Confusion Be Gone–What is Conserved When?

1.) In a collision situation, when is mechanical energy conserved?

a.) Almost never (only when you are told that the collision is *elastic*).

i.) This is generally a contrived situation. It is usually assumed when physics authors want the student to have access to the *conservation of energy* equation (most books don't include the modified conservation of energy equation, so energy considerations are not generally possible if mechanical energy isn't assumed to be conserved). About the closest we can come to a truly elastic collision is one involving the interaction of magnets.

2.) In a collision, when is mechanical energy conserved (yes, I'm repeating!)?

a.) In real life, NEVER! In physics class, *only* when you are told that the collision is elastic (remember, an elastic collision is a contrived circumstance that exists in the world of physics only to simplify a situation and make the math doable).

3.) In a scenario in which a collision has occurred, when *might* mechanical energy be conserved?

a.) Before the actual collision (i.e., if potential energy has been converted into kinetic energy, or vice versa).

i.) Example: Before a collision, if a block slides down an incline or swings down from a higher position to a lower position pendulum style, mechanical energy will be conserved *before* the collision (assuming there is no friction or other non-conservative forces).

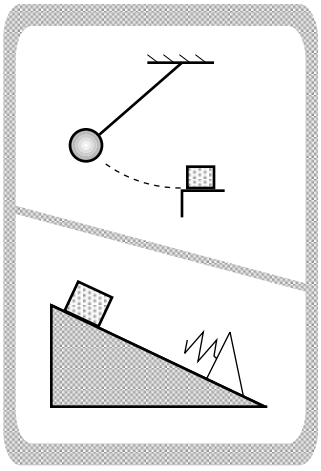


FIGURE 7.10c

b.) After the actual collision (i.e., if kinetic energy has been converted into potential energy, or vice versa).

i.) Example: A block strikes a spring that is attached to a body that is free to move (versus being pushed up against a rigid wall). Energy will

NOT be conserved through the collision (there is sound, heat, and possible molecular deformation during the hit), but just after the collision, the total mechanical energy of the whole system will stay conserved (i.e., kinetic energy will be lost as the spring compresses and spring potential energy increases) if there is no friction or other non-conservative forces acting.

ii.) How does MOMENTUM fit into the above example? Momentum will be conserved *through the entire spring collision* as long as the spring is attached to something (a block) that is *free to move* during the collision. In that case, the force the originally sliding block applies to the spring/block combination will be internal and will equal the force the spring/block combination applies to the originally sliding block. If, on the other hand, the spring/block combination is pushed up against a rigid structure like a wall, then a large, external, wall force will be applied to the system that will stop all motion dead, and momentum will not be conserved even through the collision.

4.) When is momentum generally *not* conserved?

a.) When an external force is applied in a particular direction.

i.) Example: For a ball thrown into the air, assuming we ignore friction, gravity is the only external force acting, and it is in the *y direction*. That means that momentum in that direction will change with time. In the *x direction*, there are no external forces and, hence, momentum will be conserved.

5.) When is momentum *not* conserved *through a collision*? That is, when is the total momentum of a system *not* the same from the time *just before* to the time *just after* a collision?

a.) When a BIG external force (hence big impulse $F_{ext}\Delta t$) is applied.

i.) Example: A ball hits a wall. The time during which the collision takes place is very small, so Δt is small, but the force generated during the collision is large enough to stop the ball dead.

6.) When is momentum conserved through a collision?

a.) It is conserved when (1) the only forces acting are internal (i.e., are the consequence of the interaction of the pieces of the system), or (2) when the external forces acting are small so that $F_{ext}\Delta t$ over the short time period of the collision is small.

i.) Example for 1: Whenever unrestrained bodies collide.

ii.) Example for 2: Gravity acting during a .01 second collision.

H.) A Bit of A.P. Nastiness:

1.) In an effort to confuse the hell out of anyone taking an AP test, the writers of those little gems have concocted an interesting kind of question designed to see just how completely you understand energy and momentum conservation. The following example will illuminate this bit of cheer.

2.) A bullet of known mass m_b is fired into a block of known mass m_k that sits on a frictionless table. The block is, itself, jammed up against (but not rigidly connected to) an ideal spring which, in turn, is placed against a wall. The spring *is* attached to the wall (see Figure 7.10d). The bullet embeds itself in the block. The block and bullet push the spring in to some known maximum displacement d before recoiling back outward. Assuming the unknown initial velocity of the bullet is v_o and the unknown final velocity of the bullet and block after separation from the recoil is v_r, determine the velocity of the bullet and block just after the bullet embeds itself but before the spring compresses appreciably.

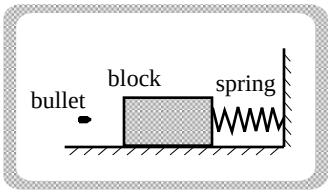


FIGURE 7.10d

a.) This is a nasty question. Why? When bright people in high school or at the university level start out dealing with physics, they have three things to worry about. They have to learn the principles, they have to understand the conceptual side of the principles (i.e., the consequences of the principles), and they have to cope with the math.

One of many intelligent ways to deal with the math is to become familiar with the characteristics that are often present when certain types of problem arise. If you are looking for the velocity of an object that is "falling" through a force field for which you happen to know the potential energy function, try the *conservation of energy*. If you are looking for the velocity of a couple of objects that have undergone a collision, try *conservation of momentum*.

This is a collision problem. You are trying to determine the velocity of the bullet/block system just after the collision. Energy is NOT conserved through the collision--a characteristic that is always true unless you are told otherwise (i.e., unless you are told it is an *elastic* collision), so what approach should you use? You'd think *conservation of momentum*.

As the problem is stated, this assumption is a bad one.

b.) If you did use the *conservation of momentum* on this problem, the resulting equation would look like:

$$m_b v_o = (m_b + m_k)v_{\text{just after}}.$$

The problem? There are two unknowns in this expression. You are looking for $v_{just\ after}$, but you don't know v_o.

c.) So what to do? As this is your first real experience with momentum and energy, and because the most superficially obvious approach isn't going to do the job, you're up the proverbial creek . . . unless you have the presence of mind to look at the information that has actually been given in the problem.

d.) What is that information? You don't know anything about the *before collision* situation, but you do know something about the *after* collision situation. Specifically, you know that the spring is deformed a maximum distance *d*. Maybe that information might help . . .

In fact, what this question is really testing is whether you are insightful enough to use what you know about the system *after the collision has taken place* to solve the problem.

e.) So let's try it. Just after the collision has taken place (but before the spring has depressed appreciably), the bullet and block have become one, have velocity $v_{just\ after}$ and have kinetic energy:

$$(1/2)(m_b+m_k)(v_{\text{just after}})^2.$$

f.) As the spring depresses after the collision, the bullet and block slow down. The amount of kinetic energy lost by the bullet and block during this part of the motion is absorbed by the spring in the form of *spring potential energy*.

g.) At the spring's maximum deflection, all of the mechanical energy in the system has converted to spring potential energy. And because the spring is ideal, *no energy was lost* while the spring is being compressed. That is, mechanical energy is conserved *after the collision*. Remembering that the spring potential energy function is $(1/2)kx^2$ or, in the case of maximum deflection, $(1/2)kd^2$, we can use this bit of insight to write out the AFTER-COLLISION *conservation of energy* expression as:

$$(1/2)(m_b+m_k)(v_{\text{justafter}})^2 = (1/2)kd^2.$$

This is nice as we know all the variables except $v_{just\ after}$.

h.) So what is the moral of the story? To do the problem *with the information provided*, we have to ignore the tried and true road sign that suggests that this is a *conservation of momentum* problem and look into the *middle* of the scenario to find the key that will unlock the door. If you've never run into a problem that requires you to think in this way, a problem like this can be a real killer. If you are prepared for this kind of twist, it is still a killer . . . but it's doable.

3.) For the chuckle of it, the next part of a problem like this would undoubtedly ask, "What is the initial velocity v_o of the bullet?" Given that we now know $v_{just\ after}$, this is where the *conservation of momentum* comes in handy. In fact, the expression that does the job is found in *Part H-2b*.

4.) And for complete amusement, the final question would undoubtedly be, "What is the velocity of the bullet and block after recoil (i.e., after leaving the spring)?" Interestingly enough, this problem requires no calculations. The spring will lose no mechanical energy as it compresses, then expands outward, so the mechanical energy of the bullet and block as they leave the spring should be the same as the mechanical energy of the bullet and block just after the collision but before the spring is compressed. (Remember, energy is conserved starting just after the collision.) That means the *velocity* of the bullet and block after recoil will be the same as the velocity of the bullet and block just after the collision. That velocity was the object of the original problem (i.e., the one done in *Part H-2g*).

I.) A Little More A.P. Nastiness . . .

1.) A bullet moving horizontally with velocity v_0 embeds itself into an immovable block of wood. As the bullet moves into the block, it experiences a retarding force that is proportional to velocity (that is, $F = -bv$, where b is a constant). Assuming very little mechanical energy is lost due to sound or heat during the impact, how deep will the bullet penetrate before coming to rest?

2.) The temptation is to try the work/energy theorem on this problem (after all, energy *was* mentioned in the problem and little was lost due to the collision).

a.) The work/energy expression yields:

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int \mathbf{F} \bullet d\mathbf{r}$$

$$= \int (-bv\mathbf{i}) \bullet (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= -\int (bv)dx.$$

b.) Unfortunately, even if we try to substitute $v = dx/dt$ into the integral, it still isn't going to fall out. So what next?

3.) Consider the impulse expression. You know the initial and final momentum, so if you could determine the net impulse the bullet experiences between the time it hits the block and the time it stops, you'd at least have an expression you might be able to solve.

a.) Noting that the final momentum is zero (the bullet is at rest at the end) and the one-dimensional force is equal to $-bv$, the impulse equation for this situation is:

$$p_o + \int F dt = p_f$$

$$\Rightarrow mv_o + \int F dt = 0$$

$$\Rightarrow mv_o = -\int (-bv) dt$$

b.) Substitute dx/dt for v, this becomes:

$$mv_o = -\int (-b\frac{dx}{dt}) dt$$

$$= b\int_{x=0}^{x=L} dx$$

$$\Rightarrow L = \frac{mv_o}{b}.$$

Tricky, eh?

J.) Exotica--The *Center of Mass* Reference Frame:

1.) There are times when *creative problem solving* can make life a lot easier. Most physics texts present one such approach using what is called *the center of mass frame-of-reference.*

2.) Even though there may be considerable motion and interaction within a system during an experiment, you and I normally observe the action from a frame of reference that is *stationary* relative to the room in which the experiment is being performed. That frame, our normal observation-frame, is called *the laboratory frame-of-reference.*

3.) It is possible to consider problems from a frame of reference other than the *lab frame*. For instance, you could look at an interaction from the perspective of the *center of mass*, almost as though you were sitting at the center of mass watching the motion of the system's particles from that vantage point. Doing so elicits a number of observations:

a.) Observation 1: Although the *center of mass* may move with velocity v_{cm} relative to the *laboratory frame-of-reference*, the *center of mass* is STATIONARY in the *center of mass frame-of-reference*. In other words, examining a problem from the *center of mass's point of view* assumes the *center of mass* is motionless with all else moving around it.

b.) Observation 2: We know that $Mv_{cm} = m_1v_1 + m_2v_2 + m_3v_3 + \ldots$, where v_{cm} is the *velocity* of the *center of mass* relative to the frame of reference in which one is observing the motion. As the velocity of the *center of mass* in the *center of mass frame* is zero, the sum of all the *momenta* of the system as viewed in the *center of mass frame* will also be zero (this fact sometimes makes problem solving from the perspective of the *center of mass frame* easy).

c.) Observation 3: Consider two colliding billiard balls: Figure 7.11a shows the *lab frame* positions of the balls at a number of points-in-time before and after the collision. It also shows the position of the *center of mass*, relative to the lab-frame, at those times.

Figure 7.11b shows the ball-positions from the *center of mass frame-of-reference*. Notice that in the *center of mass frame*, the ball's motion is

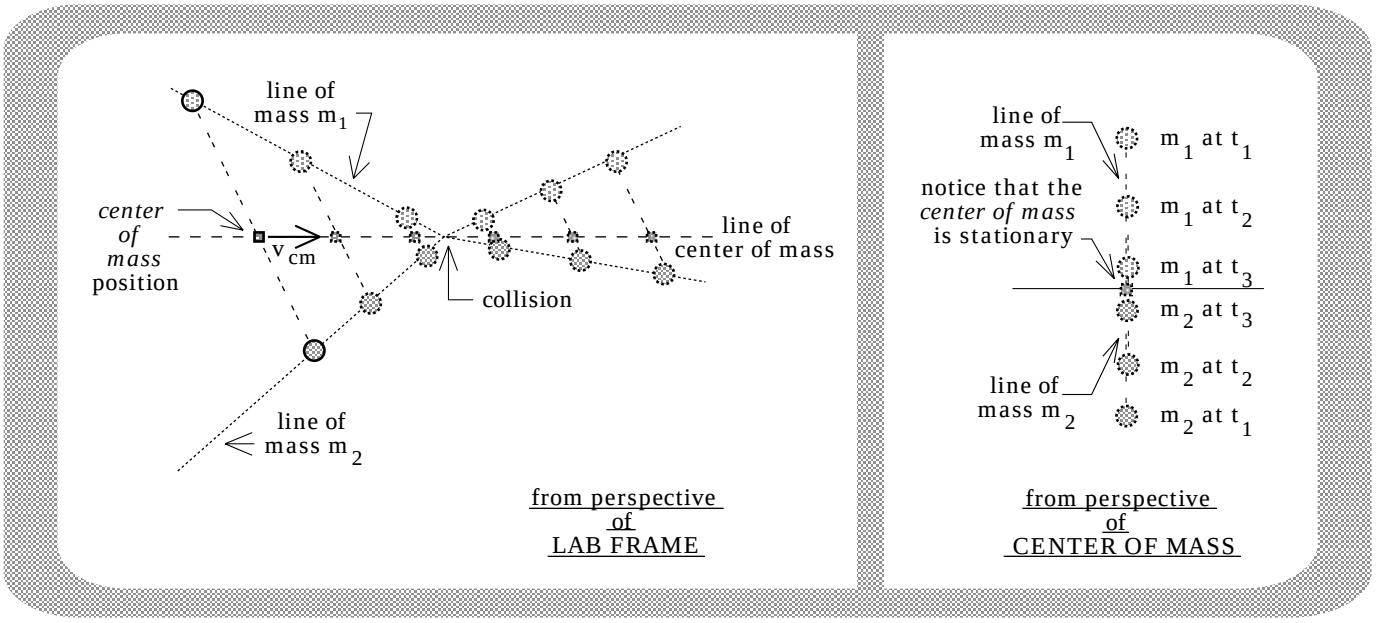


FIGURE 7.11a FIGURE 7.11b

linear--another simplification that can make problem solving in the *center of mass* frame easier.

d.) Observation 4: If we know the *center of mass's* velocity in the *lab frame* (call this $\boldsymbol{v}_{cm}$) and the velocity of a particle as viewed from the *center of mass frame* (call this $\boldsymbol{v}_{part.rel.\ to\ cm}$), the velocity of the particle in the *lab frame* (call this $\boldsymbol{v}_{lab}$) will be:

$$\mathbf{v}_{\text{lab}} = \mathbf{v}_{\text{part. rel. to cm}} + \mathbf{v}_{\text{cm}}.$$

e.) Observation 5: It should be obvious from the sketch that the velocities observed in the *lab frame* are greater than the velocities observed in the *center of mass frame*. One consequence: the mechanical energy of the system as determined in the *lab frame* will be <u>*greater*</u> than the mechanical energy of the system as determined in the *center of mass frame*.

Note: It turns out that the *total kinetic energy* in the lab frame equals the *total kinetic energy* of the pieces as measured in the *center of mass frame*, plus the kinetic energy of a particle whose mass is the total mass of the system and whose velocity is the *lab frame's* center of mass velocity. Mathematically, this is:

$$\text{KE}_{\text{lab}} = \text{KE}_{\text{pieces in cm frame}} + \text{KE}_{\text{cm}}.$$

f.) Observation 6: The *kinetic energy expression* stated above is not particularly important here, but its consequence is. You cannot assume that the energy calculated in the *lab frame* is the same as the mechanical energy determined in the *center of mass frame*.

g.) With all this information, the problem-solving technique is simple. If a velocity problem looks difficult in the lab frame:

i.) Transform to the *center of mass frame* (that is, look at the problem from the point of view of the system's *center of mass*);

ii.) Determine the velocity of the particle-of-interest as seen in the *center of mass frame*;

iii.) Take the particle's velocity as determined in the *center of mass frame* and transform it back into the *lab frame* using $\boldsymbol{v}_{lab} = \boldsymbol{v}_{part.\ rel.\ to\ cm} + \boldsymbol{v}_{cm}$.

iv.) Depending upon the problem, the technique can work easily or with great pain.

4.) An Example: Consider the following one dimensional problem: A ball of mass m moving horizontally with velocity v_o strikes a stationary, massive wall (i.e., the wall is, relative to the ball, assumed to be *infinitely* massive). Upon collision, the ball rebounds in the horizontal with velocity v_o (i.e., mechanical energy is conserved)--see Figure 7.12a and 7.12b.

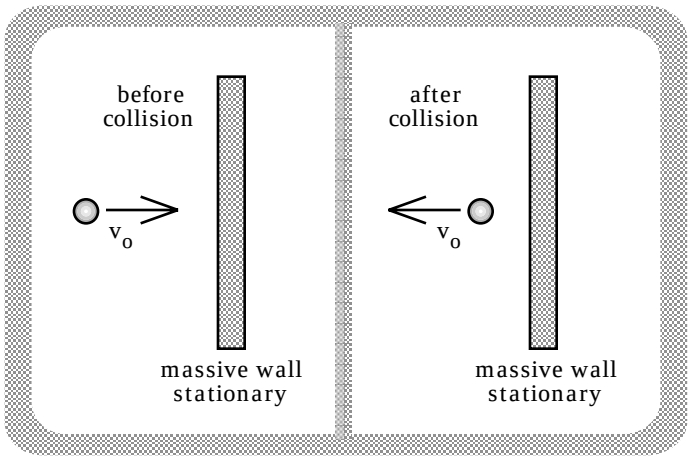


FIGURE 7.12a FIGURE 7.12b

The same wall is then made to move horizontally with velocity v_o. The same ball is projected horizontally at the wall with velocity v_o (see Figures 7.13). What is the ball's velocity after striking and bouncing off of the wall?

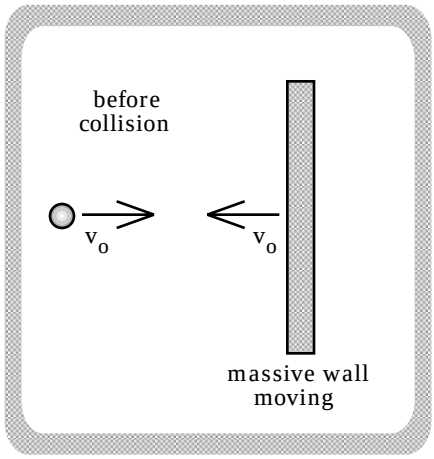


FIGURE 7.13

a.) The temptation is to say $2v_o$, but that would be a mistake.

b.) Notice:

i.) Because the wall is infinitely massive, the *center of mass* of the system *is* the wall.

ii.) When the wall is stationary, the *lab frame of reference* and the *center of mass frame of reference* are the same. That is not true when the wall is moving (i.e., when the wall moves, the c. of m. moves with it while you in the *lab frame* stay stationary).

iii.) As the *center of mass frame* is the frame in which the wall *is not moving*, and as that situation is the only one we know anything about as far as energy goes (i.e., in the frame in which the wall is not moving, the ball's mechanical energy is *conserved*), and as energy in the *lab frame* and energy in the *center of mass frame* are different, we *have* to deal with the problem from the *center of mass* point of view.

c.) From the *center of mass* perspective, the wall is stationary and the mass m is moving toward the wall with velocity $2v_o$. As mechanical energy is conserved in this frame of reference, the ball will bounce off the wall and leave with velocity $2v_o$ *relative to the wall*.

d.) But the wall, itself, is moving with velocity v_o, relative to the *lab frame*. That means that relative to the *lab frame*, the BALL is moving with velocity $2v_o$ (relative to the wall) plus v_o (the wall's velocity relative to the lab frame). In other words, transforming our solution in the *center of mass frame* ($2v_o$) back into the *laboratory frame* requires us to use:

$$\mathbf{v}_{\text{lab}} = \mathbf{v}_{\text{rel. to cm}} + \mathbf{v}_{\text{cm}}.$$

Noting that this is a one-dimensional situation, we can write:

$$v_{\text{lab}} = 2v_o + v_o.$$

e.) Bottom line: The ball leaves the moving wall with velocity $3v_o$, relative to the *lab frame*.

f.) As improbable as this solution might appear, there is an easy way to experimentally check it. Consider a fairly massive ball (a tennis ball or a large superball will do) freefalling from a height h toward a hardwood floor. As it does, a much lighter ball (a ping pong ball) follows just above it. Assume the tennis ball strikes the floor with velocity v_o. If we further assume an *elastic collision*, the tennis ball will bounce off the floor and return with an initial upward velocity of v_o. Meanwhile, the ping pong ball will have dropped from approximately the same height and will have a downward velocity that is close to v_o as it nears the floor. Because the ping pong ball is trailing the tennis ball, and because the tennis ball will execute its bounce before colliding with the ping pong ball, we now have a situation that is close to our "small ball moving with velocity v_o hits massive wall moving with velocity $-v_o$" problem.

In short, if the theory is correct the ping pong ball should collide with the massive ball, bounce off the massive ball, and leave with a velocity close to $3v_o$. Does it? Go home and try it. With three times its pre-collision velocity, the ping pong ball should have *nine* times its pre-collision kinetic energy (remember, kinetic energy is a function of velocity *squared*). That, in turn, means the ping pong ball should fly *nine times higher* than its initial height above the floor.

Note: You will not be tested on the frame of reference changes discussed in this section. It was included because it has educational value and because

approaching such problems can be fun. It is also a technique that is often used by physicists in analyzing problems. I hope you found it amusing!

QUESTIONS

7.1) Just before striking a ceiling, a .4 kilogram ball is moving 11 m/s. It strikes the ceiling at a 30^o angle relative to the vertical (see Figure I). Assuming only a small frictional force acting at the ceiling:

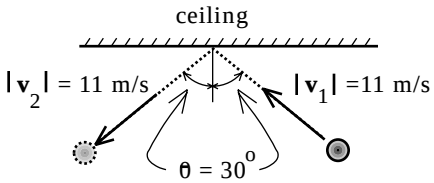


FIGURE I

a.) Is momentum conserved *through the collision*:

i.) In the *y direction*? Explain.

ii.) In the *x direction*? Explain.

b.) What *kind* of collision is it (i.e., elastic, inelastic, perfectly inelastic . . . what?)?

c.) What is the *impulse* applied to the ceiling by the ball?

d.) If the *average force* applied to the ceiling during the collision is 3200 newtons, over what *period of time* does the collision occur?

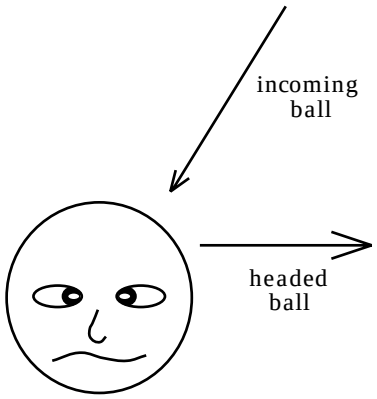
7.2) You have just made the football team. You are told to get into one of two tackling lines when you realize you have a choice: you can either be tackled by a 60 kilogram player who runs at 10 m/s or by a 120 kilogram player who runs at 5 m/s.

a.) Which player has the greater momentum?

b.) Which player has the greater kinetic energy?

c.) By whom would you prefer to be hit? Explain.

7.3) A .5 kg soccer ball comes down at a 30^o angle relative to the vertical moving at 25 m/s (this is the same as having a ball freefall approximately 100 feet). It is headed by a player, leaving the player's head at an angle of 90^o relative to the vertical moving with a velocity magnitude of 18 m/s.



a.) What was the ball's *momentum-change* off the player's head?

b.) If the collision takes .08 seconds, what is the *force* applied to the player's head (note that this is opposite the force applied to the ball)?

7.4) Determine the *y coordinate* of the center of mass of the inhomogeneous solid (i.e., a solid whose mass is *not* uniformly distributed throughout the form) shown in Figure II. Assume that the body's *surface density function* is *ky*, where *k* is a constant whose magnitude is *1* and whose units are appropriate to the situation.

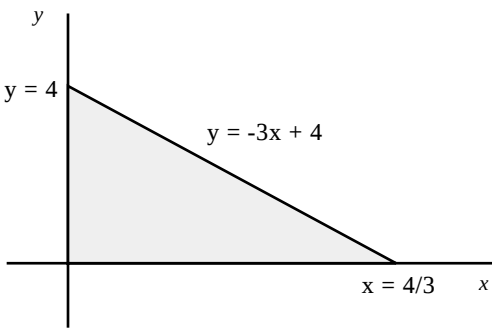


FIGURE II

7.5) A 60 kilogram bum stands on an 800 kilogram flatcar that is moving 15 m/s. The bum begins to run, finally reaching a speed of 5 m/s relative to the flat-car. How fast does the flat-car end up moving if:

a.) The bum runs in the same direction as that of the flat-car?

b.) The bum runs opposite the direction of motion of the flat car?

c.) The bum runs in a direction perpendicular to the direction of motion of the flat-car?

d.) For the situation outlined in *Part a*, compare the system's total *kinetic energy* before and after the bum began to run.

7.6) A cannon shell moving 240 m/s at an angle of 60^o with the horizontal explodes into two pieces. The first piece, comprised of two-thirds of the shell's mass, leaves the explosion with initial velocity of 260 m/s in the horizontal (i.e., at angle 0^o).

a.) What is the *velocity* of the second piece?

b.) Assuming the shell's mass is 30 kilograms, how much chemical energy was converted to kinetic energy in the explosion?

7.7) An 880 kilogram car rear-ends a stationary 1000 kilogram car whose brakes are locked. The collision is perfectly inelastic. If the coefficient of friction between each car's wheels and the pavement is .6, and if the cars slide-with-friction a distance of 1.2 meters after the collision, what must the first car's *velocity magnitude* have been just before the accident?

7.8) A man of mass 90 kilograms is ice skating out of control. While moving at 8 m/s in the *x direction*, he collides with his 55 kilogram girlfriend who is moving with velocity 10 m/s at an angle of 120^o with the *x axis* at the time. If

the collision is perfectly inelastic, determine the *velocity* (as a vector) of the couple after the collision.

7.9) A spring-gun of mass $m_g = 2$ *kilograms* uses an ideal spring with a spring-constant $k = 120$ *nt/m* to shoot a ball of mass $m_b = .04$ *kilograms* out of its barrel. At a particular point in time, the cocked gun and ball are moving backwards over a *frictionless table* with velocity $v_o = 5$ *m/s* (the word "backwards" is intended to mean that when the gun is fired the bullet will leave opposite the direction of the gun's motion). Relative to the table, what will the gun's velocity (v_g) and the ball's velocity (v_b) be just after firing? Assume the spring's *compression-distance* is $x = .15$ *meters* when the gun is cocked.

7.10) Jane (mass 40 kg) has entered a soap-box derby. The course ends by passing through a mountain via a tunnel, over a bridge, then to the finish line (a sketch--Figure III--is provided to the right).

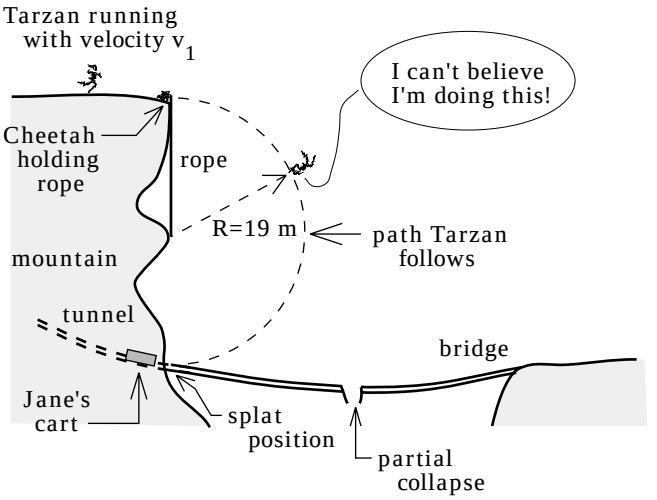


FIGURE III

Unfortunately, the bridge has collapsed. Tarzan, love-sick and definitely not-too-bright, sees the danger from above. Quickly, he makes a few calculations and realizes that if he can run fast enough, he can use the vine Cheetah is playing with to swing down from above and meet Jane's cart as it comes out of the tunnel. With the right momentum, he could stop the cart by colliding with it (again, see Figure III).

a.) Assume the vine is 19 meters long. If Jane's cart has a mass of 190 kgs (without Jane) and a speed of 11 m/s out of the tunnel, and if Tarzan's mass is 90 kg, how fast must Tarzan run "up top" to effect a dead stop of Jane's cart just as it comes out of the tunnel? Assume Tarzan does not bounce off the cart upon collision, that everything stops dead when he hits the cart, and that the collision occurs directly under the point at which Tarzan begins his swing.

b.) For the amusement of it, what is the tension in the vine just before Tarzan and the cart collide?

7.11) NOTE: This question is for anyone interested in trying a problem using a *center of mass frame of reference*. There will *not* be a problem like this one on your next test.

A satellite approaching a planet at just the right angle will pick up kinetic energy from the planet as it whips around the planet and exits in the opposite direction (see Figure IV). This interaction can be approximated as an elastic collision (i.e., a collision in which the bodies within the system change their motion *while mechanical energy is conserved* within the system).

A satellite is observed to be moving with velocity 7 km/s at a particular distance d from a planet that is, itself, moving with velocity 12 km/s. The satellite sling-shots around the planet. What is the satellite's velocity v_s once it again reaches a distance d from the planet?

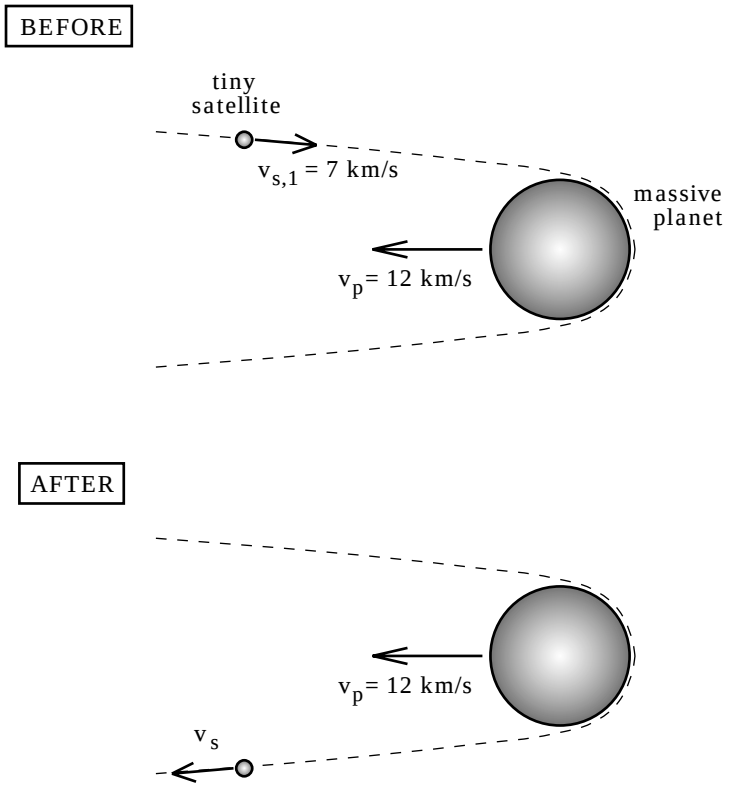


FIGURE IV

Chapter 8

ROTATIONAL MOTION -- PART I

A.) Preliminary Comments and Basic Definitions:

1.) We are about to draw an almost perfect parallel between *translational* and *rotational motion*. That is, every translational concept so far covered (i.e., kinematics, Newton's Laws, energy considerations, momentum, etc.) has its rotational counterpart. The next two chapters will focus on those parallels and the general analysis of rotational systems.

2.) The *radian* angular measure:

a.) Consider a circle. If we take its radius R and lay it onto the *circumference of the circle*, we will create an angle whose *arc length* is equal to R (Figure 8.1a). Any angle that satisfies this criterion is said to have an angular measure of *one radian*.

Put another way, a *one radian angle* subtends an arc length Δs equal to the radius of the circle (R).

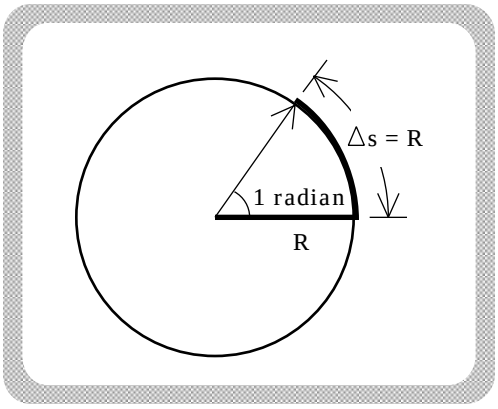


FIGURE 8.1a

b.) With this definition, a *one-half radian* angle subtends an arc length equal to $(1/2)R$ (see Figure 8.1b); a *two radian* angle subtends an arc length equal to $2R$ (see Figure 8.1c); and a general $\Delta\theta$ *radian angle* subtends an arc length Δs equal to $R\Delta\theta$ (see Figure 8.1d). In other words, the most general expression relating arc-

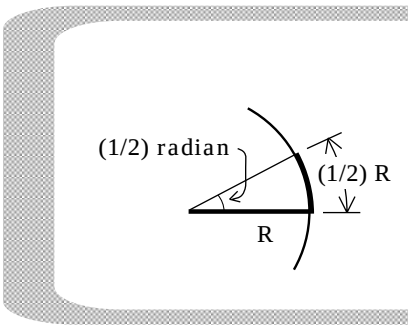


FIGURE 8.1b

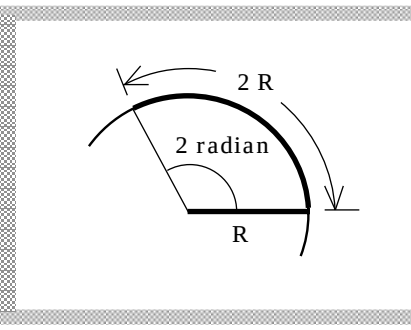


FIGURE 8.1c

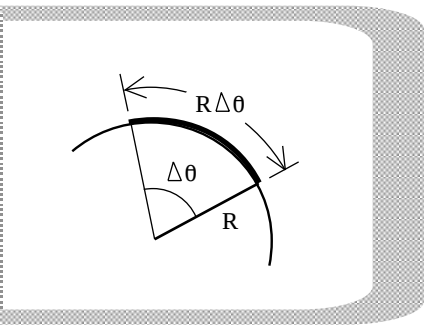


FIGURE 8.1d

length and angular measure in radians is:

$$\Delta s = R\,\Delta\theta.$$

c.) Just as *displacement* is measured in meters by coordinate variables like *x* and *y*, *angular displacement* is measured in radians by angular coordinate variables like θ_1 and θ_2.

3.) Angular Velocity:

a.) A point on a circle moves from a position defined by the angle θ_1 to a position defined by angle θ_2 in time Δt. Lines are drawn from the *center of rotation* to the point's positions at t_1 and t_2 (see Figure 8.2).

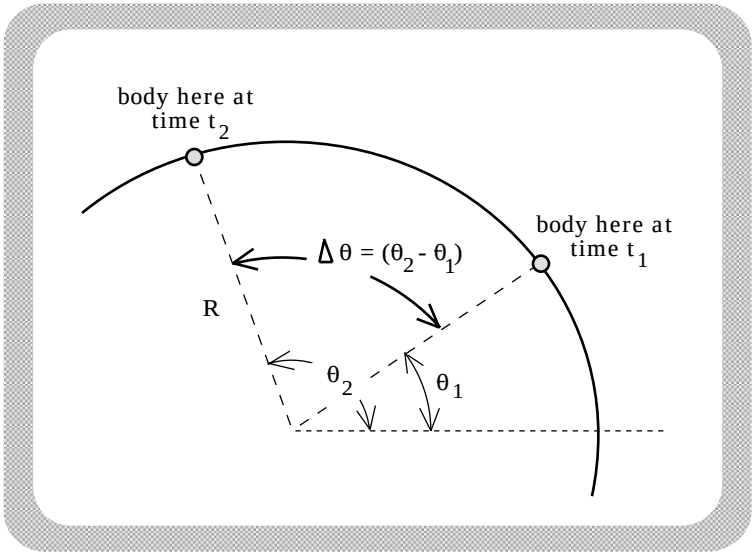


FIGURE 8.2

b.) The *average angular velocity* $\boldsymbol{\omega}_{avg}$ is a *vector* quantity that denotes the *angular displacement* (i.e., net change of *angular position* $\Delta\theta$ *per unit time* Δt) over some large time interval. With units of *radians/second*, it is mathematically defined as:

$$\boldsymbol{\omega}_{avg} = \Delta\boldsymbol{\theta}/\Delta t.$$

c.) Most elementary rotation problems assume rotational motion in the *x-y plane*. Such motion is *one dimensional* (the body isn't rotating simultaneously around *two* axes, just one--see the BIG NOTE below). As such, we can ignore the vector symbolism and write the *average angular velocity* as:

$$\omega_{avg} = \Delta\theta/\Delta t\,,$$

where $\Delta\theta$ is the net *angular displacement* of the object and Δt is the time interval over which the motion occurs.

While this appears to be a scalar equation, it is not. It matters whether the body is rotating clockwise or counterclockwise. We will account for rotational direction shortly (see BIG NOTE below).

d.) Big Note and preamble to *direction of rotation* discussion:

i.) A *TRANSLATIONAL velocity* vector is designed to give a reader three things: the *magnitude* of the velocity (i.e., the number of *meters per second* at which the object is moving); the *axis* or combination of axes along which the motion proceeds (unit vectors do this); and the *positive* or *negative* sense of the *direction* along those axes.

Example: A velocity vector $\boldsymbol{v} = -3\boldsymbol{i}$ tells us the object in question is traveling at 3 m/s along the *x axis* in the negative direction.

ii.) A *ROTATIONAL velocity* vector is also designed to give the reader three things: the *magnitude* of the rotational velocity (i.e., the number of *radians per second* through which the body moves); the *plane* in which the rotation occurs (i.e., does the rotation occur in the *x-y plane* or the *x-z plane* or some combination thereof); and the *directional sense* of the rotation (i.e., is the body rotating *clockwise* or *counterclockwise?*).

iii.) Bottom line: The *notation* used to define the *rotational velocity* vector needs to convey different information than does the *notation* used to define a *translational velocity* vector. The format used to convey the rotational information required is outlined below.

e.) Rotational direction:

i.) Consider a disk rotating in the *x-y plane* (this is the plane in which almost all of your future problems will be set). The magnitude of its angular velocity is, say, a constant $\omega = 5$ *radians/second*. Notice that although the *instantaneous, translational* direction-of-motion of each piece of the disk is constantly changing as the disk rotates, the *axis* about which the disk rotates always stays oriented in the same direction.

ii.) The DIRECTION of an *angular velocity vector* is defined as the direction of the *axis* about which the rotation occurs.

iii.) We have already decided that the direction about which our example's rotation occurs is along the *z axis*; the *angular velocity* vector for the problem is, therefore:

$$\boldsymbol{\omega} = (5 \text{ radians/second})\mathbf{k},$$

where ***k*** is the unit vector in the *z direction*.

iv.) We have just developed a clever way to mathematically convey the fact that a rotation is in the *x-y plane*. We have done so by attaching to the *angular velocity magnitude* a unit vector that defines *the axis about which the motion occurs*.

Put in a little different context, we have earmarked the *plane of rotation* by defining the direction *perpendicular to that plane* (the *z-direction* is perpendicular to the *x-y plane*).

v.) We still have not designated a way to define the *sense of the motion* (i.e., whether the rotation is clockwise or counterclockwise). Assuming we are looking at motion in the *x-y plane*, these two possibilities are covered nicely by assigning a positive or negative sign to the ***k*** *axis* unit vector being used to define the axis of rotation. That is:

BY DEFINITION, CLOCKWISE ROTATIONS IN THE *x-y PLANE* ARE DEFINED AS HAVING UNIT VECTOR DIRECTIONS OF -***k***, WHEREAS COUNTERCLOCKWISE ROTATIONS ARE DEFINED AS HAVING UNIT VECTOR DIRECTIONS OF +***k*** (see Figures 8.3a and 8.3b for a summary of this information).

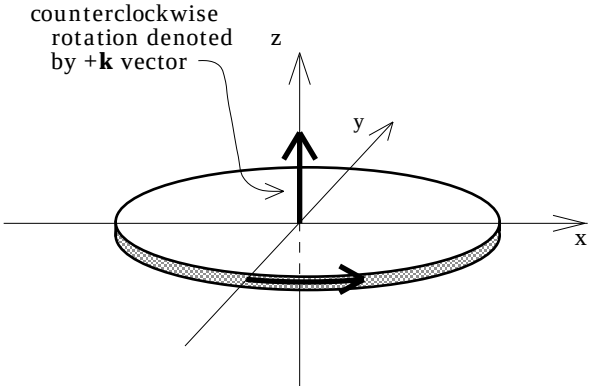


FIGURE 8.3a

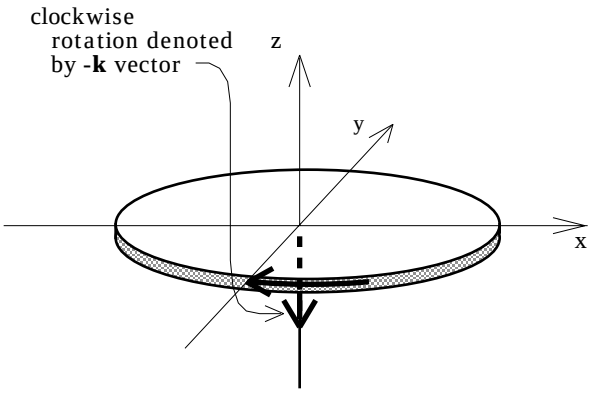


FIGURE 8.3b

Note: This formalism is not as off-the-wall as it probably seems. Rotate a screw *counterclockwise* and it will proceed upward out of the plane in which it is embedded (that is how you unscrew a screw). Define that plane with a standard, right-handed, *x-y axis* and the screw is found to unscrew in the ***+k*** *direction.* A screw rotated *clockwise* will proceed into the plane in the ***-k*** *direction.*

As long as we always use a right-handed coordinate system (the standard within mathematics these days), the notation works nicely.

vi.) Mathematicians have created a mental tool by which one can remember this rotational formalism. Called *the right-hand rule*, it follows below:

Mentally place your *right hand* on the rotating disk so that when you curl your fingers, they follow the direction of the disk's rotation. Once in the correct position, extend your thumb perpendicularly out away from your fingers (i.e., in a "hitchhiker's" position). If the thumb points *upward*, the direction of the *angular velocity* is in the ***+k*** *direction.* If you have to flip your hand over to execute the curl, your thumb will point downward *into* the plane and the direction of *angular velocity* will be in the ***-k*** *direction.*

vii.) In summary, if our disk were rotating at 5 radians per second in the *clockwise* direction in the *x-y plane*, the *angular velocity vector* would be:

$$\boldsymbol{\omega} = (5 \text{ radians/second})(-\mathbf{k}),$$

which, for simplicity, would probably be written as:

$$\boldsymbol{\omega} = -5 \text{ rad/sec } \mathbf{k}.$$

Note: As all our problems will be one-dimensional (i.e., rotation in the *x-y plane*), there is no need to include the ***k*** part of this representation when solving problems. IT IS IMPORTANT TO KEEP TRACK OF THE SIGN, THOUGH. As such, this *angular velocity* vector would normally be written as $\omega = -5 \text{ rad/sec}$.

4.) *Instantaneous angular velocity* ω for one-dimensional motion is defined as a measure of an object's *displacement per unit time* (i.e., its rate of angular travel), measured at a particular point in time.

a) Mathematically, *instantaneous angular velocity* (referred to as *angular velocity* from here on) is defined as:

$$\omega = \lim_{\Delta\theta \to 0} \left(\Delta\theta / \Delta t \right)$$
$$= d\theta / dt .$$

b.) Example: If $\theta = at^3 - 4bt$, then $\omega = d(at^3 - 4bt)/dt = 3at^2 - 4b$.

5.) *Average angular acceleration* α_{avg} for one-dimensional rotation is defined as the rate at which the <u>*angular velocity*</u> changes *per unit time*. It is mathematically defined as:

$$\alpha_{avg} = \Delta\omega / \Delta t.$$

Its units are "radians-per-second-per-second."

Note: Again, this is a vector quantity even though we have not included its associated unit vector.

a.) Example: A man has an *angular velocity* $\omega_1 = (3\ rad/sec)$ when at θ_1. Three seconds later, he is at θ_2 moving with an *angular velocity* $\omega_2 = 9\ rad/sec$ (see Figure 8.4). What is his *average angular acceleration*?

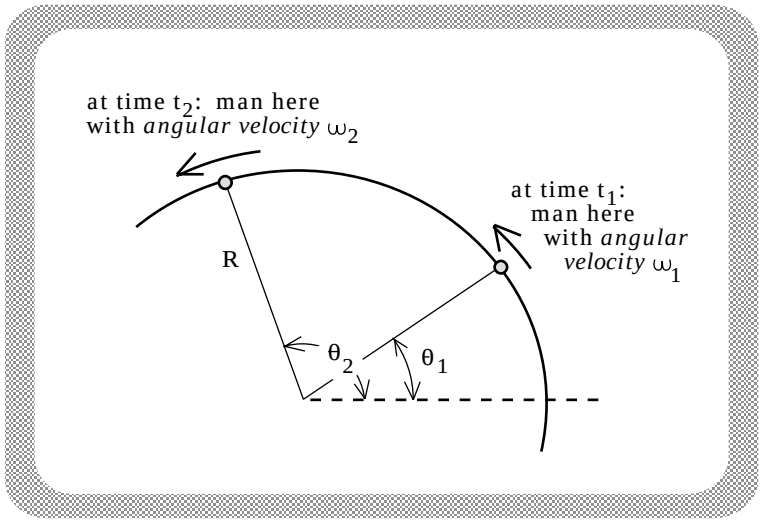


FIGURE 8.4

<u>Solution</u>:

$$a_{avg} = \Delta\omega / \Delta t$$
$$= (\omega_f - \omega_i)/(\Delta t)$$
$$= (9\ rad/s - 3\ rad/s)/(3\ sec)$$
$$= 2\ rad/s^2.$$

6.) *Instantaneous angular acceleration* (α): a measure of an object's *change-of-angular-velocity per unit time*, evaluated at a particular point in time.

a.) Mathematically, instantaneous angular acceleration (referred to as *angular acceleration* from here on) is defined as:

$$\alpha = \lim_{\Delta\theta \to 0} \left(\Delta\omega / \Delta t \right)$$
$$= d\omega / dt .$$

b.) Example: If $\omega = 3bt^2 - 4c$, then $\alpha = d(3bt^2 - 4c)/dt = 6bt$.

7.) NOTICE: For every translational parameter, we have identified a comparable rotational parameter. *Translational position* is defined using coordinates like *x* and *y*; *angular position* is defined using angular coordinates (θ*'s* measured in radians). *Velocity* is defined as *dx*/*dt* in meters/second; *angular velocity* is defined as $d\theta/dt$ in radians/second. *Acceleration* is defined as *dv*/*dt* in meters/second^2; *angular acceleration* is defined as $d\omega/dt$ in radians/second^2.

8.) Relationship between Angular Motion and Translational Motion:

a.) Consider a point moving with a constant *angular velocity* ω in a circular path of radius *R*. At time t_1, the point's *angular position* is defined by the angle θ_1. At time t_2, its angular position is θ_2 (see Figure 8.5a).

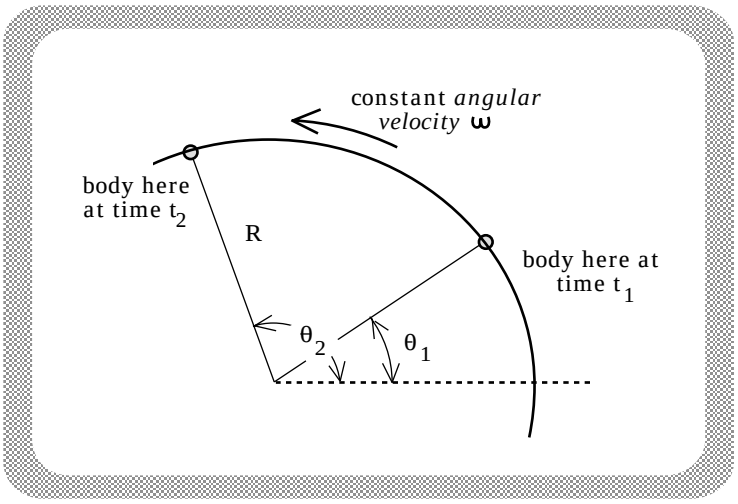


FIGURE 8.5a

b.) During the interval Δt, the point travels a translational distance equal to the arc length Δs of the subtended angle $\Delta\theta = (\theta_2 - \theta_1)$ (see Figure 8.5b on next page). We know from the definition of radian measure that that arc length is:

$$\Delta s = R\,\Delta\theta .$$

c.) Dividing both sides by the time of travel yields:

$$\Delta s/\Delta t = R\,(\Delta\theta/\Delta t).$$

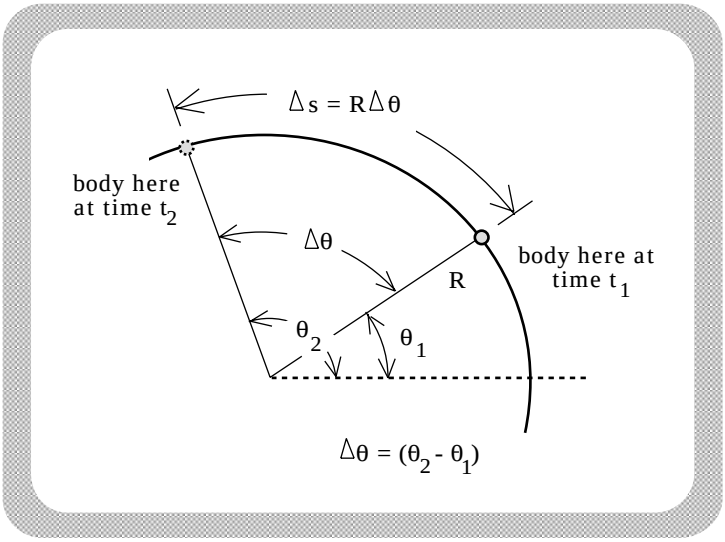


FIGURE 8.5b

i.) The left-hand side of this relationship is simply the *magnitude* of the *instantaneous <u>translational</u> velocity v* of the point as it moves along the arc (it is actually the magnitude of the *average* translational velocity, but because the point is moving with a constant angular velocity, the *average magnitude* and the *instantaneous magnitude* will be the same).

ii.) The right-hand side of the equation is the *radius of motion* times the *magnitude* of the *instantaneous <u>angular</u> velocity* (ω).

iii.) In other words, at a given instant the *magnitude* of a rotating body's *instantaneous <u>translational</u> velocity* at a given point will equal the *radius r of the motion* times the *magnitude* of the body's *instantaneous angular velocity* at that same instant. Mathematically, this is written:

$$v = r\omega.$$

d.) Through similar reasoning, the relationship between the *magnitude* of a point's *instantaneous <u>translational</u> acceleration* and the *magnitude* of its *instantaneous <u>angular</u> acceleration* at the same moment is:

$$a = r\alpha.$$

B.) Rotational Kinematic Equations:

1.) Parallels:

a.) An object moving under the influence of a *constant acceleration* has a *velocity versus time* graph that looks like the one shown in Figure 8.6. In

examining that graph at the beginning of the year, we noticed a number of useful things.

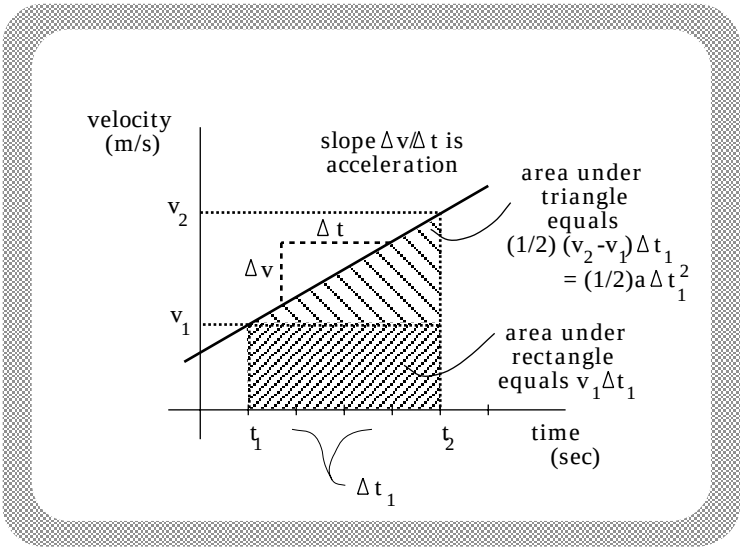


FIGURE 8.6

i.) The slope of the graph $\Delta v/\Delta t$ equals the acceleration of the object.

ii.) The *distance* traveled $x_2 - x_1$ between times t_1 and t_2 is equal to the *area under the graph.* In fact, our beginning-of-the-year endeavors noted that the area bounded by the bottom rectangle was $v_1 \Delta t_1$, the area bounded by the triangle was $(1/2)a(\Delta t_1)^2$, and that the distance traveled was $(x_2 - x_1) = v_1 \Delta t_1 + (1/2)a(\Delta t_1)^2$.

b.) Consider now a body moving in rotational motion with *constant angular acceleration* (see Figure 8.7).

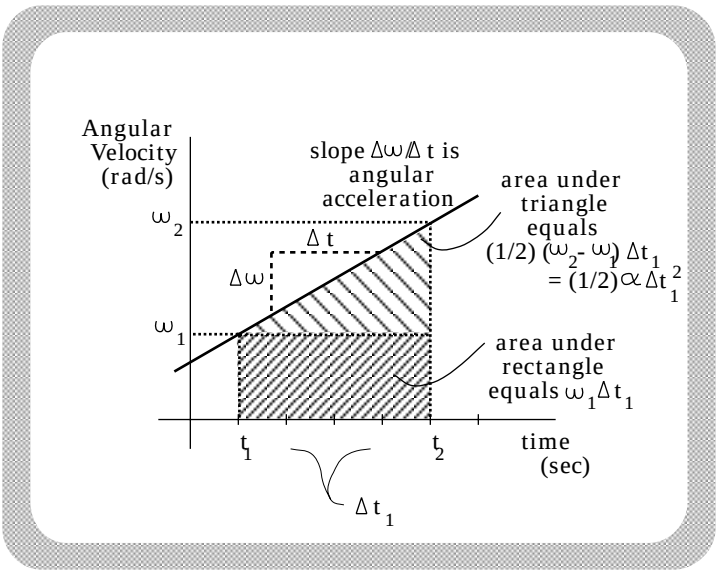


FIGURE 8.7

i.) Note that the <u>angular</u> *velocity versus time* graph for that motion looks exactly like its translational counterpart shown in Figure 8.6. It follows logically that:

ii.) The slope of that graph $\Delta\omega/\Delta t$ yields the *angular acceleration* of the motion;

iii.) The area under the graph yields the angular distance traveled $\theta_2 - \theta_1$.

iv.) In fact, with the exception of the fact that the *v's* are replaced by ω*'s* and the *a's* replaced by α*'s*, everything about Figure 8.6 holds for Figure 8.7.

c.) Having observed the parallel between the two systems of motion, it makes sense that the same reasoning that led to the translational equation $(x_2 - x_1) = v_1 \Delta t_1 + (1/2)a(\Delta t_1)^2$ will lead to its rotational image--a *rotational kinematic equation* that is exactly like its translational counterpart but with rotational parameters in place of the translational parameters. In fact, we could go through all the derivations we experienced in deriving the original set of kinematic equations and end up with all of their rotational doubles.

d.) This would be a waste of time. Instead, the general rotational kinematic equations are presented below next to their general translational counterparts.

$(x_2 - x_1) = v_1 \Delta t + (1/2)a(\Delta t)^2 \quad \Rightarrow \quad (\theta_2 - \theta_1) = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2.$

$(x_2 - x_1) = v_{avg} \Delta t \quad \Rightarrow \quad (\theta_2 - \theta_1) = \omega_{avg} \Delta t.$

$v_{avg} = (v_2 + v_1) / 2 \quad \Rightarrow \quad \omega_{avg} = (\omega_2 + \omega_1) / 2.$

$a = (v_2 - v_1) / \Delta t \quad \Rightarrow \quad \alpha = (\omega_2 - \omega_1) / \Delta t.$

$(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1) \quad \Rightarrow \quad (\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1).$

2.) Some Examples:

a.) Example 1: A turntable whose *angular velocity* is 20 rad/s *angularly accelerates* at 5 rad/s^2 for three seconds. What is its *angular velocity* at the end of that time period?

Solution: We know the initial *angular velocity*, the constant *angular acceleration*, and the *time interval* over which the acceleration occurs. With that we can write:

$$\alpha = (\omega_2 - \omega_1)/\Delta t$$

or $(5\ \text{rad/s}^2) = [\omega_2 - (20\ \text{rad/s})]/(3\ \text{sec})$

$\Rightarrow \quad \omega_2 = 35\ \text{rad/s}.$

b.) Example 2: What is the *average angular velocity* of the turntable mentioned in Example 1 above as it *angularly accelerates* from 20 rad/s to 65 rad/s in three seconds?

Solution: We know the *angular velocities* at the beginning and end of the time interval, and we know the angular acceleration is constant:

$$\begin{aligned}\omega_{avg} &= (\omega_2 + \omega_1)/2 \\ &= (65\ \text{rad/s} + 20\ \text{rad/s})/2 \\ &= 42.5\ \text{rad/s}.\end{aligned}$$

c.) Example 3: Our turntable is found to be at $\theta_2 = -26\ radians$ after having moved with an *average angular velocity* of -7 rad/s for five seconds. Where was the turntable at $t = 0$?

Solution: We know the *final* coordinate θ_2, the average angular velocity ω_{avg}, and the time of travel Δt:

$$\begin{aligned}(\theta_2 - \theta_1) &= \omega_{avg}\,\Delta t \\ ((-26\ \text{rad}) - \theta_1) &= (-7\ \text{rad/s})(5\ \text{sec} - 0\ \text{sec}) \\ \Rightarrow \quad \theta_1 &= +9\ \text{radians}.\end{aligned}$$

Note: Rotational motion treats signs just as translational motion does. Keep track of them in the equation and the equations will do everything needed to solve the problem.

d.) Example 4: A turntable capable of *angularly accelerating* at 12 rad/s^2 needs to be given an *initial angular velocity* if it is to rotate through a net 400 radians in 6 seconds. What must its initial *angular velocity* be?

Solution: We know the *angular acceleration*, the *angular distance traveled* ($\theta_2 - \theta_1$), and the *time* of travel. To determine the initial *angular velocity* ω_1:

$$\begin{aligned}(\theta_2 - \theta_1) &= \omega_1\,\Delta t + (1/2)\,\alpha(\Delta t)^2 \\ (400\ \text{rad} - 0) &= \omega_1(6\ \text{sec}) + (1/2)(12\ \text{rad/s}^2)(6\ \text{sec})^2 \\ \Rightarrow \quad \omega_1 &= 30.7\ \text{rad/s}.\end{aligned}$$

e.) Example 5: A turntable *angularly accelerates* from rest to 110 rad/s in 350 radians. What is its *angular acceleration*?

Solution: We know the *initial* and *final angular velocities* and the *angular distance traveled* ($\theta_2 - \theta_1$). To get the *angular acceleration*, we use:

$$(\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \alpha = [(\omega_2)^2 - (\omega_1)^2]/[2(\theta_2 - \theta_1)]$$

$$= [(110 \text{ rad/s})^2 - (0)^2]/[2(350 \text{ rad} - 0)]$$

$$= 17.29 \text{ rad/s}^2.$$

C.) A Plug for Rotational Parameters:

1.) Why rotational parameters? Why hassle with an "entirely new parallel system" when the old translational systems seem to do the job just fine? The answer is, "Simplicity!"

2.) Consider a rotating disk. Every point on the disk moves with some *translational velocity*. But as anyone who has ever played "crack the whip" knows, the farther out from the axis of rotation, the greater the translational velocity. Remember, $v_p = R_p \omega$.

3.) What is true but is not so obvious is that although the *translational velocity* of various pieces of the disk will differ, the angular velocity of each piece will be the same NO MATTER WHICH AXIS YOU CHOOSE TO MEASURE THAT ANGULAR VELOCITY ABOUT.

Confused? Consider the following two scenarios:

a.) You are sitting in a chair attached to the center of a disk. The chair is constrained to face in the same direction at all times (as the disk turns, the chair does not turn--you find you are always looking at the *Point X* shown in Figure 8.8a on the next page). The disk rotates at a constant rate through one complete revolution in, say, two seconds. What is the disk's *angular velocity* from the perspective of an axis through the *center of mass* (i.e., from where you are sitting)?

b.) The *angular velocity* will equal the *number of RADIANS through which the disk travels PER UNIT TIME.* As seen by you, one revolution is equal to 2π radians and the angular velocity is:

$$\omega_{\text{about cm}} = (2\pi)/(2 \text{ sec}) = \pi \text{ rad/sec.}$$

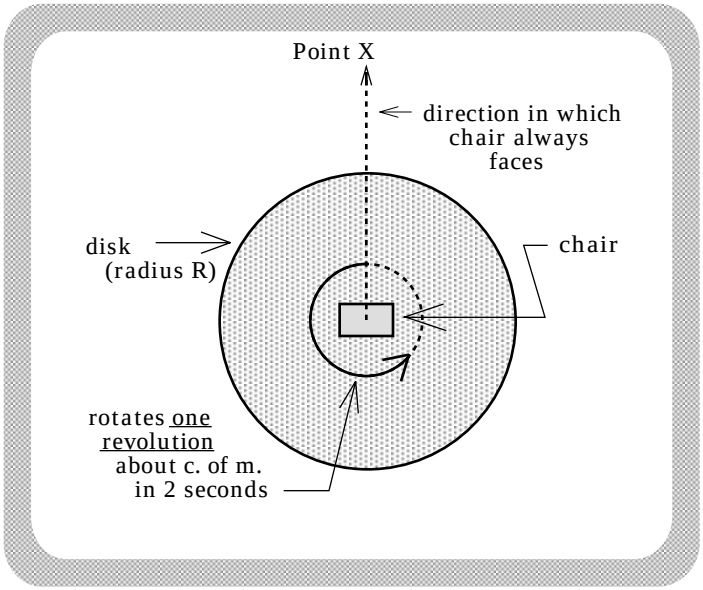


FIGURE 8.8a

c.) Your friend has a similar chair situated on the disk's perimeter (*Point P* in Figure 8.8b). While you are experiencing the rotation of the disk, he is experiencing the same rotation, with one big exception. Being completely self-involved, he assumes that all things revolve around him. So as the disk moves, *he sees its center rotating about himself* and not vice versa. The sketch in Figure 8.8b shows the situation.

From this perspective, how does the disk seem to rotate? It seems to make one complete revolution (2π radians) around *Point P* in 2 seconds. That means the disk's *angular velocity <u>about an axis through a point on the perimeter</u>* equals:

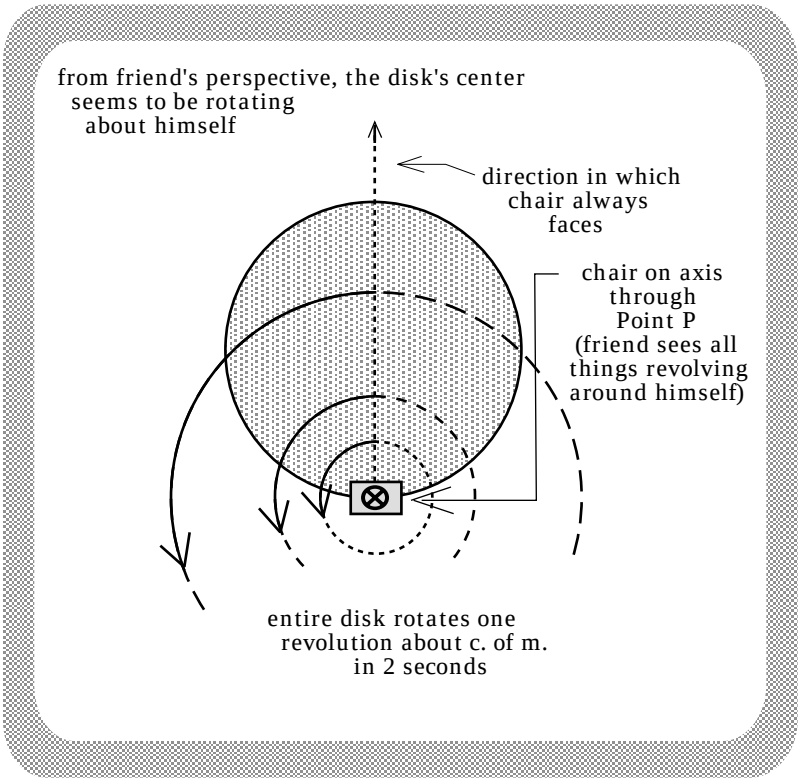

from friend's perspective, the disk's center seems to be rotating about himself
direction in which chair always faces
chair on axis through Point P (friend sees all things revolving around himself)
entire disk rotates one revolution about c. of m. in 2 seconds

FIGURE 8.8b

$$\omega_{\text{pt.P}} = (2\pi \text{ rad})/(2 \text{ sec}) \qquad (= \pi \text{ rad/sec}).$$

d.) Bottom line: The *angular velocity* of a rotating object is the same no matter what axis is used to reference the motion. The same is true of *angular acceleration* and *angular displacement*. If you know the *value of* or have *an expression for* a rotational variable about one axis at a given instant, you know that variable at that instant about *all* axes on the rotating body.

D.) Rotational Inertia (Moment of Inertia):

1.) Massive bodies have a definite tendency to resist changes in their motion. Put a truck and a feather in space, blow hard on both, and you'll find the feather is quite responsive while the truck just sits there. Why? Because the truck has more inertia--it resists changes in its motion considerably more than does the feather. The mass of a body is a quantitative measure of a body's relative tendency to resist changes in its motion. That is, saying the body has 2 kilograms of mass means that it has twice as much inertia as does a 1 kilogram mass.

2.) Rotating bodies have *rotational* inertia. That is, they tend to resist changes in their rotational motion. Rotational inertia is mass related--the more the mass, the greater the rotational inertia--but it is also related to how the mass is distributed relative to the *axis of rotation*. The more the mass is spread out away from the *axis of rotation*, the more rotational inertia. We need to determine a quantitative expression for the *rotational inertia* of a massive object. We will do so by considering the *rotational kinetic energy* of a spinning disk (you will not be expected to *reproduce* the following derivation).

3.) Consider a disk of mass M and radius R rotating about its central axis with a constant *angular velocity* w. How much *kinetic energy* is wrapped up in the disk's motion?

a.) Begin by defining the i^{th} bit of mass m_i located a distance r_i meters from the *axis of rotation* and moving with an *instantaneous translational velocity* v_i (Figure 8.9 looks down on the disk from above). The kinetic energy of that bit of mass will be:

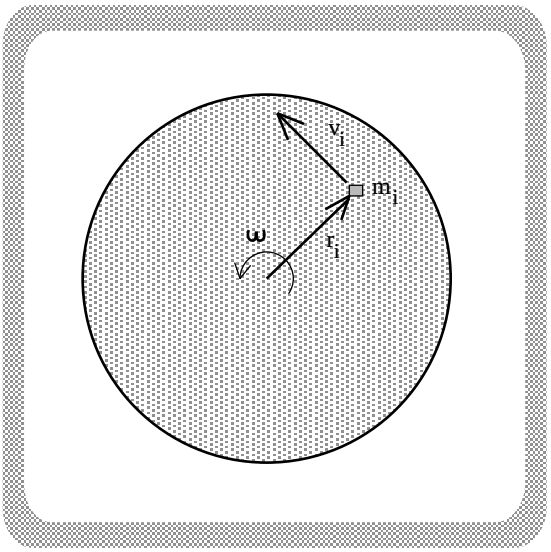


FIGURE 8.9

$$(KE)_i = (1/2)m_i v_i^2.$$

b.) The *total* kinetic energy of the entire disk will be the KE sum of all of the bits. That is:

$$KE = \sum (KE)_i = \sum [(1/2)m_i v_i^2].$$

c.) As we are dealing with a rotating body, it might be useful to incorporate our rotational parameters into the above expression. Remembering that the *translational velocity* v of an object moving with *angular velocity* ω a distance R units from the axis of rotation is $v = R\omega$, m_i*'s* velocity in rotational parameters must be $v_i = r_i\omega$. Substituting into the kinetic energy equation yields:

$$\begin{aligned} KE &= \sum [(1/2)m_i(r_i\omega)^2] \\ &= 1/2\ [\sum m_i r_i^2]\,\omega^2. \end{aligned}$$

d.) This last expression is interesting because it looks very much like a kinetic energy equation, rotational style. That is, normal kinetic energy has an equation that looks like $(1/2)m_i v_i^2$. We would expect a rotational version to have the same form: 1/2 times some *mass related term* times the *angular velocity* ω squared. That is exactly what we have above.

e.) Notice that the mass related term in the expression above is equal to $\sum m_i r_i^2$. This term is called *the* <u>*moment of inertia*</u> of the body (in this case, about one of its central axes). Its symbol is I and its size provides us with a relative measure of the body's *rotational inertia.*

4.) The *moment of inertia* equation derived above is for systems of individual, discrete masses. Examples follow:

a.) Example 1: Consider two 3 kg masses connected by a very light (read that "massless") bar of length 4 meters (see Figure 8.10). Determine the *moment of inertia* of the system about an axis through the system's *center of mass.*

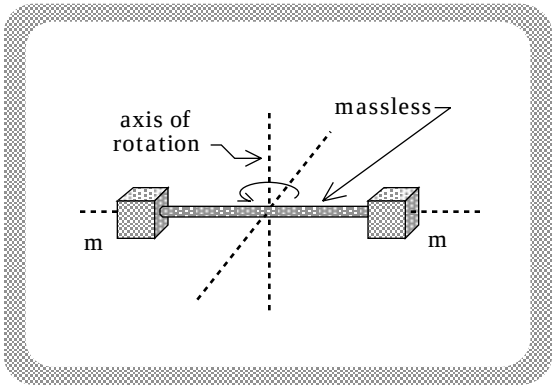


FIGURE 8.10

Solution:

$$I = \sum m_i r_i^2$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$= (3\text{ kg})(2\text{ m})^2 + (3\text{ kg})(2\text{ m})^2$$
$$= 24\text{ kg}\cdot\text{m}^2.$$

b.) Example 2: Using the *bar and mass* set-up presented in Example 1 above, determine the *moment of inertia* for the system about an axis through *one of the masses* (this axis is denoted in Figure 8.11).

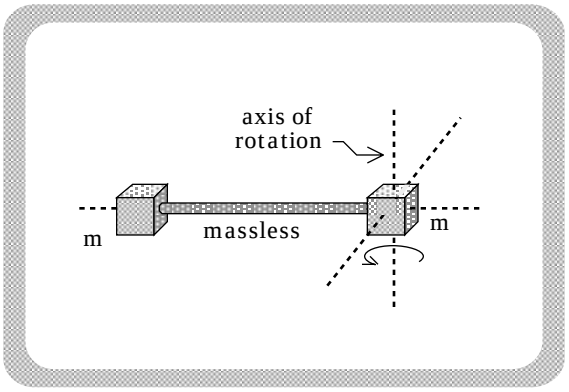


FIGURE 8.11

Solution:

$$I = \sum m_i r_i^2$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$= (3\text{ kg})(0\text{ m})^2 + (3\text{ kg})\ (4\text{ m})^2$$
$$= 48\text{ kg}\cdot\text{m}^2.$$

c.) Example 3: Consider the same set-up as above. Determine the *moment of inertia* about an axis located 2 meters to the right of the right-most mass (see Figure 8.12). Note that the massless rod would have to be extended out to the right to accommodate such a situation.

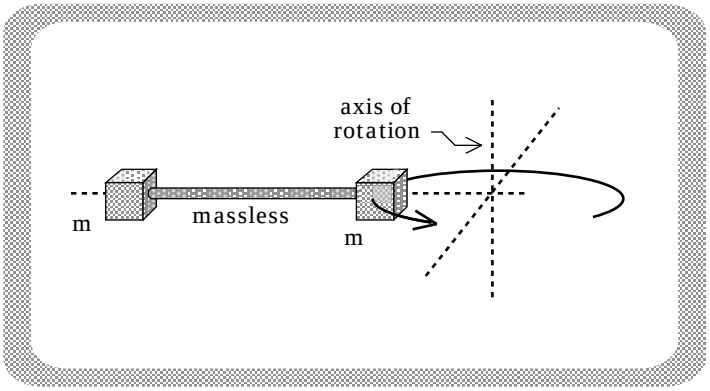


FIGURE 8.12

Solution:

$$I = \sum m_i r_i^2.$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$= (3\text{ kg})(2\text{ m})^2 + (3\text{ kg})(6\text{ m})^2$$
$$= 120\text{ kg}\cdot\text{m}^2.$$

5.) As the *axis of rotation* moves farther and farther from the *center of mass*, the moment of inertia increases. In fact, the moment of inertia will always be a *minimum* about an axis through the *center of mass*. There is a formula that allows one to determine the *moment of inertia* about any axis *parallel* to an axis

through the *center of mass*. Called "the PARALLEL AXIS THEOREM," it states that the *moment of inertia* about any axis *P* is:

$$I_p = I_{cm} + Md^2,$$

where I_{cm} is the known *moment of inertia* about a *center of mass axis* parallel to *P*, *M* is the total mass in the system, and *d* is the *distance* between the two parallel axes.

a.) Example 4: In Example 1 above, we calculated the *moment of inertia* about the *center of mass* of our *bar and masses* system. Using the *parallel axis theorem*, determine the moment of inertia about an axis through one of the masses (Figure 8.11).

Solution:

$$\begin{aligned} I_p &= I_{cm} + Md^2, \\ &= (24\ \text{kg}\cdot\text{m}^2) + (6\ \text{kg})(2\ \text{m})^2 \\ &= 48\ \text{kg}\cdot\text{m}^2. \end{aligned}$$

Again, this is exactly the value calculated in Example 2.

b.) Example 5: Determine the *moment of inertia* for our *bar and masses* system about an axis 2 meters to the right of the right-most mass (Figure 8.12):

$$\begin{aligned} I_p &= I_{cm} + Md^2, \\ &= (24\ \text{kg}\cdot\text{m}^2) + (6\ \text{kg})(4\ \text{m})^2 \\ &= 120\ \text{kg}\cdot\text{m}^2. \end{aligned}$$

This is exactly the value calculated in Example 3.

6.) THINK ABOUT WHAT THE MATHEMATICAL OPERATION WE HAVE BEEN EXAMINING IS ACTUALLY ASKING YOU TO DO. It says, *move out a distance "r" units from the axis of interest. If there is mass located at that distance out, multiply that mass quantity by the square of the distance "r." Do this for all possible "r" values, then sum.*

The general *moment of inertia* equation derived above works fine for systems involving groups of individual masses, but it would be cumbersome for continuous masses like the disk with which we began. To determine the *moment of inertia* for structures whose mass is extended out over a continuous volume requires Calculus. Specifically, we must solve the integral:

$$I = \int r^2 dm,$$

where *dm* is the mass found a distance *r* units from the axis of choice.

The approach for solving this integral is similar to the approach used when de-termining *center of mass* quantities for extended objects in the last chapter.

a.) Example: Determine the *moment of inertia* about an axis through the center of mass of a flat, homogeneous disk of mass *m* and radius *R* (assume the axis is perpendicular to the face of the disk).

b.) Figure 8.13 shows a hoop of differential mass *dm* with differential thickness *dr* drawn a distance *r* units from the disk's center of mass.

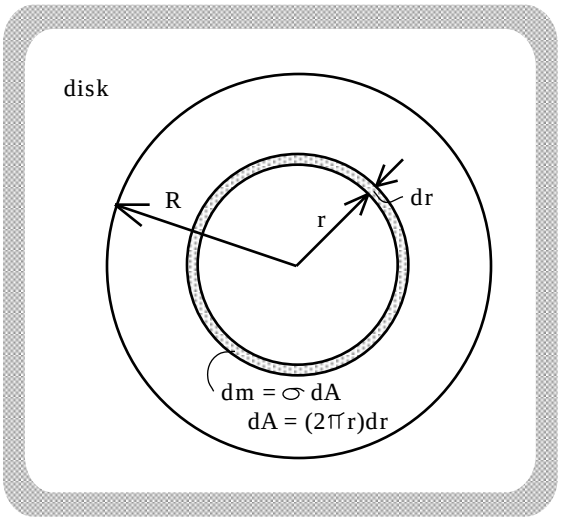


FIGURE 8.13

c.) We can define a *mass per unit area* function σ in two ways: macroscopically and microscopically.

i.) Macroscopically (i.e., as a whole):

$$\sigma = M/[(\pi R^2)].$$

ii.) Microscopically:

$$\sigma = dm/dA,$$

where *dm* is the differential mass within the differential hoop of differential area *dA*. With this, we can write:

$$dm = \sigma dA.$$

iii.) Looking at Figure 8.13, we can also write:

$$\begin{aligned} dA &= (\text{circumference of hoop})(\text{hoop thickness}) \\ &= (2\pi r)\, dr. \end{aligned}$$

iv.) That means:

$$dm = \sigma dA$$
$$= \sigma[(2\pi r)dr].$$

d.) We are now ready to evaluate the *moment of inertia* integral:

$$I = \int r^2 dm$$
$$= \int_{r=0}^{R} r^2[\sigma(2\pi r)dr].$$

Substituting $\sigma = M/[(\pi R^2)]$ and pulling out the constants yields:

$$I = \frac{2\pi M}{\pi R^2}\int_{r=0}^{R} r^3 dr$$
$$= \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_{r=0}^{R}$$
$$= \frac{2M}{R^2}\left[\frac{R^4}{4} - \frac{0^4}{4}\right]$$
$$= \frac{1}{2}MR^2.$$

This is the moment of inertia of a solid disk about an axis perpendicular to the disk's face and through its center of mass.

e.) There is another way to do this integral. Instead of defining an *area density function* σ related to the mass *behind* a given area on the disk's face, we can define a *volume density function* ρ related to the amount of mass that is actually wrapped up in that volume.

This density function can be expressed in two ways:

i.) Macroscopically, the *volume density* can be written as the total mass divided by the total volume, or:

$$\rho = M/[\pi R^2 t],$$

where M is the disk's total mass, πR^2 is the area of the disk's face, and t is the thickness of the disk.

ii.) Microscopically, the *volume density* can be written as the differential mass divided by the differential volume it occupies, or:

$$\rho = dm/dV,$$

where *dm* is the differential mass wrapped up in the cylindrical shell (a hoop with depth) and *dV* is the differential volume of that shell. Taking the differential face-area $2\pi r dr$ times the hoop thickness *t*, we get, $dV = 2\pi rt(dr)$. Putting everything together:

$$\begin{aligned}\rho &= (dm)/(dV)\\ &= (dm)/[2\pi rt(dr)].\end{aligned}$$

iii.) Substituting in for ρ (i.e., $\rho = M/\pi R^2 t$) and solving for *dm*, we get:

$$\begin{aligned}dm &= \rho\, dV \qquad\qquad \text{(Equation A)}\\ &= \rho\,[2\pi rt(dr)]\\ &= [M/(\pi R^2 t)][2\pi rt(dr)]\\ &= (2Mr/R^2)dr.\end{aligned}$$

This leaves us with the integral:

$$I = \frac{2M}{R^2}\int_{r=0}^{R} r^3 dr,$$

which is the same integral we found when we used the *surface-area* approach.

Bottom Line: When dealing with a symmetrically shaped, homogeneous structure, either a volume density or area density function will work.

7.) Another example: Determine the *moment of inertia* of a rod of length *L* and mass *M* about an axis through the rod's center of mass and perpendicular to the rod's length.

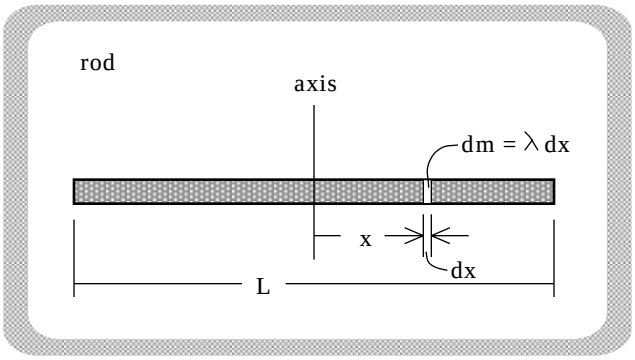


FIGURE 8.14

a.) Figure 8.14 shows a differential mass *dm* within a differential length *dx* located a distance *x* units from the rod's center of mass.

b.) Note that we could have defined either a *volume* or *area density function* for this structure. The reason we didn't? The mass is essentially distributed LINEARLY out from the axis of interest. As such, it is easier to define a *linear density function* λ (i.e., a function that tells you the *mass*

per unit length along the rod). From such a function, we can determine an expression for *dm*.

c.) As before, the linear density function λ can be expressed in two ways: microscopically and macroscopically.

i.) Looking at the entire structure macroscopically, the *linear density* can be written as:

$$\lambda = M/L,$$

where *M* is the rod's total mass and *L* is its total length.

ii.) Looking microscopically, the *linear density* can be written as:

$$\lambda = dm/dx,$$

where *dm* is the differential mass wrapped up in the differential length *dx*. Putting everything together, we get:

$$\begin{aligned} dm &= \lambda\, dx \\ &= (M/L)dx. \end{aligned}$$

iii.) Letting *r* become *x*, the *moment of inertia* integral yields:

$$\begin{aligned} I &= \int r^2 dm \\ &= \int_{x=-L/2}^{L/2} x^2 \left[\frac{M}{L}dx\right] \\ &= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} \\ &= \frac{M}{L}\left[\frac{(L/2)^3}{3} - \frac{(-L/2)^3}{3}\right] \\ &= \frac{M}{L}\left[\left(\frac{L^3}{24}\right) + \left(\frac{L^3}{24}\right)\right] \\ &= \frac{1}{12}ML^2. \end{aligned}$$

This is the *moment of inertia* about an axis perpendicular to, and through the center of mass of, a rod of mass M and length L.

Note: We could have used a *volume* or *area density function* on this problem and all would have come out the same. The extra variables needed to define these functions would have canceled out in the integral, just as was the case in the earlier example.

Bottom line: It really doesn't matter which kind of function you use. You will end up with the same *moment of inertia* integral with each of the approaches, assuming you do each correctly.

8.) A list of commonly used *moment of inertia* expressions is provided in the table shown on the next page.

Note: Do NOT memorize the *moment of inertia* expressions found in the chart. They are provided for convenience only. If a *moment of inertia* term is required on your next test, it will either be provided or you will be asked specifically to derive it.

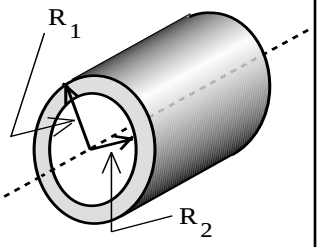
9.) Parting shot: General definitions, the concept of *moment of inertia*, and the *rotational kinematic equations* all have their place in your understanding of rotational motion. Nevertheless, the real players-with-power are the rotational versions of *Newton's Second Law*, *energy considerations*, and the concept of *angular momentum*. Although the chapter you are now completing is important in the sense that it presents background, the warm-up is over. The next chapter will get to the vegetables of the matter.

NOTE: SEE NEXT PAGE FOR *MOMENT OF INERTIA* CHART.

MOMENT OF INERTIA EXPRESSIONS

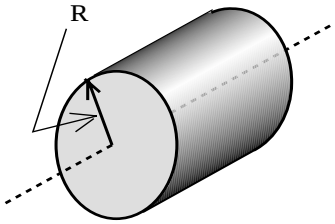
FOR VARIOUS FORMS

Ring or Annular Cylinder about central axis



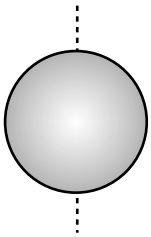
$I = (1/2)\, M(R_1^2 + R_2^2)$

Solid cylinder (or disk) about cylinder's central axis



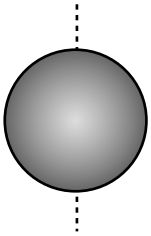
$I = (1/2)\, MR^2$

Thin spherical shell about any central axis



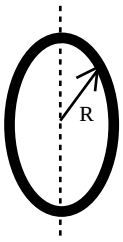
$I = (2/3)\, MR^2$

Solid sphere about any central axis



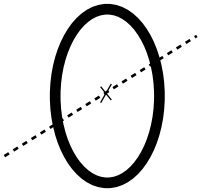
$I = (2/5)\, MR^2$

Hoop about diameter



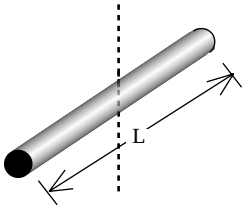
$I = (1/2)\, MR^2$

Hoop about central axis



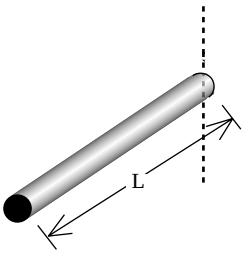
$I = MR^2$

Thin rod about axis through rod's center and perpendicular to central axis



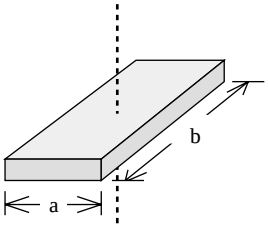
$I = (1/12)\, ML^2$

Thin rod about axis at rod's end and perpendicular to central axis



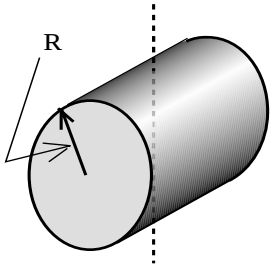
$I = (1/3)\, ML^2$

Slab about axis through center and perpendicular to slab's face



$I = (1/12)\, M(a^2 + b^2)$

Disk or Solid Cylinder about central diameter



$I = (1/4)\, MR^2 + (1/12)\, ML^2$

QUESTIONS

8.1) Note: Although this question is somewhat complex, it has been included to test your ability to determine *moment of inertia* quantities for complex structures. Try it, but don't spend hordes of time on it . . . and if you get it set up but can't do the integral, don't worry about it.

A half disk has a radius R and area density ky, where k is a constant having the appropriate units and a magnitude equal to *one*. Determine the *moment of inertia* for the body:

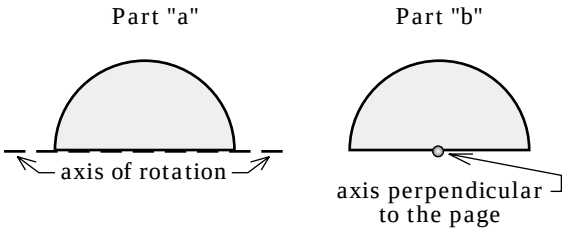


FIGURE I

a.) About a horizontal axis running along the bottom of the hemisphere (i.e., along the diameter), and

b.) About an axis perpendicular to the face and through the center of the circle defining the hemisphere's arc.

8.2) An automobile whose wheel radius is .3 meters moves at 54 km/hr. The car applies its brakes uniformly, slowing to 4 m/s over a 50 meter distance.

a.) Show that 54 km/hr is equal to 15 m/s.

b.) Show that a 15 m/s *car speed* corresponds to a *wheel angular velocity* of 50 radians/second and that 4 m/s corresponds to 13.33 rad/sec.

c.) Show that a *translational displacement* of 50 meters corresponds to a *wheel angular displacement* of 166.7 radians.

d.) Using rotational kinematics, determine the *angular acceleration* of one wheel.

e.) Using the information gleaned in *Part d*, determine the *translational acceleration* of the car.

f.) Knowing $\Delta\theta$, use rotational kinematics to determine the *time interval* required for the execution of the slow-down.

g.) Knowing ω_2, use rotational kinematics to determine the *time interval* required for the slow-down. Does your solution match the one determined in *Part f*?

h.) Determine the *average angular velocity* of one wheel during the slow-down.

i.) Using rotational kinematics, determine the *angular displacement* of the wheels during the first .5 seconds of the slow-down.

j.) Determine *how far* the car traveled during the first .5 seconds of the slow-down.

k.) Without using the time interval, determine the *wheel's angular velocity* after the first .5 seconds of the slow-down.

l.) Determine the angular displacement of one wheel between times $t = .5$ *seconds* and $t = .7$ *seconds*.

m.) Once the car has slowed to 4 m/s, it begins to pick up speed. Over a 3 second period, it reaches a *wheel angular velocity* of 20 rad/sec. Using rotational kinematics, determine *how far* the car will move during that time period.

8.3) The earth has a *mass* of 5.98×10^{24} kilograms, a *period* of approximately 24 hours (the *period* is the time required for one rotation about its axis), and a radius of 6.37×10^{6} meters.

a.) What is the earth's *angular velocity*?

b.) What is the *translational velocity* of a point on the equator?

c.) What is the *translational velocity* of a point on the earth's surface located at an angle 60^{o} relative to a line from its center through its equator?

d.) Assuming it is a solid, homogeneous sphere, what is the earth's *moment of inertia* about its axis?

Chapter 9

ROTATIONAL MOTION -- PART II

A.) Torque:

1.) So far, we have developed rotational counterparts for displacement, velocity, acceleration, and mass. It is now time to consider the rotational counterpart to force.

When a net force is applied to an object, the object accelerates (Newton's Second Law). *Torque* is the rotational counterpart to *force* in the sense that when a net torque is applied to a body, the body *angularly* accelerates. While force is applied *in a particular direction*, torque is applied *about a point* (the point of interest is usually on the body's axis of rotation). Torque calculations were briefly discussed in Chapter 1 (the idea of a torque was used there as an example of a vector *cross product* operation). We will go into more depth here.

2.) The easiest way to understand the concept of a torque is with an example.

a.) A force ***F*** is applied to a wrench a distance ***r*** from the axis of rotation (see Figure 9.1). From experience, it should be obvious that:

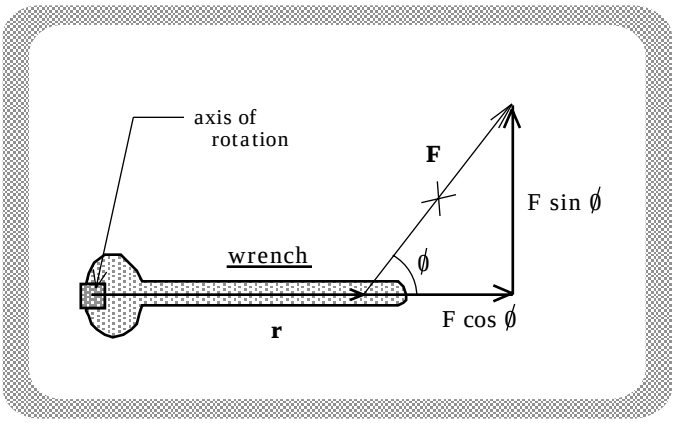


FIGURE 9.1

i.) The greater $|\mathbf{r}|$ is, the less difficult it is to *angularly accelerate* the bolt;

ii.) The greater $|\mathbf{F}|$ is, the less difficult it is to *angularly accelerate* the bolt; and

iii.) The *force component* that makes the bolt angularly accelerate is the component *perpendicular* to the line of ***r*** (i.e., $|\mathbf{F}| \sin \phi$).

b.) As *ease of rotation* is related to $|\boldsymbol{r}|$ and $|\mathbf{F}|sin\ \phi$, the product of those two variables is deemed important enough to be given a special name--*torque* ($\boldsymbol{\Gamma}$). In short, the *magnitude* of the *torque* applied by $\boldsymbol{F}$ about some point will be $|\boldsymbol{\Gamma}| = |\boldsymbol{r}x\boldsymbol{F}|$. As a *vector*, torque is defined as:

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}.$$

Note: It is not unusual to find physics texts making statements like, "a force $\boldsymbol{F}$ applies a torque about the axis of rotation." This can be confusing because, by definition, torques are not applied about axes--they are applied about points. A more accurate way to make the statement would be to say, "a force $\boldsymbol{F}$ applies a torque about a point that is both in the plane of the paper and on the axis of rotation." Unfortunately, although this is technically correct, it is also wordy and cumbersome. As a consequence, physicists shorten such statements to, "a force $\boldsymbol{F}$ applies a torque about the axis of rotation."

There is nothing wrong with this shorthand description as long as you understand the assumption being made when torque calculations are termed this way.

Bottom line: From here on out, you will be expected to know what "take the torque about the axis of rotation" means.

3.) In the first chapter we found that a *cross product* is a vector manipulation involving two vectors (say $\boldsymbol{r}$ and $\boldsymbol{F}$). It generates a third vector whose *magnitude* is numerically equal to the product of:

a.) The *magnitude* of one of the vectors (say $|\boldsymbol{r}|$ in this case), and

b.) The *magnitude* of the second-vector's-component that runs *perpendicular* to the first vector (in this case, $|\boldsymbol{F}|sin\ \phi$).

c.) Assuming the vector information is in polar notation, the *magnitude* of the torque calculation will be the *magnitude* of a *cross product*, or:

$$\begin{aligned} |\boldsymbol{\Gamma}_F| &= |\mathbf{r}\mathrm{x}\mathbf{F}| \\ &= |\mathbf{r}|\ |\mathbf{F}| \sin \phi, \end{aligned}$$

where ϕ is the angle between the *line of* $\boldsymbol{r}$ and the *line of* $\boldsymbol{F}$.

4.) The *direction* of the *cross product* is *perpendicular* to the plane defined by the two vectors. In the case of a torque produced by an ***r*** and ***F*** vector in the *x-y plane*, this direction is along the *z axis* in the ***k*** *direction*. That is fortunate. Remembering that the direction of an *angular velocity* and *angular acceleration* vector is along the axis of rotation, a torque that makes an object rotate in the *x-y plane* should have a direction perpendicular to the *x-y plane* (i.e., in the direction of the axis of rotation about which the angular acceleration is taking place). That is exactly the direction the *cross product* gives us.

5.) There are three ways to calculate a *cross product* and, hence, a torque. All three will be presented below in the context of the following problem: A 10 newton force is applied at a 60^o angle to a giant wrench 3 meters from the axis of rotation (see Figure 9.2). How much torque does the force apply about a point on the central axis of the bolt (i.e., on the axis of rotation) in the plane of the wrench?

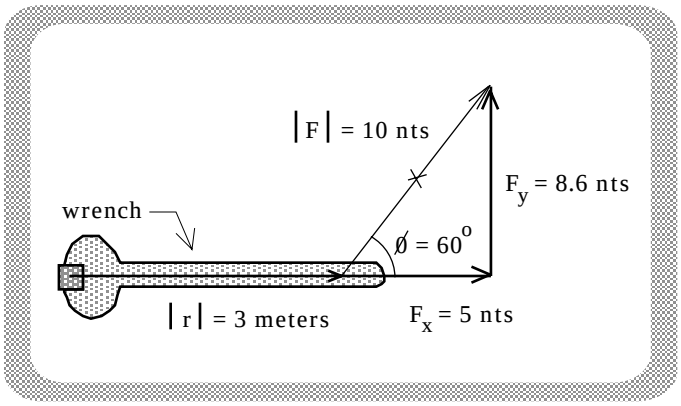


FIGURE 9.2

a.) The "definition" approach: Take the definition of a cross product and apply it to the situation. Doing so yields:

$$\begin{aligned} |\boldsymbol{\Gamma}_F| &= |\mathbf{r}\text{x}\mathbf{F}| \\ &= |\mathbf{r}| \quad |\mathbf{F}| \quad \sin\phi, \\ &= (3\text{ m})\,(10\text{ nts})\sin 60^o \\ &= 25.98\text{ nt}\cdot\text{m}. \end{aligned}$$

The direction is determined using the right-hand rule. Doing so yields a ***+k*** *direction*. As our rotation is one-dimensional (i.e., there is only one axis about which the rotation occurs) in the *x-y plane*, we don't need to include the ***k*** *unit vector*. We do need to include the "+" sign (it tells us that the torque will attempt to *angularly accelerate* the object in the *counterclockwise* direction). The end result is, therefore:

$$\Gamma_F = +25.98\text{ nt}\cdot\text{m}.$$

b.) The "$r_\perp$" approach: We know that the *magnitude* of a cross product is equal to the *magnitude of one vector* times the *perpendicular*

component of the second vector (i.e., the component of the second vector perpendicular to the line of the first vector). If we let ***F*** be the first vector, the "perpendicular component of the second vector" will be the component of ***r*** perpendicular to the *line of* ***F***. Calling this term $r_\perp$, we have:

$$\left|\boldsymbol{\Gamma}_F\right| = \left(r_\perp\right)|\mathbf{F}|.$$

Note 1: This approach is so commonly used that most texts give $r_\perp$ a special name. They call it *the moment arm*. Using that term, we write, "the *torque* about *Point P* is equal to the *force* times the *moment arm* about *Point P*."

Why is the $r_\perp$ *approach* used so often? Read Note 2!

Note 2: Physically, $r_\perp$ is the *shortest distance* between the *point about which the torque is being taken* (usually on the axis of rotation) and the *line of the force*. As it is often easy to determine the *shortest distance* between a *point* and a *line*, this method of calculating torques is very popular.

Note 3: Having extolled the virtues of the $r_\perp$ *approach*, it should be pointed out that in this particular problem, the easiest way to determine the torque is by using either the *definition approach* or the approach that will be presented last. Be that as it may, $r_\perp$ is what we are concerned with here!

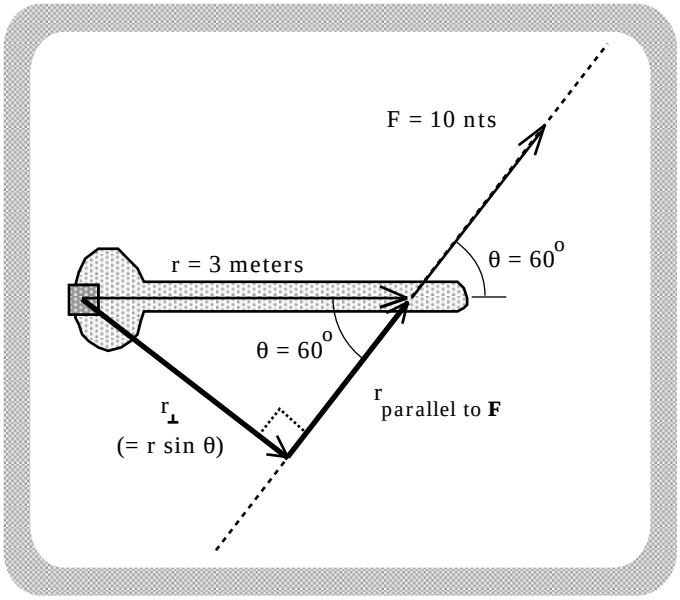


FIGURE 9.3

CONTINUING: Consider Figure 9.3. The line of ***F*** has been extended in both directions, allowing us to see the shortest distance between "the *axis of rotation* and the *line-of-the-force*" (i.e., $r_\perp$). With that and a little geometry, we find that:

$$|\boldsymbol{\Gamma}_F| = |\mathbf{r}\text{x}\mathbf{F}|$$
$$= (r_\perp) |\mathbf{F}|$$
$$= [(3\text{ m})(\sin 60^o)]\ (10\text{ nts})$$
$$= 25.98\text{ nt·m}.$$

Note: The *direction* is determined using either the right-hand rule or your knowledge about *clockwise* versus *counterclockwise* rotations. The final solution is +25.98 nt·m.

c.) The "$F_\perp$" approach: We know that the *magnitude* of a cross product is equal to the *magnitude of one vector* times the *perpendicular component of the second vector* (i.e., the component of the second vector perpendicular to the line of the first vector). If we let ***r*** be the first vector, the "perpendicular component of the second vector" will be the component of ***F*** that is perpendicular to the *line of* ***r***. Calling this component $F_\perp$, we have:

$$|\boldsymbol{\Gamma}_F| = (F_\perp)|\mathbf{r}|.$$

Note 1: This is the flip-side of the $r_\perp$ *approach* and it works in approximately the same way. Extend the line of ***r*** until you can see the component of ***F*** perpendicular to that line. With that information, you simply multiply the magnitude of ***r*** by $F_\perp$.

Note 2: This approach is most useful whenever you are *given* the force component perpendicular-to-the-line-of-**r**. Our problem is a good example of such a situation. The vector ***r*** is in the *x direction*. We know $F_\perp$ because it was given to us (look back at Figure 9.2). With that unit vector information, the $F_\perp$ *approach* falls out nicely.

Bottom line: If you are given information in *unit vector notation*, think $F_\perp$ *approach*. It won't always work, but when it does it will work easily.

Continuing: As $F_\perp$ is F_y, we can write:

$$|\boldsymbol{\Gamma}_F| = |\mathbf{r}\text{x}\mathbf{F}|$$
$$= (F_\perp) |\mathbf{r}|$$
$$= [8.6\text{ nts}]\ (3\text{ m})$$
$$= 25.8\text{ nt·m}.$$

Note: The direction is determined using either the right-hand rule or your knowledge about *clockwise* versus *counterclockwise* rotations. The final solution is +25.98 nt·m.

d.) Even though the $r_\perp$ *approach* is often used, there is really no one approach that is better than any other. For some problems, the $r_\perp$ approach is a horror. KNOW THEM ALL. It's better to have a choice than to get hung with a problem that doesn't seem to easily work out using the only approach you have learned!

B.) Rigid Body Equilibrium Problems:

1.) There are two kinds of <u>*equilibrium*</u>: dynamic equilibrium in which a body is moving but not accelerating, and static equilibrium in which a body is at rest and not accelerating. The common denominator is *no acceleration*.

Put another way, if one has equilibrium:

a.) The *sum of the forces* acting in the *x direction* must add to zero (if that weren't the case, we would see *translational* acceleration in the *x-direction*);

b.) The *sum of the forces* in the *y direction* must add to zero; and

c.) The *sum of the torques* acting about <u>any point</u> must add to zero (if that weren't the case, we would see *angular* acceleration).

2.) Example: A ladder of length L is positioned against a wall. The wall is *frictionless* and the floor is *frictional*. A man of mass m_m stands on the ladder a distance $L/3$ from the top. If the ladder meets the floor at an angle θ with the horizontal, and if the ladder's mass is m_L, determine the forces acting at the floor and the wall. See Figure 9.4.

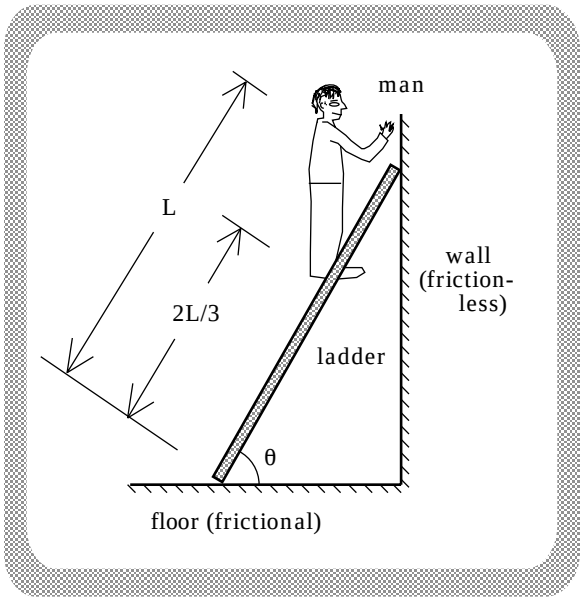


FIGURE 9.4

a.) Preliminary Comment #1: In looking at the ladder's contact with the floor, it should be obvious that there is both a *normal* and a *frictional*

force acting at that point. Assume, for the moment, that that fact is *not* obvious.

In that case, all we know is that the floor must be providing a *net force* F_{floor} acting at some unknown angle ϕ (note that ϕ is not θ here). From experience, we know that unknown forces are easy to deal with, but the math can get dicey when unknown angles are injected into a situation (angles are usually attached to *sine* and *cosine* functions which can make solving simultaneous equations difficult). It would be nice if we could deal with a *force-at-the-floor* problem without having to deal with the unknown angle.

That can be cleverly done by noticing that the *force-at-the-floor* must have *x* and *y components*. Calling the *horizontal component H* and the *vertical component V*, we can solve for those variables. We will still have two unknowns (*H* and *V* versus F_{floor} and ϕ) but we will have traded off for a tidier problem.

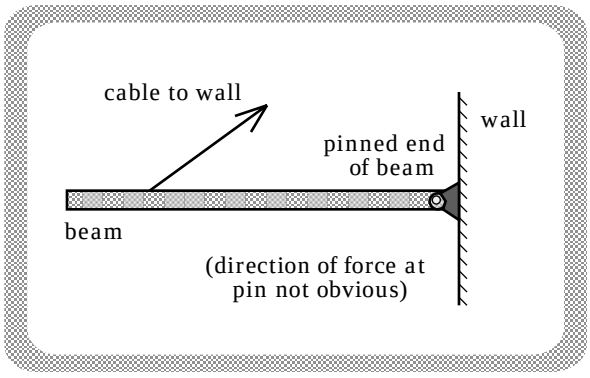


FIGURE 9.5

Note: Another example of a situation in which this ploy will be useful: Consider a lab comprised of a *beam* pinned so as to rotate about an axis at its end (see Figure 9.5). You know *absolutely nothing* about the magnitude and direction of the force acting on the beam at the pin, but you are asked to theoretically determine what that force and angle should be under the circumstances embodied within the set-up. This is a prime example of a situation in which solving for the *components* is preferable to hassling with the actual force vector and its angle.

b.) Preliminary Comment #2: Because the wall is frictionless, the force acting at the wall is strictly a *normal force*. As such, we will call that force *N*. Note also that the ladder's weight $m_L g$ is concentrated at the ladder's *center of mass* at *L*/*2*. This is all shown in the *free body diagram* presented in Figure 9.6.

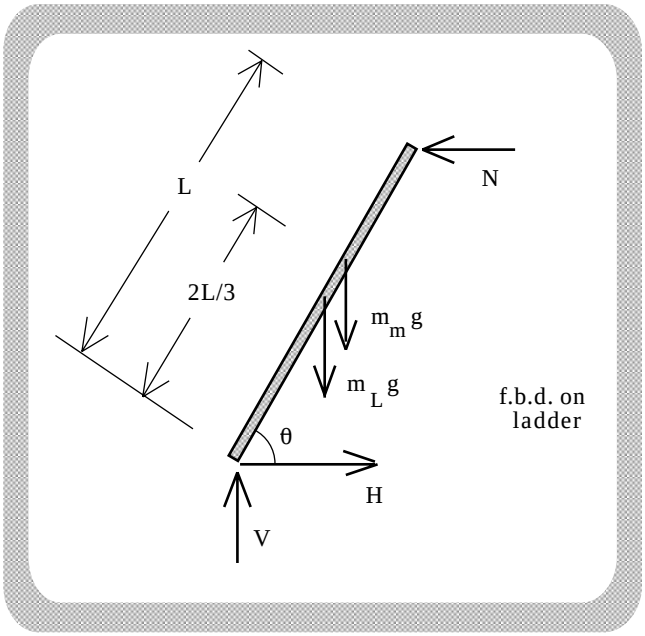


FIGURE 9.6

Solution:

c.) All the acting forces are in the *x* and *y directions*, so there is no need to worry about breaking forces into their component parts. We begin by writing:

$\underline{\sum F_x}$:

$$-N + H = (m_m + m_L)\, a_x \qquad (= 0 \text{ as } a_x = 0)$$
$$\Rightarrow \quad N = H.$$

$\underline{\sum F_y}$:

$$-m_m g - m_L g + V = (m_m + m_L)\, a_y \qquad (= 0 \text{ as } a_y = 0)$$
$$\Rightarrow \quad V = (m_m + m_L)\, g.$$

d.) We have three unknowns and two equations. The final equation comes from *summing the torques* about any point we choose. For the sake of amusement, let's choose the ladder's center of mass. Using $r_\perp$:

$\underline{\sum \Gamma_{cm}}$:

$$N\,[(L/2)\sin\theta] - m_m g\,[(L/6)\cos\theta] - V\,[(L/2)\cos\theta] + H\,[(L/2)\sin\theta] = I_{cm}\alpha$$
$$= 0 \text{ as } \alpha = 0$$

$$\Rightarrow \quad N = \left[(m_m g/6)\cos\theta + (V/2)\cos\theta - (H/2)\sin\theta\right]/\left[(1/2)\sin\theta\right]$$

Note 1: The equation we have generated above has four torque calculations instead of five--the torque due to the weight of the beam produces no torque about the *center of mass* as that force acts *through* the center of mass. *Forces that act through the point about which the torque is being taken produce no torque about that point.*

Note 2: This last equation is comprised of *three* unknowns. To solve it, we will have to go back to our original two equations, lift the derived values for *V* and *H* (in terms of *N*) and plug them into this last equation. The end result will be a very messy equation to solve. Once *N* is found, we will then have to go back, plug *N* into the *H* and *V* equations and solve some more.

It would have been so much easier to have generated a "final" equation that had only one unknown in it (say, *N*). *We could have done just that if we had summed the torques about the floor!*

Doing so yields:

$\underline{\sum \Gamma_{floor}}$:

$$N\,[L \sin\theta] - m_m g\,[(2L/3)\cos\theta] - m_L g\,[(L/2)\cos\theta] = I_{floor}\,\alpha = 0$$
$$\Rightarrow \quad N = [(m_m g)(2/3)\cos\theta + (m_L g)(1/2)\cos\theta] / \sin\theta.$$

This is a smaller equation (you had to do torque calculations for only three forces) and has only *one* unknown.

Bottom Line: Take your torques about whichever point will eliminate as many unknowns as possible (assuming you don't eliminate them all).

C.) Rotational Analog to Newton's Second Law:

1.) Just as a *net force* motivates a body to *translationally accelerate*, a *torque* motivates a body to *angularly accelerate*. For translational motion, Newton's Second Law states that the sum of the forces acting in a particular direction will equal the *mass* of the object times the object's *acceleration.* Mathematically, this takes the form:

$\underline{\sum F_x}$:

$$(F_{1,x}) \pm (F_{2,x}) \pm (F_{3,x}) \pm \ldots = \pm m a_x.$$

For rotational motion, Newton's Second Law states that the *sum of the torques* acting about any point must equal the *moment of inertia* (the mass-related rotational inertia term) about an appropriate axis through that point times the object's *angular acceleration*. Mathematically, this looks like:

$\underline{\sum \Gamma_p}$:

$$(\Gamma_{F_1}) \pm (\Gamma_{F_2}) \pm (\Gamma_{F_3}) \pm \ldots = \pm I_p\,\alpha$$

The easiest way to see the consequences of this law is by using it in a problem.

2.) Example: Determine the angular acceleration α of the beam shown in Figure 9.7a (next page). Assume you know its length *L*, its mass *m*, and the fact that the moment of inertia of a beam about its *center of mass* is $(1/12)mL^2$.

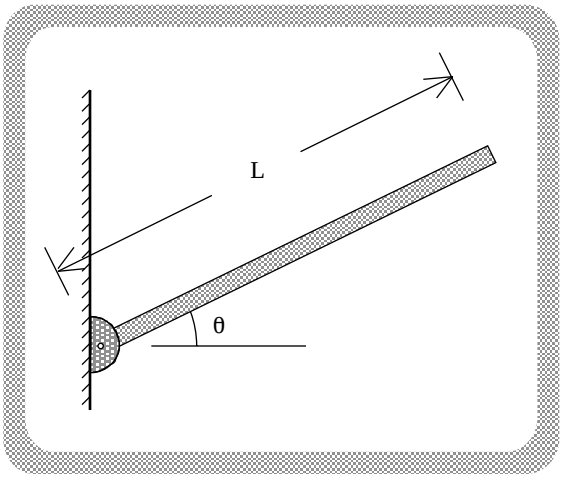


FIGURE 9.7a

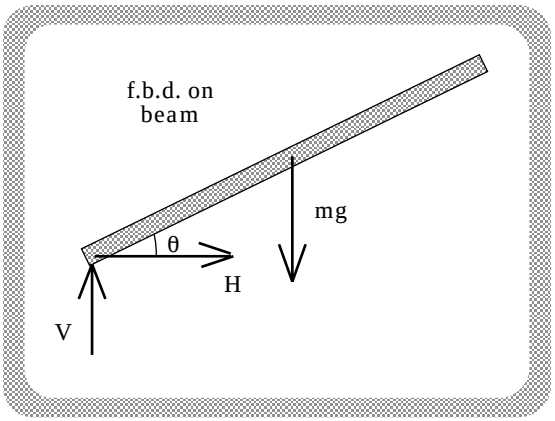


FIGURE 9.7b

a.) Using the f.b.d. shown in Figure 9.7b and the Parallel Axis Theorem, the sum of the torques about the axis of rotation (i.e., the pin) is:

$$\underline{\Sigma \Gamma_{pin}}:$$

$$-mg\,(L/2)\cos\theta = -\,I_{pin}\,\alpha$$

$$= -\,[\quad I_{cm} \quad + \quad md^2 \quad]\,\alpha$$

$$= -\,[(1/12)\,mL^2 + m(L/2)^2]\,\alpha$$

$$\Rightarrow \quad \alpha = 3\,(g\cos\theta)/(2L).$$

b.) Notice that the *angular acceleration* is a function of the beam's angular position θ. As θ changes while the beam rotates, the *angular acceleration* changes. Conclusion: If you had been asked to determine, say, the *angular velocity* of the beam at some later point in time, you would NOT be able to use rotational kinematics to solve the problem.

D.) Rotation And Translation Together--Newton's Second Law:

1.) Determine the *acceleration* of the hanging mass shown in Figure 9.8 if it is released and allowed to accelerate freely. Assume you know the mass of the hanging

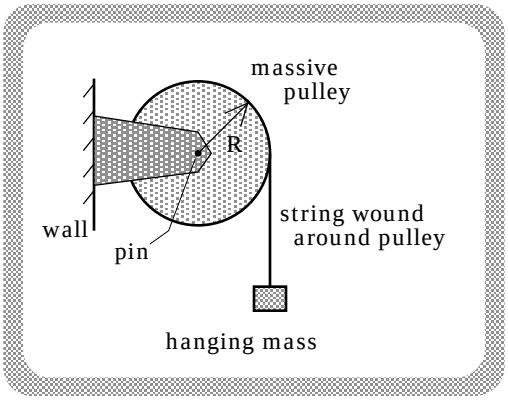


FIGURE 9.8

weight m_h, the pulley's mass m_p, radius R, and the moment of inertia about its *center of mass* $I_{cm} = (1/2)\, m_p R^2$ (we are taking the pulley to be a uniform disk).

a.) We are looking for an *acceleration.* This should bring N.S.L. to mind almost immediately. Using that approach with the f.b.d. shown in Figure 9.9 yields:

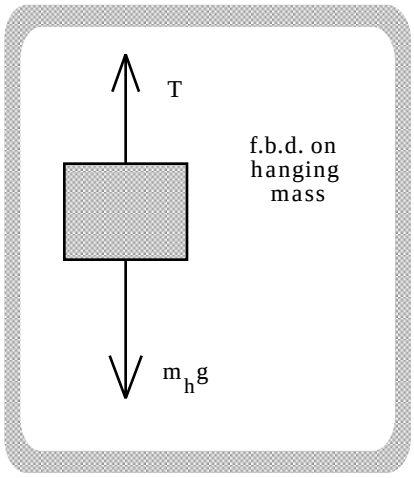


FIGURE 9.9

$\underline{\sum F_y}$:

$$T - m_h g = -m_h a$$
$$\Rightarrow \quad T = m_h g - m_h a \qquad \text{(Equ. A).}$$

There are two unknowns in this equation, T and a. We need another equation.

b.) Figure 9.10 shows a *free body diagram* for the forces acting on the pulley. Summing the torques about the pulley's pin yields:

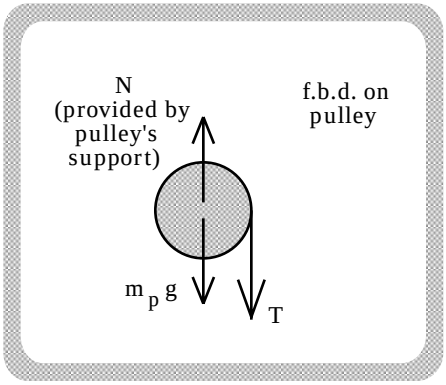


FIGURE 9.10

$\underline{\sum \Gamma_{cm}}$:

$$-TR = -I_{cm}\,\alpha$$
$$= -[(1/2)\, m_p R^2]\,\alpha$$
$$\Rightarrow \quad T = [(1/2)\, m_p R]\,\alpha.$$

c.) We can further simplify this equation by remembering that the *translational acceleration a* of a point on the PULLEY'S EDGE a distance R units from the axis of rotation (this will also be the *acceleration of the string*) is related to the *angular acceleration* α of the pulley by:

$$a = R\,\alpha.$$

Using this to eliminate the α yields:

$$T = [(1/2)\, m_p R]\,(a/R).$$
$$= (1/2)\, m_p\, a \qquad \text{(Equation B).}$$

d.) Putting *Equation A* and *Equation B* together yields:

$$(1/2)\, m_p\, a = m_h g - m_h a$$
$$\Rightarrow \quad a = (m_h g)/(m_h + m_p/2).$$

2.) A trickier problem: Consider the Atwood Machine shown in Figure 9.11. If the pulley is *massive* and has the same characteristics (i.e., mass, radius, moment of inertia, etc.) as the one used in the problem directly above, determine the *magnitude of the acceleration* of the masses as they freefall. (Assume m_1 is more massive than m_2).

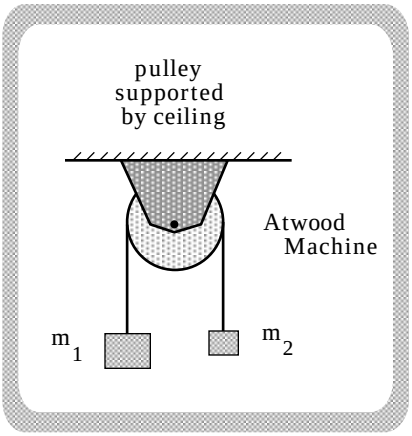


FIGURE 9.11

Note: Notice that the f.b.d. in Figure 9.12a depicts a peculiar situation. If both tensions are *T*, the net torque acting about the pulley's *center of mass* is zero. That is, if both tensions are equal, they produce torques that are *equal in magnitude* and *opposite in direction* and, hence, cancel one another out. With no *net torque* acting on the pulley, it will not *angularly accelerate*. And with no angular acceleration, it will not rotate.

This clearly is an unacceptable situation--everyone knows that pulleys rotate. The problem? When a pulley is *massive*, the tension in a rope draped over it will not be equal on both sides, assuming the system is angularly accelerating. A more accurate depiction of the forces acting on a massive pulley is, therefore, seen in Figure 9.12b.

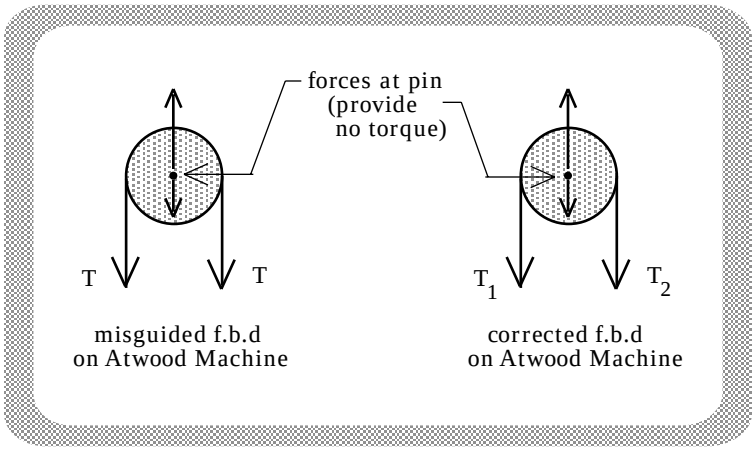


FIGURE 9.12a

FIGURE 9.12b

a.) Figure 9.13a shows the f.b.d. for mass m_1:

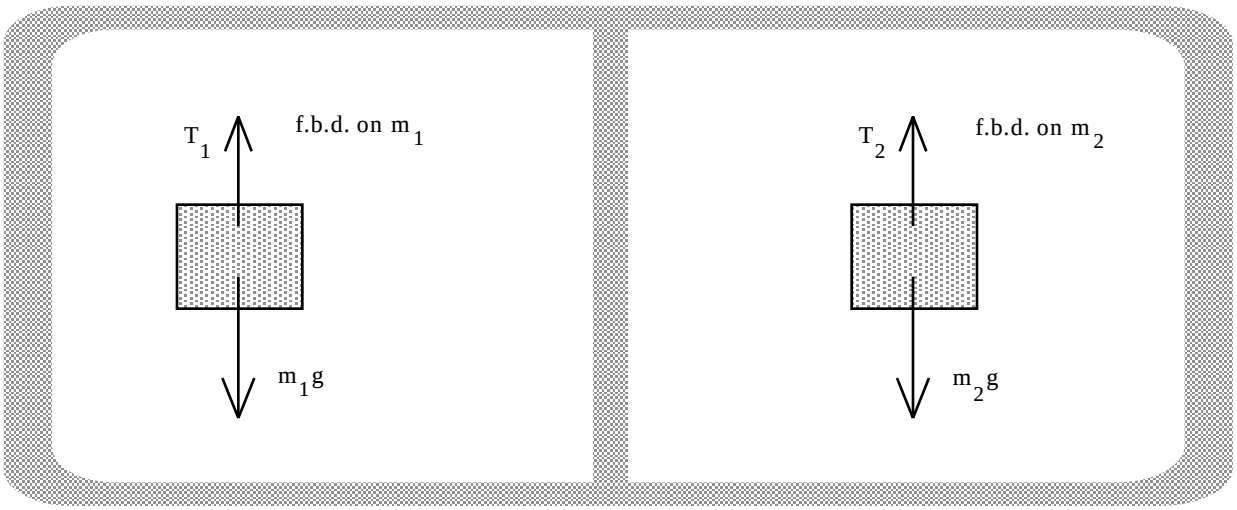


FIGURE 9.13a FIGURE 9.13b

b.) Newton's Second Law yields:

$\underline{\sum F_y}$:

$$T_1 - m_1 g = - m_1 a$$
$$\Rightarrow \quad T_1 = m_1 g - m_1 a.$$

Call this *Equation A* (note the sign in front of the acceleration term).

c.) Figure 9.13b shows the f.b.d. for mass m_2. N.S.L. yields:

$\underline{\sum F_y}$:

$$T_2 - m_2 g = m_2 a$$
$$\Rightarrow \quad T_2 = m_2 g + m_2 a.$$

Call this *Equation B*.

d.) At this point, we have three unknowns (T_1, T_2, and α) and only two equations. For our third equation, we need to look at the pulley.

e.) Figure 9.14 shows the f.b.d. for the pulley. The rotational counterpart of N.S.L. yields:

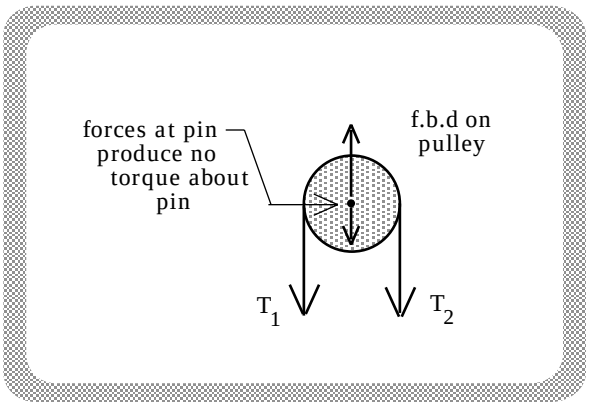


FIGURE 9.14

$\underline{\Sigma \Gamma_{cm}}$:

$$T_1R - T_2R = I_{cm}\,\alpha$$
$$= [(1/2)\, m_pR^2]\,\alpha$$

$$\Rightarrow \quad T_1 - T_2 = [(1/2)\, m_pR]\,\alpha$$

Call this *Equation C* (note the sign in front of the angular acceleration term).

f.) *Equation C* introduces another unknown, α. Fortunately, we know α in terms of a and R:

$$a_{string} = R\,\alpha \qquad \text{(Equation D)},$$

which leaves us with:

$$T_1 - T_2 = [(1/2)\, m_pR]\,(a/R)$$
$$= (1/2)\, m_pa.$$

g.) Using Equations A, B, C, and D, we get:

$$T_1 - T_2 = (1/2)\, m_p\, a$$
$$(m_1g - m_1a) - (m_2g + m_2a) = (1/2)\, m_p\, a$$
$$\Rightarrow \quad a = [m_1g - m_2g]/[m_1 + m_2 + m_p/2].$$

3.) Rotation with a twist: Consider a hollow ball of radius R and mass m rolling down an incline of known angle θ (Figure 9.15). What is the *acceleration* of the ball's *center of mass* as the ball rolls down the incline?

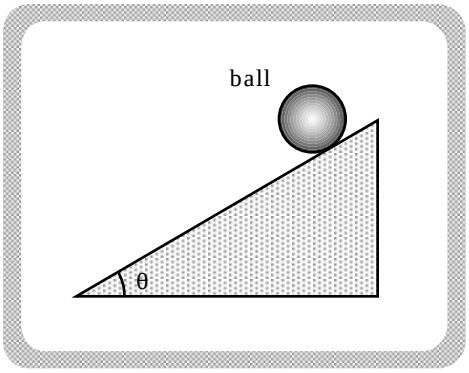


FIGURE 9.15

a.) The *free body diagram* for the forces and force-components acting on the ball is shown in Figure 9.16. The axis has been placed along the direction of the

translational acceleration of the ball's *center of mass*--i.e., along the line of the incline. Noting that there must be *rolling friction* in the system, a summing of the forces along that line yields:

$$\underline{\Sigma F_x}:$$
$$f_r - mg\sin\theta = -ma_{cm}.$$

Call this *Equation A*.

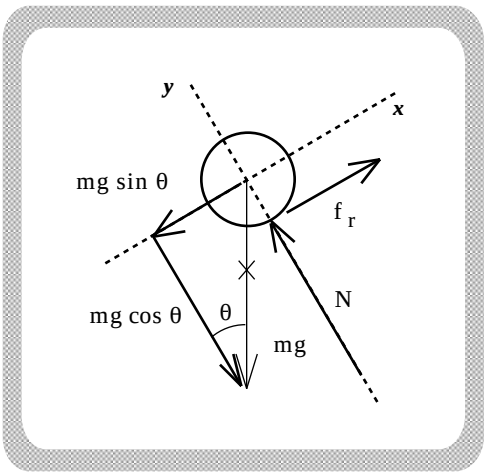


FIGURE 9.16

b.) This would be easy if we knew something about the *rolling friction* between the ball and the incline. As we do not have that information, we haven't a clue as to the *magnitude* of the *rolling frictional force* f_r.

Stymied, let's consider the rotational counterpart to N.S.L.

c.) Noticing that both the *normal force* and the *force due to gravity* pass through the *center of mass*, summing the *torques* about the *center of mass* yields:

$$\underline{\Sigma \Gamma_{cm}}:$$
$$f_r R = I_{cm}\alpha$$
$$= [(2/3)\, mR^2]\,\alpha$$
$$\Rightarrow \quad f_r = [(2/3)\, mR]\,\alpha \qquad \text{(Equation B).}$$

d.) We know the relationship between the *acceleration* of the *center of mass* (a_{cm}) and the *angular acceleration* about the *center of mass* is:

$$a_{cm} = R\,\alpha \qquad \text{(Equation C).}$$

This means we can write:

$$f_r = [(2/3)\, mR]\,(a/R)$$
$$= (2/3)\, ma.$$

e.) Combining *Equations A, B* and *C* yields:

$$f_r - mg\sin\theta = -ma_{cm}$$
$$\Rightarrow \quad (2/3)ma - mg\sin\theta = -ma_{cm}$$
$$\Rightarrow \quad a_{cm} = [g\sin\theta]/[1 + (2/3)]$$
$$= (3/5)g\sin\theta.$$

Note: Knowing the *translational acceleration* of the center of mass, we can determine the *angular acceleration* of the ball using $a_{cm} = R\alpha$:

$$a_{cm} = R\,\alpha$$
$$\Rightarrow \quad \alpha = a_{cm} / R$$
$$= [(3/5)g\sin\theta] / R.$$

E.) A Weird But Effective Alternate Approach:

1.) There exists an altogether different way of looking at problems in which a body rolls without slipping. The following endeavors to present the rationale behind the needed perspective.

2.) Consider an incline so slippery that a ball is sliding down its face without rolling at all (see Figure 9.17).

a.) Relative to the incline, there will be *sliding motion* between the *bottom of the ball* (i.e., the point on the ball that touches the incline--*Point P* in Figure 9.17) and the incline itself.

b.) Put another way, at any given instant the point-on-the-ball that happens to be in contact with the incline will have a velocity along the line of the incline (in the *x direction*) relative to the stationary incline.

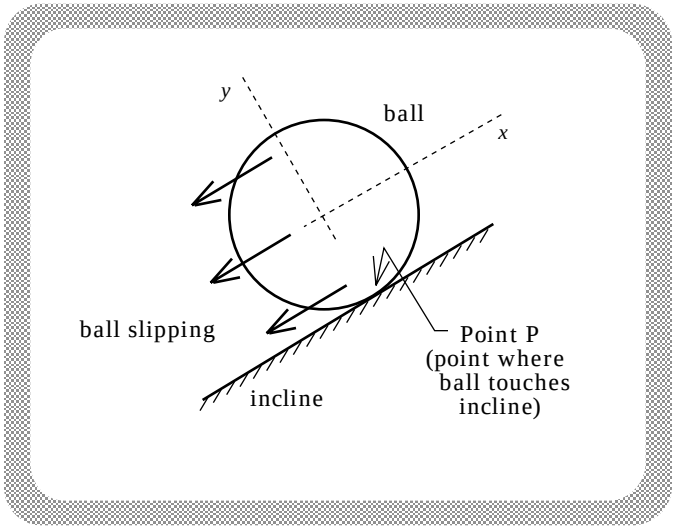


FIGURE 9.17

c.) Bottom line: *Point P* moves; the incline does not.

3.) A ball that rolls without slipping, on the other hand, will experience *no relative motion* in the *x direction* between *the point that happens to be in contact with the incline* (*Point P* in Figure 9.18) and the STATIONARY incline.

a.) This not-so-obvious fact is justified as follows: If the ball is NOT SLIDING over the incline's surface (i.e., as long as we are *not dealing with the case cited in Part 2* above), the velocity of the stationary *incline* and the velocity of the *NON-SLIDING* point-of-contact-with-the-incline must be the same.

b.) Bottom line: If the incline's *velocity* in the *x direction* is zero, *Point P's instantaneous velocity* in the *x direction* must also be zero.

4.) Let us now take a few moments to re-examine the rolling situation in which there is no slippage between the ball and the incline.

a.) Relative to the incline, the *top* of the rolling ball (by *top*, we are talking about the point on the ball farthest from the incline's surface--the point whose *y coordinate* is greatest; this is labeled *Point A* in Figure 9.18) is instantaneously translating faster than the *center of mass* of the ball, and the *center of mass* is instantaneously translating faster than the *bottom* (i.e., *Point P*).

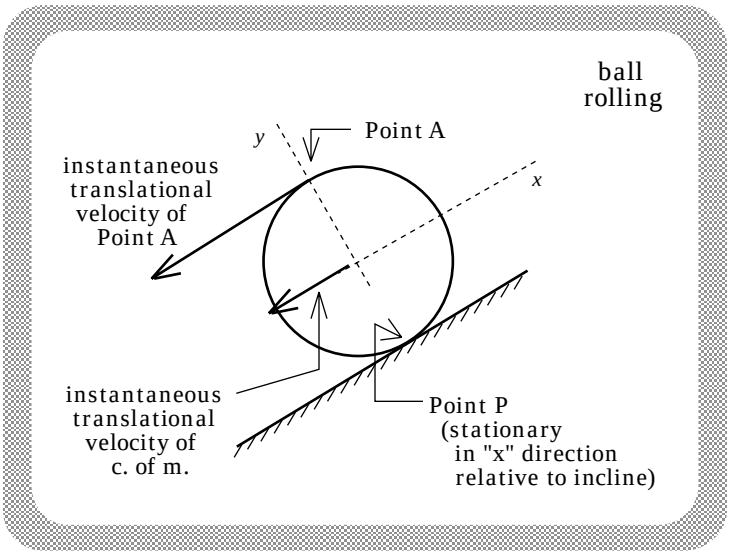


FIGURE 9.18

b.) In fact, the *velocity* of *Point A* is twice that of the *center of mass*.

c.) As stated above, *Point P* is NOT MOVING AT ALL instantaneously in the *x direction*, relative to the stationary incline (if this isn't clear, ask in class or look at the *Note* at the end of the chapter).

5.) Consider another situation. The ball in Figure 9.18 is taken off the incline and a pin is placed through *Point P*. The pin is mounted so that the ball

can rotate freely about the pin. The ball is then allowed to freefall. What can we say about the ball as it rotates about an axis through this point on its circumference?

a.) Begin by examining Figure 9.19.

b.) Notice that *Point A* on the ball is instantaneously translating faster than the *center of mass* of the ball, and the *center of mass* is instantaneously translating faster than the *bottom* (i.e., *Point P*).

This is exactly the same characteristic as was observed in the instantaneous "rolling" situation outlined above.

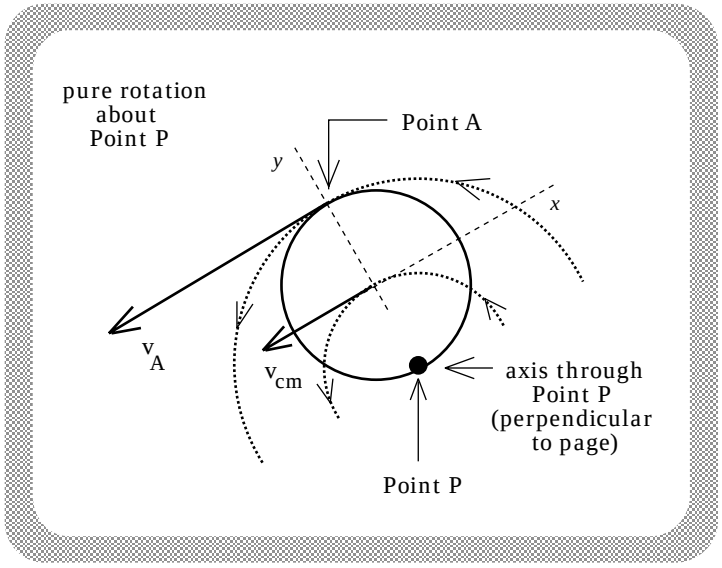


FIGURE 9.19

c.) Also, the *instantaneous velocity* of *Point A* is twice that of the *center of mass*.

Again, this is exactly the same characteristic as was observed in the *instantaneous rolling* situation outlined above.

d.) Notice that the *translational velocity* of the ball at *Point P* (i.e., at the axis of rotation) is zero.

For the last time, this is exactly the same characteristic that was observed in the *instantaneous rolling* situation outlined above.

6.) The *characteristics of motion* for a ball rolling down an incline and a ball pinned to execute a pure rotation appear to be quite similar. In fact, the question arises, "If you could not see what was supporting the ball and only got a quick look, could you be sure which of the two situations you were observing?" That is, could you tell if you were seeing:

a.) A ball rolling down an incline (i.e., *rotating about its center of mass* while its *center of mass* additionally *translates* downward toward the left); or

b.) A ball executing a pure rotation about an axis through its perimeter?

c.) The fact is, if all you got was a glance, it would be *impossible to tell the difference between the two situations*.

d.) Consequences: When dealing with a body that is both translating and rotating without slippage (i.e., executing a pure roll), an alternate way to approach the situation is to treat the moving object as though it were instantaneously executing a pure rotation about *its point of contact* with the surface that supports it (*Point P* in the sketch). Analysis to determine, for instance, the "instantaneous acceleration of the center of mass" at a particular instant will yield the same answer no matter which perspective you use. As far as the bottom line goes, they are identical.

7.) Do you believe? Let us try both approaches on the same problem and see how the two solutions compare. Reconsider the "ball rolling down the incline" problem.

The question: "What is the *angular acceleration* of a ball of mass *m* as it rolls down the incline shown in Figure 9.20?"

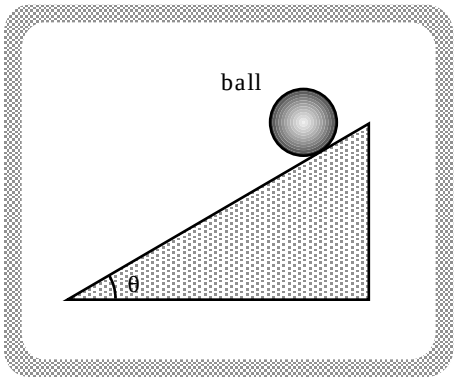


FIGURE 9.20

Note: We could as easily have decided to solve for the *instantaneous translational acceleration* of the *center of mass* instead. The two parameters are related by $a_{cm} = R\alpha$; knowing one parameter means we know the other.

a.) We have already approached this problem from the point of view of a ball rolling down an incline. The solution for the ball's *angular acceleration* about its *center of mass*, derived in *Part 3e* of the previous section, was:

$$\alpha = [(3/5)g \sin \theta]/R.$$

b.) Consider now a *pure rotation about Point P*:

i.) From the *free body diagram* shown in Figure 9.21, we begin by summing the torques about *Point P*. As the *normal* and *frictional* forces

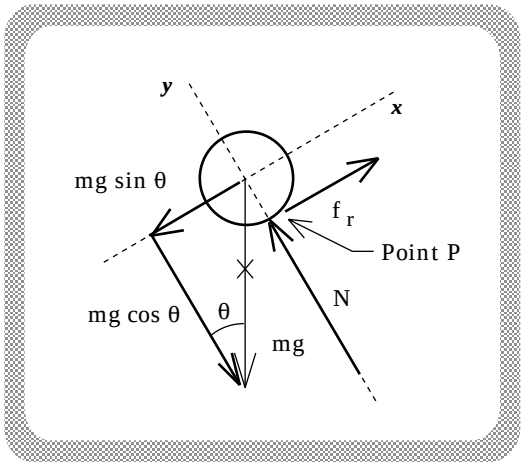


FIGURE 9.21

act through *Point P*, the $r_{\perp}$ approach yields:

$$\underline{\Sigma \Gamma_p}:$$
$$(mg)(R \sin \theta) = I_p \alpha.$$

ii.) We do not know the *moment of inertia* about *Point P*, only the *moment of inertia* about the *center of mass*. As the torque is being taken about *Point P*, we need I_p. Using the *Parallel Axis Theorem*, we write:

$$\begin{aligned} I_p &= I_{cm} + Md^2 \\ &= (2/3)mR^2 + mR^2 \\ &= (5/3)mR^2. \end{aligned}$$

Note: The variable *m* in the above equation is the *total mass* in the system (this happens to be the mass of the ball in this case); *d* is the distance between the two parallel axes (i.e., the axis through the *center of mass* and the axis through *Point P*); and *m* is the mass of the ball.

iii.) Completing the problem:

$$\underline{\Sigma \Gamma_p}:$$
$$(mg)(R \sin \theta) = I_p \alpha$$
$$(mg)(R \sin \theta) = [(5/3)\, mR^2]\, \alpha$$

$$\Rightarrow \quad \alpha = [g \sin \theta]/[(5/3)R]$$
$$= [(3/5)g \sin \theta]/R.$$

iv.) This is exactly the *angular acceleration* solution we determined using the "rolling" approach to the problem. If we additionally wanted the instantaneous translational acceleration of the *center of mass*, we could use $a_{cm} = R\alpha$, yielding:

$$\begin{aligned} a_{cm} &= R\alpha \\ &= R\,[(3/5)g \sin \theta]/R \\ &= (3/5)g \sin \theta. \end{aligned}$$

Again, this is exactly what we determined using the other method.

c.) Bottom line: There are two ways to deal with a rolling object that is not additionally sliding. You can either treat it:

i.) As a body executing a pure roll (i.e., as a body whose mass is rotating about its *center of mass* while its *center of mass* is itself translating); or

ii.) As a body instantaneously executing a *pure rotation* about its perimeter at the *point of contact* with the structure that supports it.

8.) Which way is the best? It depends upon you. The first approach is more conventional but requires the use of both the *translational* and *rotational* counterparts to *Newton's Second Law*. The second approach (the "weird" one) requires only the use of the *rotational* version of *N.S.L.*, but the torques are *not* taken about the *center of mass* so the *parallel axis theorem* must be used to determine the *moment of inertia* I_p about the appropriate axis.

My suggestion is that you use the approach that seems most sensible, given what the system is doing. If, for instance, you see pure rotation, use the rotational approach. If there is rotation and translation happening, use the other approach.

F.) Energy Considerations and Rotational Motion:

1.) Remembering back, energy considerations are useful whenever the forces in a system are conservative and the *velocity* or a *distance traveled* is the parameter of interest. Briefly, energy considerations and the *modified conservation of energy* equation were derived as follows:

a.) We began by writing out the WORK/ENERGY THEOREM (i.e., the *net work done* on an object is equal to the *change* of the body's *kinetic energy*) for a body that moves from Position 1 to Position 2 under the influence of a number of forces. The equation was:

$$W_{net} = \Delta KE$$
$$\Rightarrow \quad W_A + W_B + W_C + W_D + \ldots = \Delta KE,$$

where W_A was the work *force A* did on the object as it moved from Position 1 to Position 2, etc.

b.) We derived expressions for the work done by all the *conservative forces with known potential energy functions* as the body moved from Position 1 to Position 2:

$$W_{\text{cons force A}} = - \quad \Delta U_A$$
$$= - (U_{A,2} - U_{A,1})$$

c.) We derived a general expression for the work done by any *non-conservative force*. We did the same for any *conservative forces* for which we did not know a *potential energy function*. For both cases:

$$W_{\text{noPEfct,C}} = \mathbf{F}_C \cdot \mathbf{d} \qquad \text{etc.}$$

d.) Putting it all into the work/energy theorem (i.e., $W_{net} = \Delta KE$), we ended up with:

$$[-(U_{A,2} - U_{A,1})] + [- (U_{B,2} - U_{B,1})] + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)\, mv_2^2 - (1/2)mv_1^2.$$

e.) Rearranging by putting the "before" quantities on the left-hand side of the equation and the "after" quantities on the right-hand side, we got:

$$(1/2)mv_1^2 + U_{A,1} + U_{B,1} + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^2 + U_{A,2} + U_{B,2}.$$

f.) This was put in short-hand form:

$$KE_1 + \Sigma\, U_1 + \Sigma\, W_{\text{extraneous}} = KE_2 + \Sigma U_2.$$

g.) The last touch: We noticed that in a system of more than one body, the *total* kinetic energy in the system at a given instant is the *sum of all the kinetic energies* of all the bodies moving in the system at that instant. As such, the final form of the *modified energy conservation equation* became:

$$\Sigma\, KE_1 + \Sigma\, U_1 + \Sigma W_{\text{extraneous}} = \Sigma\, KE_2 + \Sigma\, U_2.$$

2.) This equation also works fine for *rotating* systems. There are only two changes to be made:

a.) Although there may still be *kinetic energy* due to the translational motion of bodies within the system, there can now also be *kinetic energy* due to the *rotational* motion of bodies within the system. That means the *total kinetic energy* term has a new member--*rotational kinetic energy*. As such, we need to write:

$$\Sigma \text{ KE}_1 = \Sigma \text{ KE}_{1,\text{trans}} + \Sigma \text{ KE}_{1,\text{rot}}.$$

Note: We determined at the end of the last chapter that just as the *translational kinetic energy* of an object is $(1/2)mv^2$, the *rotational kinetic energy* of an object is $(1/2)(I_{axis\ of\ rot})\omega^2$.

b.) Concerning *gravitational potential energy*: Consider a body that moves some *vertical distance h* in a gravitational field. Its change of *potential energy* will be $\pm mgh$. Why? Because the *potential energy function* for *gravity* is related to the *vertical distance traveled*. The question is, "How do you determine the *vertical distance traveled* if the motion is that of a rotating body?"

i.) Example: A pinned beam rotates from one angular position to a second (see Figure 9.22 to the right). What is its *change of potential energy* during the move? Put another way, what is *h* in the *mgh* equation that defines changes of *gravitational potential energy*?

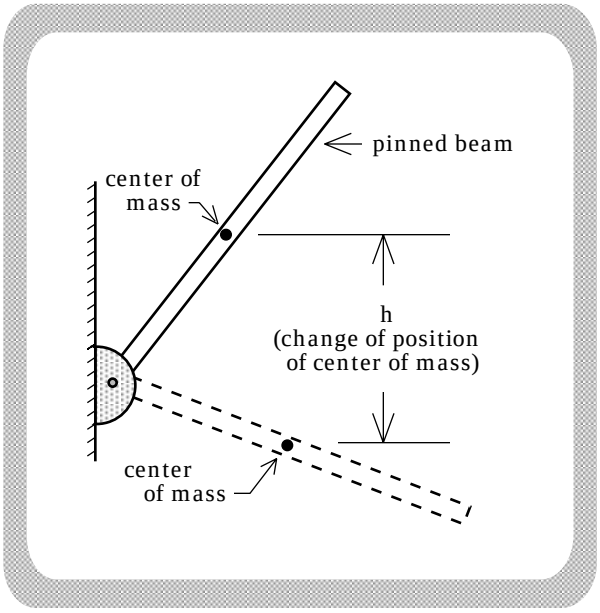


FIGURE 9.22

ii.) As shown in the sketch, *h* is defined as the *vertical displacement* of the body's <u>*center of mass*</u>.

Note: As there is *rotational* <u>*kinetic energy*</u>, is there *rotational* <u>*potential energy*</u>?

The answer to that question is *yes* and *no*. It is theoretically possible to define a *potential energy function* that tells you *how much work* a torque Γ does on a body as the body moves through some *angular displacement* $\Delta\theta$, <u>but you will not deal with such a function in this class</u>.

Nevertheless, when you use the *conservation of energy* equation you may be asked to determine the *amount of work* ($\boldsymbol{F}\cdot\boldsymbol{d}$) done as friction acts at the axle of a rotating object. In that case, *d* equals $r\,\Delta\theta$, where *r* is the distance between the *axis of rotation* and the place at which the friction acts.

G.) Energy Consideration Examples:

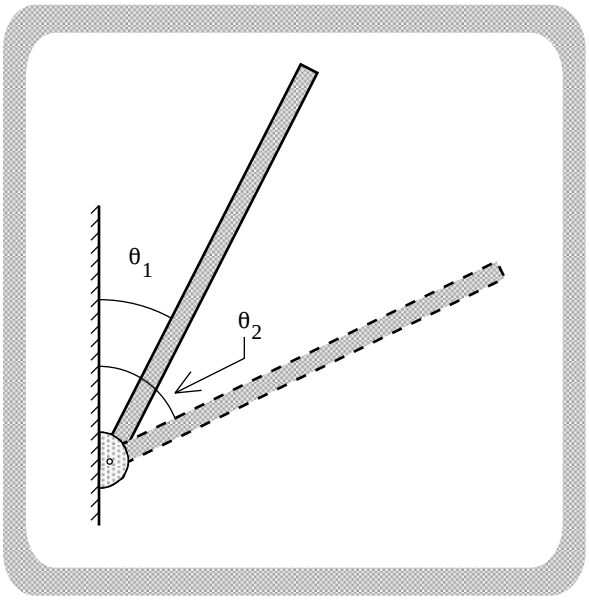


FIGURE 9.23

1.) A typical *Pure Rotation* Problem: A beam is frictionlessly pinned (see Figure 9.23). From rest, the beam is allowed to freely rotate about its pinned end from an angle $\theta_1 = 30^o$ with the vertical. If the beam's mass is *M*, its length is *L = 2 meters*, and its *moment of inertia* about its *center of mass* is $(1/12)ML^2$, what is its *angular velocity* as it passes through $\theta_2 = 70^o$ with the vertical?

a.) We are looking for a *velocity* (it is an *angular* velocity, but a velocity nevertheless). The first approach that should come to mind whenever a body falls in a gravitational field and a velocity value is requested is the *modified conservation of energy* approach. Executing that approach yields:

$$\Sigma\, KE_1 + \Sigma\, U_1 + \Sigma\, W_{extran} = \Sigma\, KE_2 + \Sigma\, U_2$$

$$0 + Mg\,(L/2)\,(\cos\theta_1 - \cos\theta_2) + 0 = (1/2)I_p\,\omega_2^{\,2} + 0\ .$$

Note 1: Notice that there was no initial *angular velocity*. That didn't have to be the case. DON'T BE LULLED INTO THE BELIEF THAT v_1 AND ω_1 WILL *ALWAYS* BE ZERO!

Note 2: Figure 9.24 illustrates how the position of the center of mass (depicted by a circle on the beam) changes during the drop. The right triangle used to determine the *final position* of the *c. of m.* relative to the pin is shown in the drawing. A similar triangle would be used to

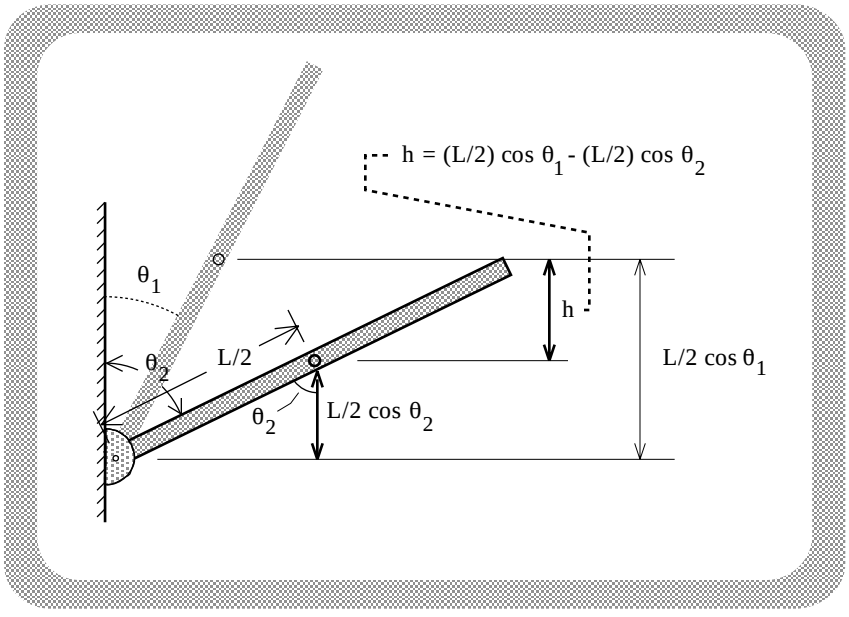


FIGURE 9.24

determine the *c. of m.'s* initial position. The difference between the two yields the drop distance *h*.

b.) We know the *moment of inertia* about the *center of mass*. We need the *moment of inertia* about the *pin*. Using the *Parallel Axis Theorem*, we get:

$$I_p = [(1/12)\, ML^2] + M(L/2)^2$$
$$= (1/3)\, ML^2.$$

c.) Putting it all together and solving, we get:

$$Mg[(L/2)(\cos 30^o - \cos 70^o)] = (1/2)(ML^2/3)\,\omega_2{}^2.$$

d.) The *M's* cancel, leaving:

$$(9.8\ m/s^2)(2\ meters/2)(.524) = .5[(2\ meters)^2/3]\,\omega_2{}^2$$
$$\Rightarrow \quad \omega_2 = 2.78\ radians/sec.$$

2.) A typical <u>*Rotation and Translation*</u> Problem: Consider a string wrapped around a massive pulley. One end of the string is attached to a hanging weight of mass *m*. The system is allowed to accelerate freely. At some instant, the hanging weight is observed to be moving with velocity $v_1 = 3\ m/s$. What will its *velocity* be after it has fallen an additional .8 meters? You may assume that the pulley's mass is $M = 4m$, its radius is *R*, and its *moment of inertia* is $3mR^2$. The system is shown in Figure 9.25.

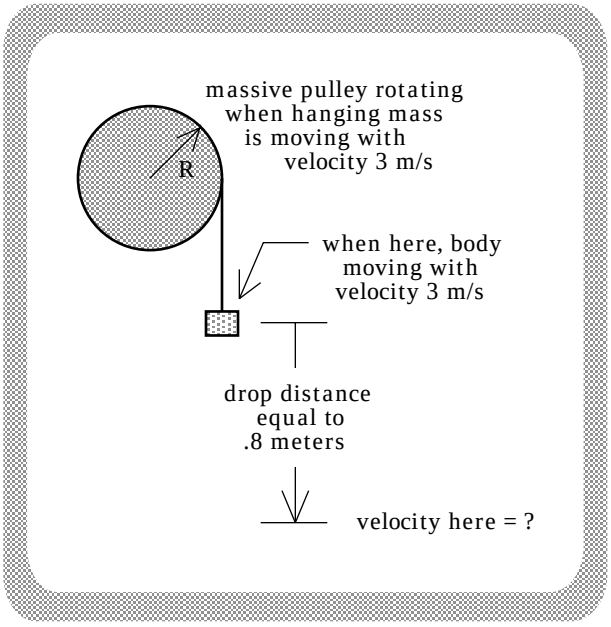


FIGURE 9.25

a.) Initially, there is *kinetic energy* wrapped up in both the rotating pulley and the hanging mass, and there is initial *potential energy* wrapped up in the hanging mass. Using the *Conservation of Energy* equation on this situation yields:

$$KE_1 + \Sigma U_1 + \Sigma W = \Sigma KE_2 + \Sigma U_2$$

$$[.5I_{pul,cm}\omega_1^2 + .5mv_1^2] + mgh_1 + 0 = [.5 I_{pul,cm}\omega_2^2 + .5 mv_2^2] + [0]$$

$$\Rightarrow \quad .5 (3mR^2)\omega_1^2 + .5mv_1^2 + mgh_1 = .5 (3mR^2)\omega_2^2 + .5 mv_2^2 .$$

Important Note: Why not include *tension in the line* in the work calculation? The short answer: Because it's an internal force. The long answer: Because the work that tension does on *m* is *-Th* while the work that tension does on the pulley is $+T(R\Delta\theta) = +Th$. The consequence of all of this is that the net work done by the tension *T* (again, an internal force) is ZERO.

b.) We know that the velocity *v* of the string (hence the velocity of the hanging mass) is the same as the velocity of the edge of the pulley. This equals $R\omega$. That means $\omega = v/R$. Using this, we can cancel the *m's*, eliminate the *w* terms and solve:

$$.5 (3mR^2)(v_1/R)^2 + .5mv_1^2 + mgh_1 = .5 (3mR^2)(v_2/R)^2 + .5mv_2^2$$

$$1.5v_1^2 + .5v_1^2 + gh_1 = 1.5v_2^2 + .5v_2^2$$

$$2v_1^2 + gh_1 = 2v_2^2$$

$$\Rightarrow \quad v_2 = [v_1^2 + gh_1/2]^{1/2}$$

$$= [(3 \text{ m/s})^2 + (9.8 \text{ m/s}^2)(.8 \text{ meters})/2]^{1/2}$$

$$= 3.59 \text{ m/s}.$$

c.) The question could as easily have asked for the *final angular velocity* of the pulley. It is the same problem with one exception: you would have eliminated the *v* terms with $v = R\omega$ instead of eliminating ω with $\omega = v/R$.

If we knew beforehand that $v_2 = 3.59\ m/s$, we would have used $v = R\omega$ to calculate:

$$\omega_2 = v_2/R$$

$$= (3.59 \text{ m/s})/R.$$

3.) A typical *Rotation and Translation Mixed In One* Problem: Figure 9.26 shows a ball rolling *up* a 30° incline. At the initial instant, the ball's *center of mass* is moving with velocity $v_1 = 8\ m/s$. How fast will its

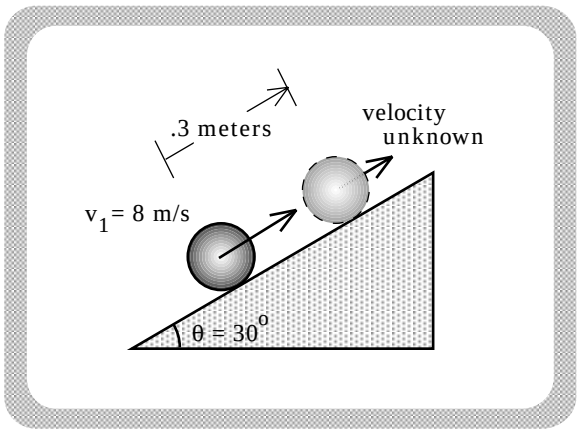


FIGURE 9.26

center of mass be moving after traveling an additional .3 meters up the incline? Assume the ball's mass is $m = .2\ kg$, its radius is $R = .1\ meters$, and its *moment of inertia* about its *center of mass* is $(2/5)mR^2$.

Note: Thinking back to the section on *angular acceleration* and Newton's Second Law, we found that any situation in which an object *rolls without slipping* can be treated either as: 1.) motion around the *center of mass* plus motion of the *center of mass* (i.e., a roll), or 2.) an instantaneous, *pure rotation* about the point of contact between the object and the support upon which it rolls (i.e., a pure rotation). We will approach this problem both ways.

The "*rotation* and *translation* of the center of mass" approach:

a.) Looking at the ball's motion when first observed, two *kinds* of motion are taking place relative to the ball's *center of mass*. The ball is rotating around its *center of mass* with angular velocity ω_1, and the ball's *center of mass* is itself moving with velocity v_1. In other words, the initial kinetic energy, as far as the center of mass of the system is concerned, is:

$$KE_{1,tot} = (1/2)I_{1,cm}\omega_1^2 + (1/2)mv_1^2.$$

b.) For the sake of ease, let us define the *gravitational potential energy* of the ball when at *Position 1* as zero.

c.) (THIS IS IMPORTANT): Rolling friction exists within the system, but rolling friction does so little work on the ball that the energy loss due to it is negligible. As such, we will approximate it to be zero. That means that there are no extraneous forces doing work on the system which, in turn, means that $\Sigma\ W_{extr} = 0$.

d.) Writing out the *conservation of energy* equation, we get:

$$\Sigma\ KE_1 + \Sigma\ U_1 + \Sigma\ W_{ext} = \Sigma\ KE_2 + \Sigma\ U_2$$
$$\{(1/2)I_{1,cm}\omega_1^2 + (1/2)mv_1^2\} + 0 + 0 = \{(1/2)I_{2,cm}\omega_2^2 + (1/2)mv_2^2\} + mgh,$$

where h is the vertical rise of the ball's center of mass.

e.) We know that $v_{cm} = R\omega$. Using that and substituting I_{cm} *for a ball* into the equation yields:

$$.5\,[(2/5)mR^2]\,(v_1/R)^2 + .5\,mv_1^{\,2} = .5\,[(2/5)mR^2](v_2/R)^2 + .5mv_2^{\,2} + mgh$$

$$(1/5)mv_1^{\,2} + .5mv_1^{\,2} = (1/5)mv_2^{\,2} + .5mv_2^{\,2} + mgh$$
$$(7/10)mv_1^{\,2} = (7/10)mv_2^{\,2} + mgh$$

$$\Rightarrow \quad v_2 = [[.7v_1^{\,2} - gh_2]/(.7)]^{\,1/2}$$
$$= [v_1^{\,2} - 1.43gh]^{1/2}.$$

f.) To find *h*, we need to use trig to determine the VERTICAL distance traveled as the ball rolled .3 meters up the incline. We know that the definition of the *sine* of an angle is equal to "the side opposite the angle divided by the hypotenuse." In this case, the "opposite side" is *h* and the hypotenuse is .3 meters. Using this, we get:

$$h = (.3 \text{ meters})\,(\sin 30^o)$$
$$= .15 \text{ meters.}$$

g.) Putting in the numbers, we get:

$$v_2 = [(8 \text{ m/s})^2 - 1.43(9.8 \text{ m/s}^2)(.15 \text{ meters})]^{1/2}$$
$$= 7.87 \text{ m/s.}$$

The "*pure rotation*" approach:

h.) Reiterating what has previously been stated, we know that if we look at an object's instantaneous motion, we can't tell whether the object is *rolling* or moving in *pure rotation* about a point on its perimeter. We've analyzed the "conservation of energy" problem outlined above from the first perspective. Now we will deal with the problem using the "pure rotation" approach.

i.) The sketch in Figure 9.27 (next page) assumes the ball is executing a pure rotation (instantaneously) about a *Point P* located at the intersection of the ball and the incline. If we take the *angular velocity* of the ball about P at that instant to be ω_1, the ball's initial *kinetic energy* will be purely rotational about *Point P* and will equal:

$KE_{1,tot} = (1/2)I_{1,p}\omega_1^2$.

A similar expression defines the ball's *kinetic energy* after traveling up the incline .3 meters.

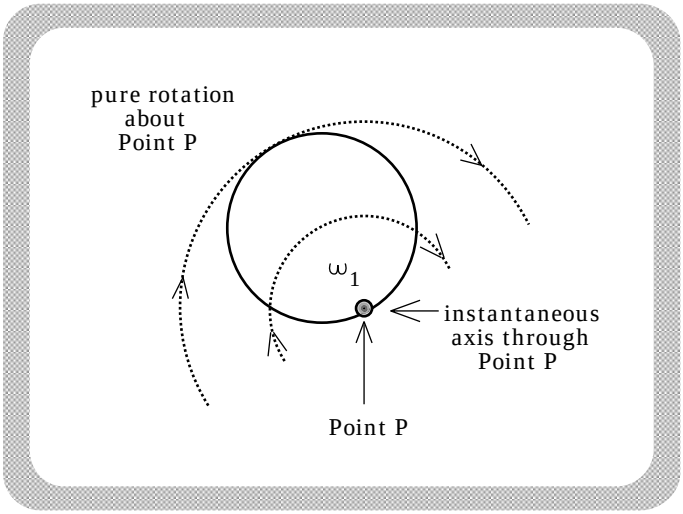


FIGURE 9.27

j.) For the sake of ease, let us define the *gravitational potential energy* of the ball at *Position 1* as zero.

k.) Writing out the *conservation of energy* equation, we get:

$$\Sigma\, KE_1 + \Sigma\, U_1 + \Sigma\, W_{ext} = \Sigma\, KE_2 + \Sigma\, U_2$$
$$(1/2)I_{1,p}\omega_1^2 + 0 + 0 = (1/2)I_{2,p}\omega_2^2 + mgh.$$

l.) We know the *moment of inertia* about the *center of mass*; we need the *moment of inertia* about the *pin*. Using the *Parallel Axis Theorem*, we get:

$$I_p = [(2/5)\, mR^2] + mR^2$$
$$= (7/5)mR^2.$$

m.) Substituting and solving yields:

$$(1/2)I_{1,p}\omega_1^2 = (1/2)I_{2,p}\omega_2^2 + mgh$$
$$(1/2)\,[(7/5)\, mR^2]\,\omega_1^2 = (1/2)[(7/5)\, mR^2]\,\omega_2^2 + mgh$$
$$(7/10)\, mR^2\,\omega_1^2 = (7/10)mR^2\,\omega_2^2 + mgh$$

$$\Rightarrow \quad \omega_2 = [.7R^2\,\omega_1^2 - gh]/(.7R^2)]^{1/2}$$
$$= [\,\omega_1^2 - 1.43gh/R^2]^{1/2}.$$

n.) To determine the velocity of the *center of mass,* we will have to use $v_{cm} = R\omega$. Doing so yields:

$$\omega_2 = [\omega_1^2 - 1.43\ gh/R^2]^{1/2}$$
$$\Rightarrow \quad v_{cm} = R\,[\omega_1^2 - 1.43\ gh/R^2]^{1/2}$$
$$= [R^2\omega_1^2 - R^2(1.43\ gh/R^2)]^{1/2}$$
$$= [v_1^2 - 1.43\ gh]^{1/2}.$$

As expected, our solutions from the two approaches are the same.

Note: WHICH APPROACH IS BEST? It depends upon the problem. The first requires more terms in the *conservation of energy* equation; the second utilizes a simpler form of the *conservation of energy* equation but requires the use of the *parallel axis theorem*.

My suggestion? Learn both approaches and use whichever seems easiest for a given problem.

H.) Comments on Test Questions: N.S.L. and ENERGY Considerations:

1.) When you are asked to determine an *acceleration* or *angular acceleration*, the first approach you should consider is *Newton's Second Law*. It won't always work, but it is one of the most powerful acceleration-involved approaches available to you.

When you are asked to determine a *velocity* or *angular velocity* in a non-collision situation, the first approach you should consider is *conservation of energy*. Again, it will not always work but it is a very powerful approach.

2.) A typical test question will have a number of parts to it. You could, for instance, be given a ball rolling down an incline and be asked to:

a.) Derive an expression for the acceleration of the system;

b.) Derive an expression for the velocity of the ball after having rolled down the incline a distance *h*;

c.) Determine the angular velocity of the ball at the point defined in *Part b*.

3.) You no longer have the cues available in previous chapters (i.e., you can no longer assume that because the chapter you are studying is, for instance, about Newton's Second Law, that the test problems will be Newton's Second Law problems only). You must now first identify the *kind* of problem you are looking at, then have the wherewithal to use the appropriate approach.

I.) Conservation of Angular Momentum:

1.) Just as a body moving in straight-line motion has *momentum* defined as the product of its inertia (its *mass*) and its *velocity*, a rotating body has *angular momentum* defined as the product of its rotational inertia (its *moment of inertia*) and its *angular velocity*. Mathematically, these two are:

$$\mathbf{p} = m\mathbf{v} \qquad \text{and} \qquad \mathbf{L} = I\boldsymbol{\omega}.$$

Note: Both *momentum* and *angular momentum* are vectors. As you will never have to worry about two or three-dimensional *angular momentum*, the only part of the vector notation you will normally use when writing out an *angular momentum* quantity is the sign. An *angular momentum* is considered "+" if it is associated with motion that is *counterclockwise* relative to the point about which the *angular momentum* is calculated (if this is a pure rotation, positive *angular momentum* would correspond to positive *angular velocity*). Negative *angular momentum* is just the opposite.

2.) Newton observed that there exists a relationship between the *net force* acting on a body and the body's *change of momentum*. In one dimension, that relationship is:

$$\mathbf{F}_{net} = d\mathbf{p}/dt$$

or, if the force is constant and the time interval large,

$$\mathbf{F}_{net} = \Delta\mathbf{p}/\Delta t.$$

A similar relationship exists between the net torque acting on a body and the body's change of *angular momentum*. That relationship is:

$$\boldsymbol{\Gamma}_{net} = d\mathbf{L}/dt$$

or, if the torque is constant and the time interval large,

$$\boldsymbol{\Gamma}_{net} = \Delta\mathbf{L}/\Delta t.$$

Big Note: If the sum of the *net external torque* is zero, the <u>*CHANGE*</u> of the system's *ANGULAR MOMENTUM* will be ZERO and the *ANGULAR MOMENTUM* will be <u>CONSERVED</u>.

3.) In dealing with torque calculations, we found that there are two general ways to determine the net torque being applied to a body.

a.) Using strictly *translational* variables, we write:

$$\boldsymbol{\Gamma}_{net} = \mathbf{r} \times \mathbf{F}_{net}.$$

b.) Using strictly *rotational* variables, we write:

$$\boldsymbol{\Gamma}_{net} = I\boldsymbol{\alpha}.$$

4.) Analogous to the *torque* situation, there are two general ways to determine the *angular momentum* of a body:

a.) Using strictly translational variables (instantaneous), we write:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

b.) Using strictly rotational variables we write:

$$\mathbf{L} = I\boldsymbol{\omega}.$$

5.) Bottom line: There are two ways to determine the *angular momentum* of a point mass. If you know the *moment of inertia* of the body about its axis of rotation and its *angular velocity*, you can use $L = I\omega$ (this also works for extended objects). If you know the body's *instantaneous momentum* (mv) and a position vector r that defines its position relative to the axis of rotation, you can use the relationship $L = |\boldsymbol{r} x \boldsymbol{p}|$.

a.) Example: Determine the *angular momentum* of an object of mass m circling with velocity magnitude v and angular velocity ω a distance R units from the axis of rotation (see sketch in Figure 9.28).

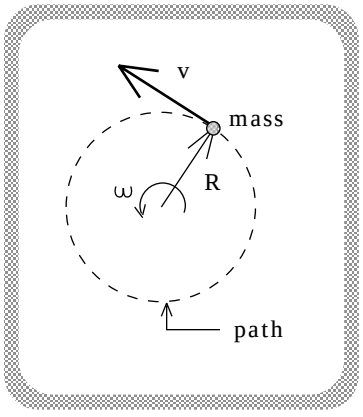


FIGURE 9.28

i.) The rotational relationship: Noting that the *moment of inertia* of a point mass a distance R units from the axis of rotation is mR^2, the magnitude of the angular momentum is:

$$L = I\omega$$
$$= (mR^2)\,\omega.$$

ii.) The translational relationship: Noting that the magnitude of the instantaneous momentum of the body is $p = mv$, and that the angle between the line of $\boldsymbol{r}$ and the line of $\boldsymbol{p}$ is 90^o, we have:

$$\begin{aligned} L &= |\mathbf{r} \times \mathbf{p}| \\ &= r\,(mv \sin 90^o) \\ &= mvR. \end{aligned}$$

Noting additionally that $v = R\omega$, we can write:

$$\begin{aligned} L &= mvR \\ &= m\,(R\omega)\,R \\ &= m\,R^2\,\omega. \end{aligned}$$

In both cases, the body's *angular momentum* is the same.

6.) Earlier, it was pointed out that when the net torque acting on a body equals zero, $\Gamma_{net} = \Delta L/\Delta t = 0$. This implies the angular momentum L does not change with time (i.e., L is constant). An expanded way of stating this is embodied in the *conservation of angular momentum* equation. Analogous to the *modified conservation of momentum* equation, this relationship for one dimensional rotational motion (i.e., rotational about a fixed axis) is written as:

$$\Sigma\, L_1 + \Sigma\, (\Gamma_{ext}\, \Delta t) = \Sigma\, L_2.$$

a.) This relationship states that in a particular direction, the *sum* of the *angular momenta* of all the pieces of a system at time t_1 will equal the sum of all of the *angular momenta* at time t_2 if there are *no external torques* acting on the system to change the net *angular momentum* during the time period. If *external torques* do exist, the final *angular momentum* increases or decreases during the time period by $\Sigma(\Gamma_{ext}\, \Delta t)$.

b.) When the $\Sigma(\Gamma_{ext}\, \Delta t)$ term is zero, angular momentum is said to be *conserved*. This occurs either when there are *no external torques* acting on the system or when *external torques* present are so small and/or act over such a tiny Δt that to a good approximation they do not appreciably alter the system's motion (hence, the system's total angular momentum).

c.) The most common use of the *conservation of angular momentum* is in the analysis of collision problems (explosion problems, for instance, are nothing more than fancy collision problems). Freewheeling collisions

happen so quickly that even if there are *external torques* acting on the system, the total angular momentum of the system *just before* the collision and *just after* the collision will be the same. In other words, angular momentum is usually conserved *through a collision*.

7.) Example #1 (situation in which a system's *moment of inertia* changes but no external torques are applied): An ice skater begins a spin with his arms out. His *angular velocity* at the beginning of the spin is $\omega_1 = 2$ *radians/sec* and his *moment of inertia* is 6 kg·m^2. As the spin proceeds, he pulls his arms in, decreasing his moment of inertia to 4.5 kg·m^2. What is his *angular velocity* after pulling in his arms?

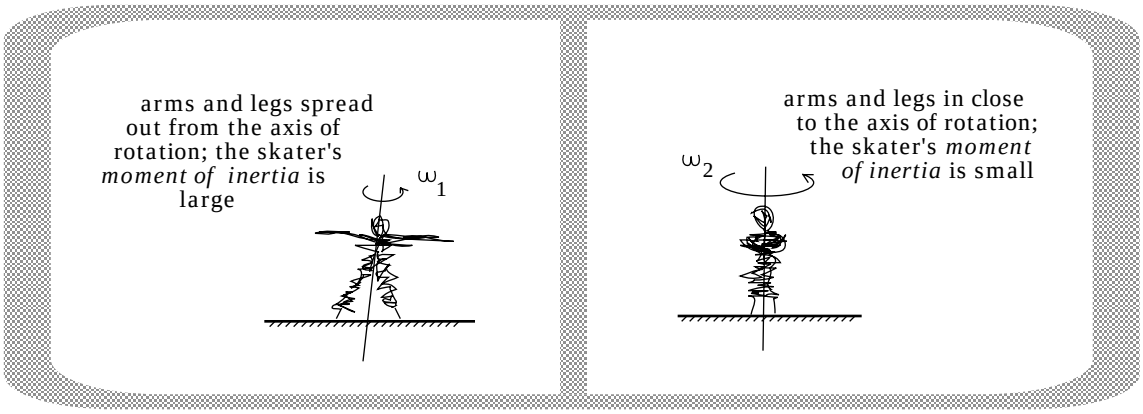

arms and legs spread out from the axis of rotation; the skater's *moment of inertia* is large
ω_1
arms and legs in close to the axis of rotation; the skater's *moment of inertia* is small
ω_2

FIGURE 9.29a FIGURE 9.29b

Solution: Figures 9.29a and 9.29b show the skater before and after pulling in his arms. The work required to do the pulling is provided by the "burning" of chemical energy wrapped up in the muscles of his body. The force applied due to that exertion provides no net torque (not only would any such torque be internal to the system if it existed, there is in fact no torque at all because the *line* of the muscle forces *acts through the axis of rotation*).

As there are no *external torques* being applied to the skater, his *angular momentum* must remain the same throughout (i.e., it is conserved).

a.) At the beginning of the spin, his *angular momentum* is:

$$\begin{aligned} L_1 &= I_1 \omega_1 \\ &= (6 \text{ kg·m}^2)\ (2 \text{ rad/sec}) \\ &= 12 \text{ kg·m}^2/\text{s}. \end{aligned}$$

b.) After his arms are pulled in, his *moment of inertia* decreases and the *angular momentum* expression becomes:

$$L_2 = I_2 \omega_2$$
$$= (4.5 \text{ kg}\cdot\text{m}^2)\ (w_2).$$

c.) Equating the two *angular momentum* quantities:

$$L_1 = L_2$$
$$12 \text{ kg}\cdot\text{m}^2/\text{s} = (4.5 \text{ kg}\cdot\text{m}^2)\ (w_2)$$
$$\Rightarrow \quad \omega_2 = 2.67 \text{ rad/sec.}$$

Note: Although angular momentum is conserved here, energy is not conserved. The skater has to use chemical energy within his muscles in pulling in his arms. A comparison of the energy before and after the pull-in shows that there is more kinetic energy in the system *after* the pull-in than *before*. (Try it. You should find that $E_1 = 12$ *joules* while $E_2 = 16$ *joules*.)

8.) Example #2 (situation in which a system's *moment of inertia* changes but no external torques are applied): A child of mass 40 kg walks from the edge of a 4 meter radius *merry-go-round* (moment of inertia $I_{m.g.r.} = 700\ kg\cdot m^2$) to a position 1.5 meters from the *merry-go-round's* center. If the system initially rotates at 3 radians/second, what is the system's *angular velocity* once the kid reaches the 1.5 meter mark? See Figures 9.30a and 9.30b for "before and after" views.

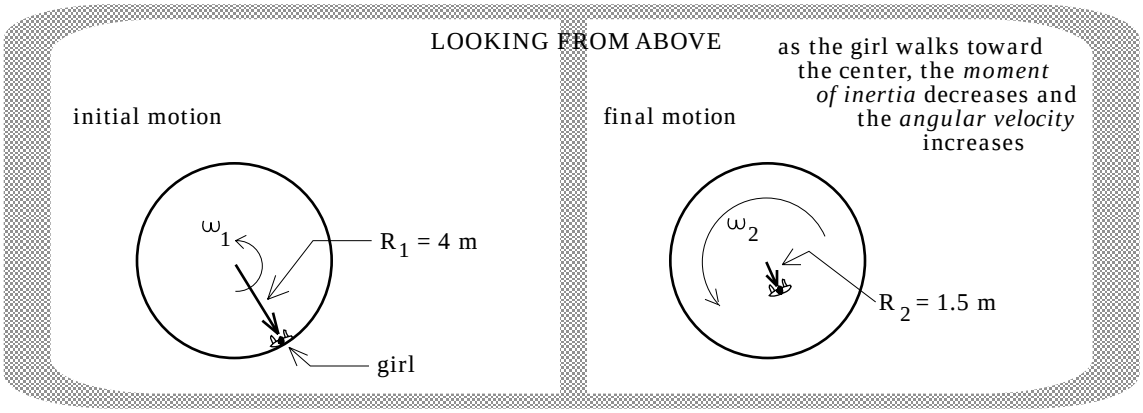


FIGURE 9.30a

FIGURE 9.30b

Solution:

a.) Once again, any change in the *angular momentum* of the *merry-go-round* will be due to a *torque* exerted by the walking kid. But according to Newton's Third Law, any torque the kid exerts on the *merry-go-round* must be matched by an equal and opposite torque exerted by the *merry-go-round* on the kid. In other words, there are only *internal torques* acting on the system. This implies that *angular momentum* is conserved.

b.) With that in mind:

$$L_{1,tot} = L_{2,tot}$$
$$[L_{1,kid} + L_{1,m.g.r.}] = [L_{2,kid} + L_{2,m.g.r.}]$$
$$[I_{1,kid}\omega_1 + I_{m.g.r.}\omega_1] = [I_{2,kid}\omega_2 + I_{m.g.r.}\omega_2]$$
$$(mR_1^2)\omega_1 + (700\ kg\cdot m^2)\ \omega_1 = (mR_2^2)\ \omega_2 + (700\ kg\cdot m^2)\ \omega_2$$

$$(40\ kg)(4\ m)^2(3\ rad/sec) + (700\ kg\cdot m^2)\ (3\ rad/sec) = (40\ kg)(1.5\ m)^2\,\omega_2 + (700\ kg\cdot m^2)\ \omega_2$$
$$\Rightarrow \quad \omega_2 = 5.1\ rad/sec.$$

Note: This makes sense. If the *moment of inertia* of the kid decreases as she gets closer to the center of the merry-go-round, the system's *angular velocity* must increase if *angular momentum* is to remain constant.

9.) Example #3 (situation in which angular momentum is conserved through a collision): A child of mass m runs clockwise with velocity v_1 right next to a merry-go-round of mass M, radius R, and moment of inertia $.5MR^2$ (i.e., the child's radius of motion is effectively R). The merry-go-round is moving counterclockwise with angular velocity ω_1, where ω_1 is not related to v_1. The child jumps on at the merry-go-round's edge. What is the final velocity of the child?

a.) The torque that changes the child's motion is produced by the child's interaction with the merry-go-round, and the torque that changes the merry-go-round's motion will be produced by its interaction with the child. In other words, the torques in the system will be internal. As such, the *total angular momentum* before the collision and after the collision must be the same.

b.) With that in mind:

$$L_{1,tot} = L_{2,tot}$$
$$[L_{1,kid} + L_{1,m.g.r.}] = [L_{2,kid} + L_{2,m.g.r.}]$$
$$[-mv_1R + (.5MR^2)\omega_1] = [mv_2R + (.5MR^2)\omega_2].$$

Note 1: We are assuming that the merry-go-round slows with the collision, but that it continues in the counterclockwise (i.e., positive) direction. That means the child reverses direction with the collision.

Note 2: The child's initial angular momentum is associated with clockwise motion. As such, the angular momentum is *negative*. If you don't believe me, do ***rxp*** and determine the appropriate sign for the cross product.

As $v_2 = R\omega_2$ (remember, $v_1 \neq R\omega_1$) we can write:

$$[-mv_1R + (.5MR^2)\omega_1] = [mv_2R + (.5MR^2)(v_2/R)].$$

Canceling out R terms and solving, we get:

$$v_2 = [-mv_1 + (.5MR)\omega_1]/[.5M + m].$$

J.) Simultaneous Sliding and Rolling:

1.) There is a class of problems characterized by the fact that within the set-up, partial sliding occurs (that is, sliding and rolling happening at the same time). The difficulty with such problems is that nothing is conserved and, if there is rolling and translating occurring during the slide, there is no known relationship between the body's center of mass velocity and its angular velocity (i.e., $v_{cm} \neq R\omega$). An example will highlight this kind of situation.

2.) Example: When a bowling ball is thrown by an accomplished bowler, the ball will often start out with no spin at all (i.e., will initially execute a pure slide), picking up spin as friction between the floor and the ball does work on the ball, making it roll. In such cases, there is a period of time during which the ball partially rolls and partially slides. With that in mind, consider the following scenario: A bowling ball is thrown with an initial velocity of v_o, initially executing a pure slide. Almost immediately, it begins to both slide and roll (friction acts immediately). Only after some time does the ball finally execute a pure roll (see Figure 9.31 on the next page).

The question?

a.) Justify the statement, "Nothing is conserved during the collision."

b.) Determine the ball's velocity v_1 when it finally begins to execute a pure roll. This should be in terms of v_o.

Solution:

i.) *Energy* is not conserved as friction (a non-conservative force) is acting. Furthermore, we can not use the *extended conservation of energy equation* because we don't have enough information to determine the magnitude of the frictional force or, for that matter, the distance over which sliding friction acts.

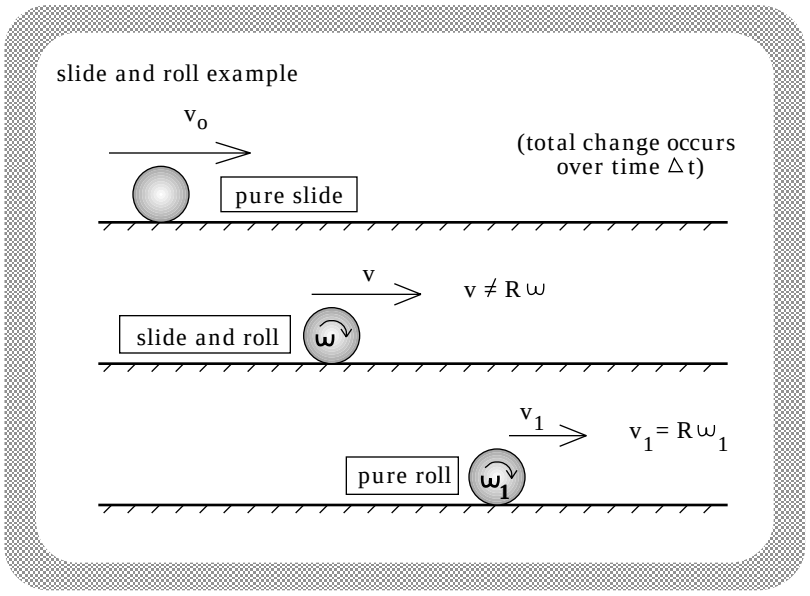


FIGURE 9.31

Momentum is not conserved as there is an external force acting on the ball (friction from the floor is not a part of *the system*). Even if that isn't obvious, the initial momentum is mv_o while the final momentum is mv_2. As the problem is stated, the two *cannot* be the same.

Angular momentum is not conserved as there is an external torque acting on the ball due to friction (in fact, there *is* no initial angular momentum as the ball starts out with a pure slide).

ii.) Because we can't use any of the conservation theorems, we must go back to first principles, which is to say Newton's Second Law (see f.b.d. in Figure 9.32). If we assume an average frictional force f_{avg} acts on the ball over the time it takes for the ball to go from a pure slide to a pure roll, and if we assume that time equals Δt,

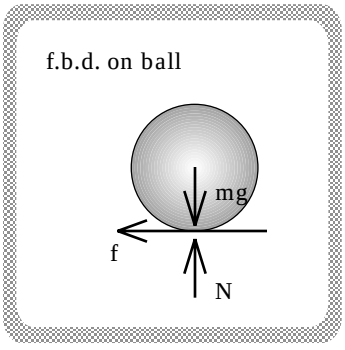


FIGURE 9.32

we can express the acceleration in terms of *velocity* and *time* and write:

$$\underline{\textstyle\sum F_x}:$$

$$-f_{avg} = ma$$

$$= m[(v_1 - v_o) / \Delta t]$$

$$\Rightarrow \quad -f_{avg}\Delta t = m(v_1 - v_o).$$

Note: As the average force is negative, we would expect a negative sign in front of the acceleration term. That sign was not unembedded in our equation because v_1 is less than v_o. As such, the difference $v_1 - v_o$ will itself be negative and the required negative will not be lost.

The moment of inertia of a ball is $I_{ball} = (2/5)mR^2$. Summing the torques acting on the ball over the pure-slide to pure-roll time interval (i.e., Δt), we get:

$$\underline{\textstyle\sum \Gamma_{cm}}:$$

$$-(f_{avg}R) = -I\alpha$$

$$= -[(2/5)mR^2][(\omega_1 - 0) / \Delta t]$$

$$\Rightarrow \quad f_{avg}\Delta t = [(2/5)mR^2](\omega_1)/R$$

$$= [(2/5)mR^2](v_1/R)/R$$

$$= (2/5)mv_1.$$

Note: In this case, we had to unembed the negative sign associated with the negative *angular acceleration* because the final *angular velocity* was, itself, negative (versus the situation we had above where the two velocity terms were both positive but their difference created the negative).

Adding the two impulse equations (remember, $F\Delta t$ is impulse), we get:

$$-f_{avg}\Delta t = m(v_1 - v_o)$$

added to

$$f_{avg}\Delta t = (2/5)mv_1$$

yields

$$0 = (2/5)mv_1 + mv_1 - mv_o$$

$$= (7/5)v_1 - v_o$$

$$\Rightarrow \quad v_1 = (5/7)v_o.$$

3.) How can you identify a problem like this? Whenever you have translation and/or rotation coupled with slippage, you know that: $v \neq R\omega$; the frictional force acting to pull the system out of slippage will always be non-conservative; the frictional force may be external (in this case, it was); and because the conservation theorems do not hold in such situations, it's back to the basics (i.e., N.S.L.).

K.) Parting Shot and a Bit of Order:

1.) For every translational parameter, there is a rotational parameter. If you are unsure what the *rotational kinetic energy* equation is, for instance, think about the *translational kinetic energy* equation and substitute in *I's* for *m's* and ω*'s* for *v's*.

2.) Aside from forces, there are only three or four parameters you will ever be asked to determine on, say, a semester final: accelerations (angular or translational), velocities (angular or translational), distances traveled (angular or translational), and/or time of travel.

As things stand, you have a number of approaches that can generate equations that will allow you to solve for any or all of the parameters listed above. All you have to do is acquire the ability to look at a problem, decide the appropriate approach to use, and generate the needed equations.

----------xxxxx----------

Note from Section E: Is the instantaneous velocity of the contact point of a rolling object really zero? To the right is a series of snapshots of a point on a ball that is rolling with constant angular velocity. Consider what happens when *the point* approaches and comes in contact with the floor. In the *y-direction*, the point transits from moving downward to moving upward. At that transition (i.e., at the contact point), the *y-component* of the point's velocity must be zero. In the *x-direction*, the net horizontal distance traveled by the point as it approaches contact gets smaller and smaller (i.e., it's slowing down), then gets larger and larger after making contact (i.e., it's speeding up). At that transition (i.e., at the contact point), the *x-component* of the point's velocity is zero. In short, the net instantaneous velocity of the point really *is* zero when it touches the ground.

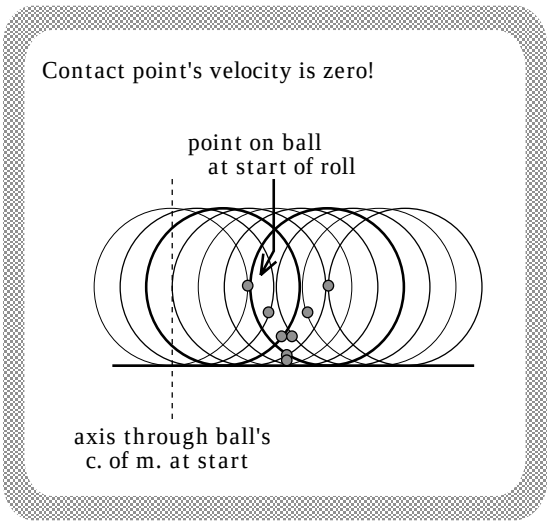


FIGURE 9.33

QUESTIONS

9.1) A *frictional wheel* of mass $m = 8\ kg$, radius $R = .6$ *meters*, and moment of inertia $(1/2)mR^2$ (i.e., that of a disk) is mounted horizontally on a fixed, massless axle. The wheel is initially at rest. A rope is wrapped around the wheel's circumference and a 12 newton force is applied. The axle, whose radius is .2R, provides a 7 newton *frictional force* as the wheel rubs against it (see Figure 0). If the rope-force F acts for 5 seconds:

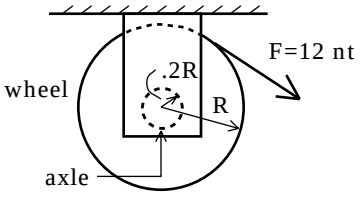


FIGURE 0

a.) Derive an expression for the *angular acceleration* of the wheel during that interval. Once the expression has been determined algebraically, put in the numbers.

b.) What is the *instantaneous translational acceleration* of a point a distance (2/3)R from the wheel's central axis during that interval?

c.) Using the rotational version of Newton's Second Law, derive an expression for the *angular momentum* of the wheel at the end of the period. Put in the numbers once done.

d.) Knowing the *angular momentum* at the end of $t = 5\ seconds$ (i.e., from *Part c*), determine the *angular velocity* of the wheel at the end of the period.

e.) If the *angular displacement* over the 5 second period is approximately 55.15 radians, determine the *angular velocity* at the end of the acceleration using *energy considerations.*

f.) Using *rotational kinematics*, show that the *angular displacement* during the acceleration interval is approximately 55.15 radians.

g.) Use *rotational kinematics* to verify your solution to *Part e.*

9.2) A block of mass $m_1 = .4\ kilograms$ sits on a frictional table (coefficient of kine-tic friction $m_k = .7$). A massless string is attached to the block, threaded over a massive pulley (mass $m_p = .08\ kg$; radius $R_p = .1875\ meters$; and moment of inertia $I_{cm} = .5mR^2$ about the pulley's *center of mass* equal to 1.4×10^{-3} $kg \cdot m^2$), and attached to a hanging mass $m_h = 1.2\ kg$ (see Figure I). If the hanging weight is allowed to freefall from rest:

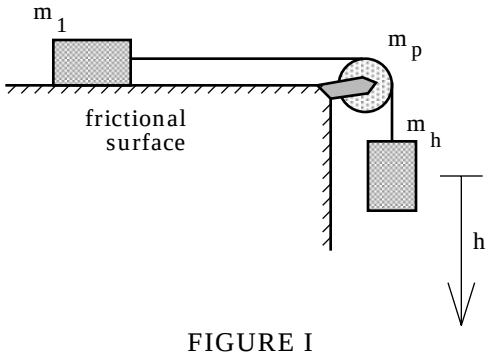


FIGURE I

a.) Derive an expression for the *angular acceleration* of the pulley during the freefall. Put in the numbers when you are finished.

b.) What is the hanging mass's *acceleration* during the freefall?

c.) Derive an expression for the *angular velocity* of the pulley after the hanging weight has dropped a distance equal to 1.5 meters. Do not use kinematics. Put in the numbers after you have finished the derivation.

d.) What is the *translational velocity* of the hanging mass after having dropped a distance equal to 1.5 meters (i.e., when the system is in the configuration outlined in *Part c*)? Don't make this hard. It isn't!

e.) After falling a distance of $h = 1.5$ *meters*, what is the *translational acceleration* of the hanging mass?

f.) Determine the *angular momentum* of the pulley after the hanging weight has fallen a distance $h = 1.5$ *meters*.

9.3) A beam of mass $m_b = 7$ *kg* and length $L = 1.7$ *meters* is pinned at a wall and sits at 30^o with the horizontal (see Figure II). A hanging mass $m_h = 3$ *kg* is attached to the beam's end, and a wire oriented at a 60^o angle with the horizontal is attached 2L/3 units up from the pin.

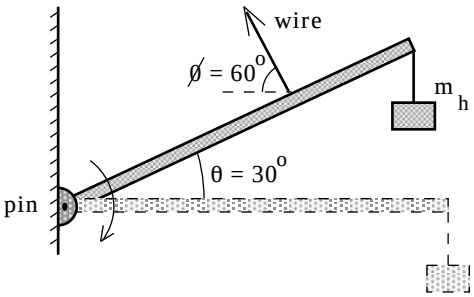


FIGURE II

a.) Derive expressions for the tension T in the wire and the *force components* acting at the pin. Once derived, put in the numbers.

b.) Assuming the *moment of inertia* through the beam's *center of mass* and perpendicular to the beam's length (i.e., into the page) is equal to $(1/12)mL^2$, derive an expression for:

i.) The *moment of inertia* of the beam *about its pin*;

ii.) The *moment of inertia* of the hanging mass about the pin;

iii.) The *moment of inertia* of the entire system about the pin.

c.) The wire is cut. Derive an expression for the initial *angular acceleration* of the beam.

d.) Derive an expression for the *instantaneous translational acceleration* of the beam's *center of mass* just after the wire is cut.

e.) Derive an expression for the beam's *angular velocity* once it has reached a horizontal position. Put in the numbers once done and *do not use kinematics*.

f.) Determine the *translational velocity* of the *beam's center of mass* once it reaches a horizontal position.

g.) Determine the *angular momentum* of the system once the beam has reached the horizontal.

h.) Is *angular momentum* conserved? Explain.

9.4) A merry-go-round has a mass $m = 225\ kg$ and a radius $R = 2.5$ *meters*. Three equally spaced children push it from rest tangent to its circumference until its *angular velocity* and theirs is .8 radians/second. At that point in time, they all hop on. If we approximate the merry-go-round as a disk; if the children each have a mass equal to $m_c = 35\ kg$; and if they push with 15 newtons of force each:

a.) Without using kinematics, determine the *number of radians* through which the children ran during the push-period. (Hint: Think energy! Remember also that $\Delta s = R\Delta\theta$).

b.) Once on, the children proceed to walk from the outer-most part of the merry-go-round to a point $r = 1$ *meter* from the center. Determine the *angular velocity* of the merry-go-round and children once they are at $r = 1$ *meter*.

c.) What quantities are conserved as the children move? Explain.

d.) What quantities are not conserved as the children move? Explain.

e.) Compare the *kinetic energy of the system* when the children were at the outer-most part of the merry-go-round and when they were at $r_1 = 1$ *meter*. Do these calculated energy values make sense in light of your response to *Parts c* and *d*? Comment.

9.5) The freefalling spool shown in Figure III is actually two *wheels* of radius $R_w = .04$ *meters* separated by an *axle* whose radius is $R_a = .015$ *meters*. If the mass of the system is .6 kg and the *moment of inertia* about the system's *central axis* is $I_{cm} = 1.2x10^{-4}\ kg{\cdot}m^2$:

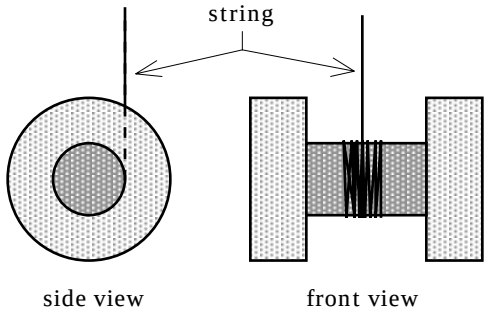


FIGURE IIIa

a.) Derive an expression for the *angular acceleration* of the system using I_{cm} (the reason for so delineating will become evident when you read the next two parts). Put the numbers in at the end.

b.) Determine the *moment of inertia* of the system about an axis *parallel* to the central axis and .015 meters below it. Call this moment of inertia I_a.

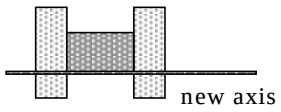


FIGURE IIIb

c.) Derive an expression for the *angular acceleration* of the system using I_a. Does this expression match the one derived in *Part a*?

d.) Determine the *angular velocity* of the system after the system's *center of mass* has fallen $d = .18$ *meters*. Assume the motion starts from rest, *do not* use kinematics, and note that there is a tension acting here (does this last point matter?).

e.) Determine the *velocity* of the *center of mass* after the system has fallen a distance d.

9.6) A loaded beam of mass m and length L is supported at one end by a pin and at the other end by a force $\boldsymbol{F}$ (see Figure IV). If the *force per unit length* on the beam is defined by the function $x = kx$, where k is a constant with the appropriate units and x is measured from the pin:

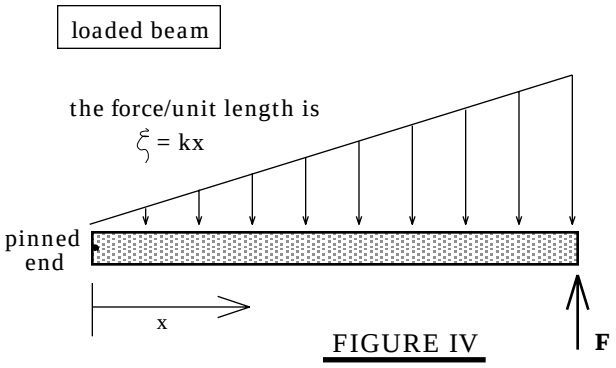


FIGURE IV

a.) What are k*'s* units?

b.) If $\boldsymbol{F}$ is removed, the beam will swing down. Derive an expression for the beam's angular acceleration just as it begins to move (i.e., just after $\boldsymbol{F}$ is removed).

9.7) Two bodies of mass m_2 each are attached to either end of an effectively massless rod of length d. The rod is frictionless and is pinned at its center (see Figure V to the right). A falling wad of putty whose mass is m_1 has velocity v_o just before colliding with the far right mass as shown, sticking to that mass upon contact. If $m_1 = .9$ *kg*, $m_2 = 2$ *kg*, $d = 1.2$ *meters,* and $v_o = 2.8$ *m/s,* determine:

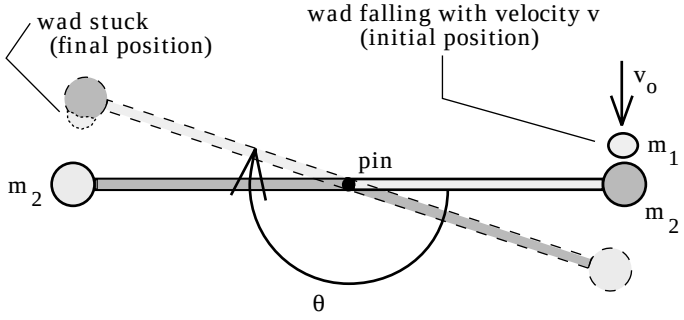


FIGURE V

a.) The magnitude of the *angular velocity* of the rod just after the collision;

b.) The amount of *energy loss* that occurred during the collision, and;

c.) The net *angular displacement* of the system from the time just before the collision to the time when the system came to rest (assume the system does not rotate through a complete revolution).

9.8) A mass m (take it to be a point mass) slides down a frictionless, circular incline of radius R and collides with a pinned meter stick of mass $5m$ initially hanging in the vertical. After the collision, the rod rotates through an angle θ before coming to rest (see Figure VI). Assuming $R = .4d$, determine θ if:

a.) The mass m stays at rest after the collision, and;

b.) The mass sticks to the rod.

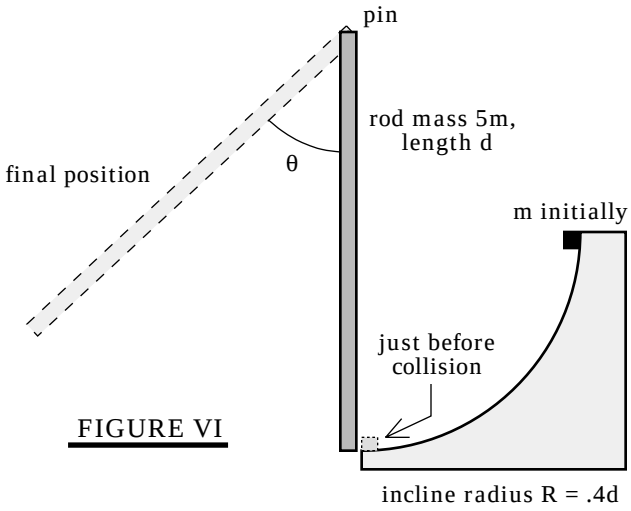


FIGURE VI

9.9) Disk #1 has radius r and mass m. It initially rotates about a frictionless pin with an angular velocity of ω_o. Disk #2 has radius R and mass M. Although it is also supported by a frictionless pin, it is initially at rest. At a given instant, the pin upon which M rests is moved so that the outer edge of the two disks come in contact (see Figure VII). Initially, there is slippage between the two, but finally the two come into a pure roll relative to one another (note that at that time, the velocity of the edge of each disk will be the same). In terms of ω_o, determine the final angular velocity ω_3 of disk #1.

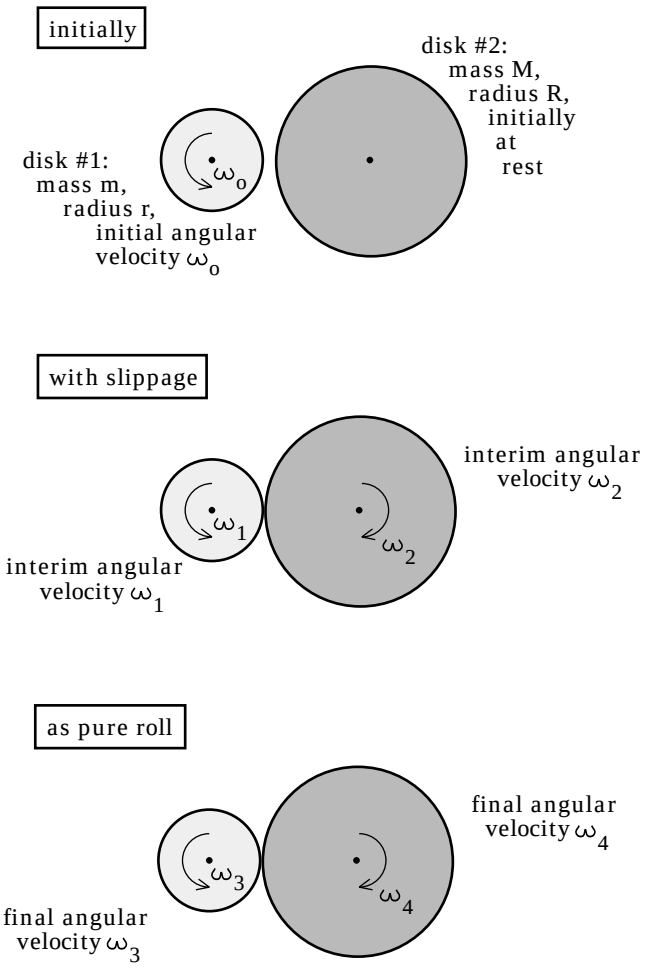


FIGURE VII

Chapter 10

GRAVITATION

A.) Introduction–A Little History:

1.) The study of gravitation is a particular favorite of elementary physics texts because it embraces all of classical Newtonian physics in one giant, special case. As such, prepare to meet your old favorites: Newton's Second Law, centripetal acceleration, conservation of energy, exotic potential energy functions, conservation of angular momentum . . . all the goodies.

2.) One special topic within this chapter of special topics is the work of Johannes Kepler. In 1601, Kepler essentially absconded with the celestial observations of just-deceased Tycho Brahe (Brahe was the last great astronomer to make observations without the use of a telescope). It took years, but Kepler's analysis of Brahe's data laid the mathematical foundation for what are now known as Kepler's three laws of planetary motion. We will spend a little time examining these laws later.

3.) The main event in this chapter will be centered on the work of Sir Isaac Newton. In 1665, Newton began to muse about the moon and its orbit around the earth. A brief summary of his preliminary thoughts on the famous apple is presented below (you won't be tested on this summary):

a.) The acceleration of a freefalling apple near the earth's surface is 9.8 m/s^2, where the apple is approximately 4000 miles from the earth's center. The centripetal acceleration of the moon as it orbits the earth is .002722 m/s^2 (this he determined knowing the moon's *period of revolution*, the *distance from the earth to the moon*, and the fact that the moon's acceleration is centripetal), where the moon is approximately 240,000 miles from the earth's center.

b.) Force is proportional to acceleration (this was to become his second law), which means that:

$$F_{apple} \propto a_{apple}$$

and

$$F_{moon} \propto a_{moon}.$$

c.) Since gravitational force decreases as one gets farther away from the earth, the relationship between gravitational force and the distance *r* from the earth must be inversely proportional. Assuming that *r* might be raised to an exponent other than one, it must be true that:

$$F_{apple} \propto 1/(r_{apple})^n$$

and

$$F_{moon} \propto 1/(r_{moon})^n.$$

d.) Putting the observations in *Parts b* and *c* together, we conclude that:

$$a_{apple} \propto 1/(r_{apple})^n$$

and

$$a_{moon} \propto 1/(r_{moon})^n.$$

e.) Taking the ratio of the two expressions, then putting in numbers yields:

$$\frac{a_{apple}}{a_{moon}} \propto \frac{\frac{1}{(r_{apple})^n}}{\frac{1}{(r_{moon})^n}}$$

$$\Rightarrow \quad \frac{a_{apple}}{a_{moon}} \quad \propto \quad \left(\frac{r_{moon}}{r_{apple}}\right)^n$$

$$\Rightarrow \quad \frac{\left(9.8 \ \frac{m}{s^2}\right)}{\left(.002722 \ \frac{m}{s^2}\right)} \propto \left(\frac{(240{,}000 \text{ mi})}{(4000 \text{ mi})}\right)^n$$

$$\Rightarrow \ 3600\alpha(60)^n.$$

f.) The exponent *n* that allows this to be an equality is *n = 2*.

g.) With that observation, Newton concluded that the gravitational force exerted between any two objects (the earth and moon or the earth and an apple or whatever) must be inversely proportional to the square of the distance between the two objects (i.e., $F \alpha 1/r^2$).

Note: Newton originally abandoned this analysis for a number of years because the then-accepted distance between the earth and moon was incorrect and, hence, his calculation yielded an *n* value that was not a whole number. It

was not until a more accurate measurement was made that he went back to his theory and, lo and behold, things worked out.

B.) Newton's Gravitational Law (in general):

1.) From the observations and analysis summarized in the previous section, Newton concluded that any two masses will be attracted to one another due to a force he called *gravity*.

Note: He didn't like this idea very much. Why? Because he couldn't see any good reason why two bodies, simply by virtue of the fact that they happen to be massive, should be attracted to one another. He put forth the idea only because attraction seemed to exist and because he couldn't come up with a better explanation for the phenomenon. In fact, it wasn't until Einstein that an alternative explanation for gravitational effects (eg., objects falling toward the earth when released, the moon orbiting the earth, etc.) was presented to the world.

We use Newton's theory of gravity because it is a model that works. Nevertheless, it is generally conceded that gravitational forces do not, in reality, exist (gravitational *effects* exist but the force postulated by Newton . . . no!). We'll talk about Einstein's view later.

2.) The magnitude of Newton's gravitational force is proportional to the mass of the two attracting bodies and inversely proportional to the square of the distance between the *center of mass* of each.

a.) Newton's relationship in its most general form is:

$$\mathbf{F}_{grav} = G\frac{m_1 m_2}{r^2}(-\mathbf{r}),$$

where m_1 and m_2 are the masses involved, r is the distance between the center of masses of the two bodies, and $\boldsymbol{r}$ is a unit vector directed along the line between the two bodies (this is called a *radial* unit vector).

Note: A *radial* unit vector is drawn *from the field producing mass* toward the mass experiencing the field (see Figure 10.1a). The NEGATIVE SIGN in front of the radial unit vector in Newton's expression means that the gravitational force is *opposite* the direction of $\boldsymbol{r}$. That is, it is an attractive force.

People get confused with the negative sign when they use the force function in conjunction with a Cartesian coordinate system because a negative sign doesn't mean the

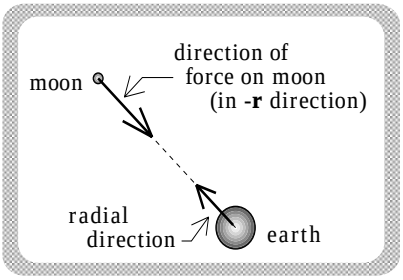


FIGURE 10.1a

same thing in that system as it does in a polar spherical system (i.e., the system in which radial vectors exist).

Bottom Line: When doing gravitational problems, use Newton's expression to determine the force *magnitude*, then decide whether the force direction is positive or negative *relative to the coordinate system being used.*

Example: The earth is placed at the origin of a Cartesian coordinate system and the moon is placed on the *x axis* to the left of the earth (see Figure 10.1b). In that case, the magnitude of the attractive gravitational force on the moon is $Gm_e m_m/x^2$, and the direction of the attractive force is *positive*.

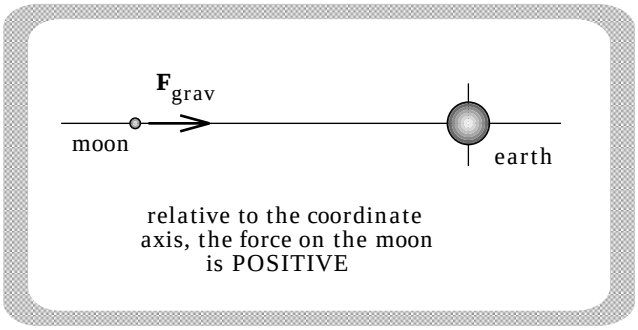


FIGURE 10.1

b.) The proportionality constant G, called the *Universal Gravitional Constant*, is equal to 6.67×10^{-11} nt·m^2/kg^2.

c.) Gravitational forces are always attractive.

Note: Newton correctly reasoned that if a gravitational-type force did exist, the net force on an apple would be the *vector sum* of all the individual gravitational forces exerted by all the individual pieces of matter that make up the earth (see Figure 10.2).

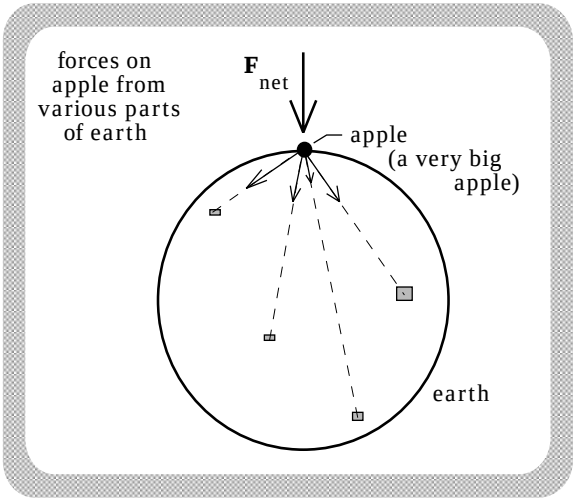


FIGURE 10.2

Newton circumvented the problem of vectorially adding billions of tiny force quantities by assuming that the net gravitational effect from all of the pieces was the same as if all the earth's mass was centered at the earth's *center of mass*. In that case, all he had to worry about was the point-mass apple and the point-mass earth.

Unfortunately, there existed no mathematics at the time from which he could justify that assumption. His solution? He took a few years off from his physics pursuits and created Calculus. That's right, folks, that mathematical discipline you have been banging your head into for the last year or two--it didn't exist before Newton. He made it up so he could finish a gravitation problem on which he was working (actually, Leibniz independently created his own version at about the same time).

3.) To see how well this force function matches experimental observation, consider the following: What is the gravitational force between the earth and a 100 kg man standing on the earth's surface?

a.) The mass of the earth is $5.97\text{x}10^{24}$ kg while its radius is $6.37\text{x}10^{6}$ meters. According to Newton's force function, the magnitude of the force felt by the man is:

$$\begin{aligned} F_g &= G\,[\,m_{man}\,m_e\,/\,r^2\,] \\ &= (6.67\text{x}10^{-11}\ \text{nt}\cdot\text{m}^2/\text{kg}^2)[(100\ \text{kg})(5.97\text{x}10^{24}\ \text{kg})/(6.37\text{x}10^{6}\ \text{meters})^2] \\ &= 9.81\text{x}10^{2}\ \text{newtons}. \end{aligned}$$

b.) As the radius of the earth is enormous, this force will be approximately the same whether the man is located on the earth's surface or some small distance above the earth's surface (i.e., a few hundred meters). That is why we say the gravitational force on a body near the earth's surface is, for all intents and purposes, a constant.

c.) Because the gravitational force on a body will be nearly constant near the earth's surface, we can define a function that allows us to determine that force more easily. Specifically, as the gravitational force is proportional to the mass m_{man}, we can write:

$$F_{grav} = m_{man}g,$$

where g is the proportionality constant required to make the proportion an equality.

i.) Let's determine g by using the force value calculated with Newton's general gravitational function. According to our calculation, a 100 kg man will feel a force due to the earth's gravitational attraction equal to $9.81\text{x}10^{2}$ newtons (in my country, we call this *the man's weight*).

ii.) If that be the case, we can write:

$$\begin{aligned} F_g &= m_{man}\,g \\ \Rightarrow\quad 9.81\text{x}10^{2}\ \text{nt} &= (100\ \text{kg})g \\ \Rightarrow\quad g &= 9.8\ \text{m/s}^2. \end{aligned}$$

iii.) This constant is equal to the experimentally observed value for the *gravitational acceleration* of an object located near the surface of the earth.

4.) An additional example: How far from the earth's surface must an astronaut in space be if she is to feel a gravitational acceleration that is half what she would feel on the earth's surface?

a.) This is a tricky question in one respect only. The gravitational expression defined by Newton had a distance term in it, but the distance was defined as that between the *center of masses* of the two bodies. Our question asks for the distance r between the astronaut and the earth's surface, not between the astronaut and the earth's center of mass (i.e., $r + r_e$, where r_e is the radius of the earth). Using the appropriate distance, we can write:

$$\underline{\Sigma F_c}:$$
$$GmM/(r+r_e)^2 = ma_1$$
$$r = (GM/a_1)^{1/2} - r_e.$$

b.) We know that a_1 is one-half the gravitational acceleration here on Earth, or 4.9 m/s^2. We know that the universal gravitational constant $G = 6.67x10^{-11}\ nt{\cdot}m^2/kg^2$, the mass of the earth $M = 5.97x10^{24}\ kg$, and the radius of the earth $r_e = 6.37x10^6\ meters$. Using all this information, we can write:

$$r = [(6.67x10^{-11}\ nt{\cdot}m^2/kg^2)(5.97x10^{24}\ kg)/(4.9\ m/s^2)]^{1/2} - (6.37x10^6\ m)$$
$$= 2.64x10^6\ meters.$$

This is approximately four-tenths of the radius of the earth.

C.) Kepler's Laws:

Note: Kepler determined all of his laws from observational data. The mathematical justifications presented below were generated much later.

1.) Kepler's First Law: Planets move in *elliptical orbits* having the *sun at one focal point*.

a.) This is called the Law of Orbits.

b.) Justification: Using the *conservation of angular momentum* and the *conservation of energy* with a *potential energy function* appropriate to an inverse-square force field, it is possible to derive an expression for $r(\theta)$ (i.e., the distance between the sun and a planet as a function of angular position within the orbit). Although it is not a pleasant integral to evaluate, doing so yields a position function $r(\theta)$ that turns out to be the equation of an ellipse.

2.) Kepler's Second Law: A line joining any planet to the sun sweeps out *equal areas in equal times* (i.e., $dA/dt = constant$).

a.) This is called the Law of Areas.

b.) Proof (although you won't be expected to reproduce this, it is instructional to see how it is achieved):

i.) The area of a triangle is one-half its *base* times its *height*. The differential area of the triangle shown to the right in Figure 10.3 is:

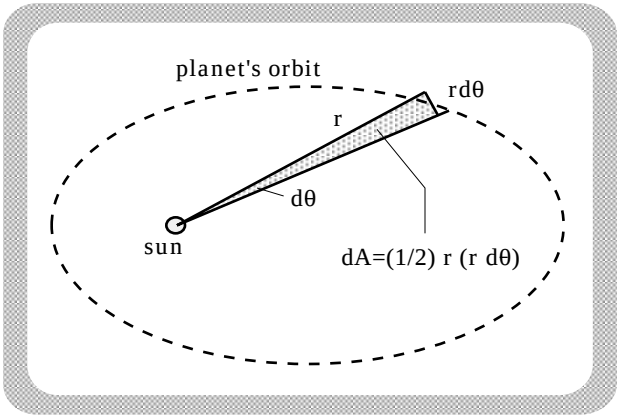


FIGURE 10.3

$$dA = (1/2)(\text{height})(\text{base})$$
$$= (1/2)\ r\ [r(d\theta)]$$
$$= .5\, r^2(d\theta).$$

ii.) IF the area swept out by a planet were to *increase at a constant rate* (i.e., dA/dt is constant) no matter where the planet was in its orbit, then in a specific time interval (say, 10 seconds) the area swept out would always be the same.

Put another way, if we can show that dA/dt is a constant, we will have proved Kepler's Second Law.

To start:

$$dA/dt = d[.5\, r^2 d\theta]/dt$$
$$= .5(2r)(dr/dt)d\theta + .5r^2(d\theta/dt).$$

iii.) A derivative implies that Δt goes to zero. When that happens, $d\theta$ goes to zero. Noting that $d\theta/dt = \omega$, our expression becomes:

$$\begin{aligned} dA/dt &= .5(2r)(dr/dt)d\theta + .5r^2(d\theta/dt). \\ &= .5(2r)(dr/dt)(0) + .5r^2(\omega) \\ &= .5r^2(\omega). \end{aligned}$$

iv.) We know that both r and ω are varying in time, but does their product vary in time? To see, consider the planet's *angular momentum*.

$$\begin{aligned} |\mathbf{L}| &= |\mathbf{r}\mathrm{x}\mathbf{p}| \\ &= r\ (mv)\ \sin\ \theta. \end{aligned}$$

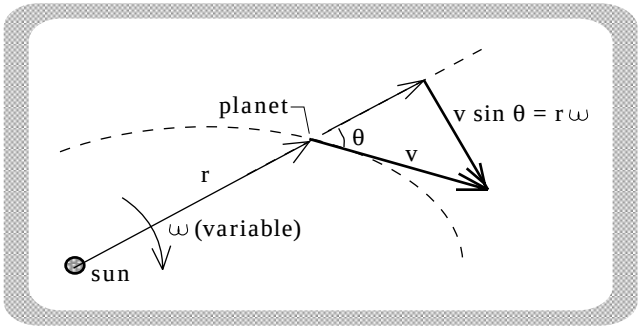


FIGURE 10.4

v.) The component of velocity *perpendicular to r* is $v\ sin\ \theta$. It is also equal to $r\omega$ (see Figure 10.4). As such, we can write:

$$\begin{aligned} |\mathbf{L}| &= r\ [m(v\ \sin\ \theta)] \\ &= r\ [m\quad (r\omega)\ \ ] \\ &= mr^2\omega. \end{aligned}$$

vi.) *Angular momentum* for planetary motion *is constant*. How do we know this? The only force exerted on the planet is gravitational. That means the force is directed radially along the line between it and the sun (we are assuming there are no other large celestial objects--planets included--exerting a gravitational force on the planet). If F is parallel to r, the angle between the line of the two vectors is zero and the cross product $\boldsymbol{rxF}$ is zero. In other words, there are no external torques acting on the planet and, hence, its *angular momentum* must be constant.

vii.) The planet's *rate of change of area* is:

$$dA/dt = (.5)r^2\omega,$$

whereas its CONSTANT *angular momentum* is:

$$L = (m)r^2\omega.$$

That is, both are proportional to the same quantity $r^2\omega$.

viii.) Conclusion: The *angular momentum* of a planet is proportional to $r^2\omega$ and is a CONSTANT. As dA/dt is also proportional to $r^2\omega$, it must also be a CONSTANT. Thus, Kepler's Second Law is secure.

Note: As was stated at the beginning, this proof is *not* something on which you will be tested.

3.) Kepler's Third Law: The *square of the period* of any planet about the sun is proportional to the *cube of the planet's mean distance* from the sun.

a.) This is called the Law of Periods.

b.) Theoretical justification (the ins-and-outs of the following ARE IMPORTANT--you will see problems based on it on your next test):

i.) Consider two celestial objects (a planet and its moon or two stars in a binary star system . . . whatever) of arbitrary mass M and m moving in circular orbits. Assume also that $M > m$ (see Figure 10.5).

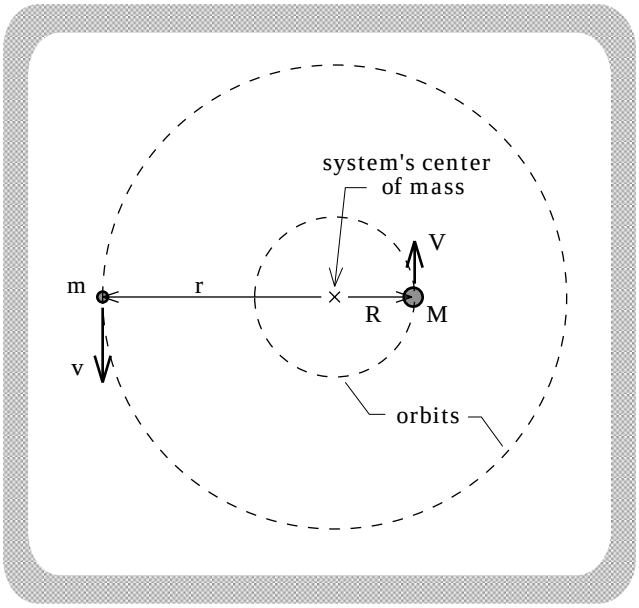


FIGURE 10.5

ii.) In such cases, the two bodies will *not* orbit one another but rather will orbit around the *center of mass* of the two-body system (our moon appears to be orbiting a stationary Earth only because the earth is so massive that the *center of mass* of the earth-moon system is *inside* the earth about two-thirds of the way from the earth's center).

The analysis we are about to do is general to all elliptical orbits even though we have chosen a circular path for simplicity.

iii.) The only force accelerating mass *m* is the gravitational pull provided by mass *M*. Noting that that force is centripetal, we can use N.S.L. to write:

$$\underline{\textstyle\sum F_c}\textbf{:}$$
$$GmM/(r+R)^2 = ma_c$$
$$= m(v^2/r).$$

Note: There will be times when the distance term in the centripetal force expression and the distance term in the gravitational force expression will, to a very good approximation, be the same. *This will not always be the case*, as the above situation shows.

iv.) We can determine *m's* velocity by noting that in the time required to orbit once (i.e., in a time interval equal to the planet's period *T*), the object moves a distance equal to the circumference of its path, or *2πr meters*. The ratio of *distance traveled* to *time* gives us the desired velocity magnitude. Doing so yields:

$$v = (2\pi r)/T.$$

v.) Substituting this into our force equation yields:

$$GmM/(r+R)^2 = m(v^2/r)$$
$$= m[(2\pi r/T)^2/r].$$

vi.) Dividing out the *m* terms and rearranging yields:

$$T^2 = [4\pi^2/GM]\ [r(r+R)^2].$$

vii.) The question? What happens if *M* is much larger than *m*? As *r>>R* (">>" means *much greater than*), our equation becomes:

$$T^2 = [4\pi^2/GM]r^3.$$

viii.) This is Kepler's Third Law: When dealing with planets (i.e., relatively small celestial objects) orbiting the sun (i.e., a much larger celestial object), the square of the period (T^2) will be proportional to the cube of the mean distance (r^3) between the sun and the planet (in the case of elliptical orbits, this distance *r* equals the length of the semi-major axis).

Note: Kepler's Third Law is the only one of his laws that is not exact. It is predicated on the assumption that one object is large while the other is small in comparison. In our solar system, the only planet that is even remotely close to the sun's mass is Jupiter . . . and it is only a few percent of the mass of the sun.

4.) The APPROACH outlined above is very powerful in generating useful equations when dealing with orbital motion. Knowing that $T^2 \alpha r^3$ is nice, but your goal should be to understand how the result was acquired.

a.) The technique was simple: Use N.S.L. and the fact that the motion is *centripetal* to generate an expression for the velocity of the body in its orbit.

b.) Note that the *period* and *velocity* are related by $v = (2\pi r)/T$.

D.) Gravitation Inside a Massive Object:

1.) A seeming paradox arises when one tries to use Newton's gravitational force expression to determine the force on a mass m_1 when inside the earth (i.e., for $r < R_e$). Specifically, if the magnitude of that force is:

$$F_g = Gm_1m_e/r^2,$$

what happens when the mass is at the earth's center and $r = 0$? Mathematically, the above force expression explodes.

2.) The problem is solved by observing a peculiarity about inverse square functions.

a.) Assume a tunnel has been drilled completely through the earth. An object of mass m is placed in the tunnel a distance r from the earth's center. What is the gravitational force acting on the body?

b.) Before attempting to do this problem, draw a horizontal line through the object and rotate it to generate two sections, one below and one above the body. Once done, construct the sphere upon which the object sits (call its radius r). The set-up is shown in Figure 10.6 on the next page.

c.) What can we say about the gravitational force acting on the body due to the mass above the line (note that this mass is all *outside* the inner sphere)?

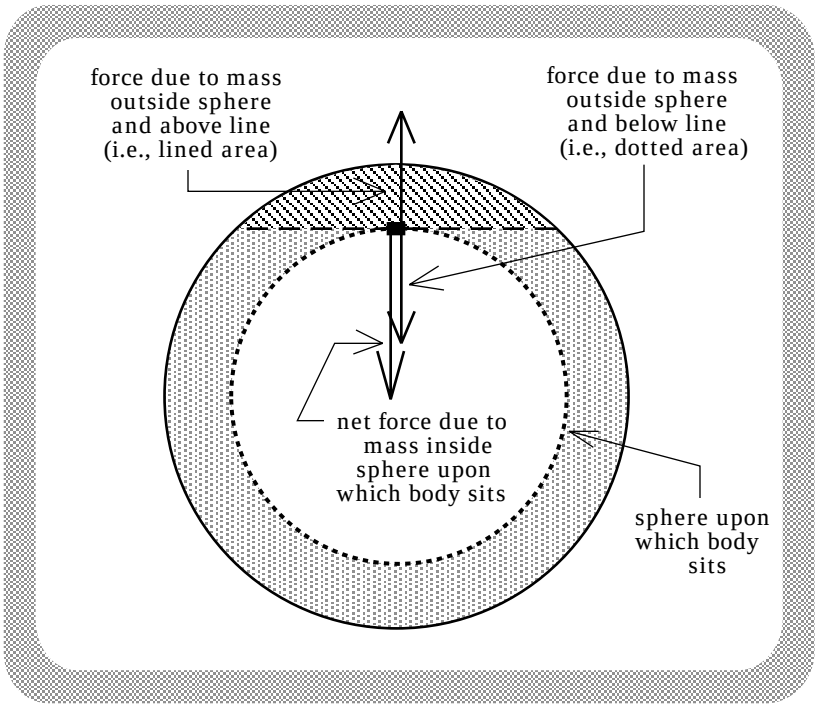

force due to mass outside sphere and above line (i.e., lined area)
force due to mass outside sphere and below line (i.e., dotted area)
net force due to mass inside sphere upon which body sits
sphere upon which body sits

FIGURE 10.6

i.) The direction of the net force will be vertically upward.

ii.) The magnitude of the force will be the consequence of a relatively small amount of mass that is located, on average, relatively close to the object.

d.) What can we say about the gravitational force acting on the body due to the mass in the lower section and *outside the sphere*?

i.) The magnitude of this force will be downward; and

ii.) Its magnitude will be the consequence of a relatively large amount of mass that is located, on average, relatively far from the object.

e.) Although it might not be obvious, the downward force due to the mass in the lower section *outside the sphere* will be equal in magnitude to the upward force due to the mass in the upper section *outside the sphere*.

f.) How can this be? It is a consequence of the inverse square law. Assuming symmetry, the net gravitational force on an object will be generated only by the mass found *inside the sphere upon which the object rests*. If the gravitational force function had not been of the form $1/r^2$, this would not be true (for those with nothing better to do, you might find it amusing to try to prove this).

Note: This holds true even when the object is on the earth's surface. In that case, the mass *inside the sphere upon which the object rests* is the total mass of the earth.

3.) With this observation, all we have to do is determine the amount of mass inside the sphere upon which the object rests, then use that mass in our gravitational expression:

$$F_{grav} = G(m_{inside\ sphere})m_{obj}/r^2.$$

4.) The easiest way to determine m_{inside} is to calculate the *proportion* of mass inside the sphere (i.e., $V_{inside}/V_{total} = (4/3)\pi r^3/(4/3)\pi r_e^3$), then multiply that by the total mass of the earth m_e. The solution is $m_{inside} = (r^3/r_e^3)m_e$. The problem with this approach, at least from an instructional perspective, is that it only works for homogeneous bodies. The approach presented below works for any homogeneous or inhomogeneous body as long as there is both symmetry and a known *volume density function* for the system.

a.) Assume the earth is homogeneous with a constant volume density function ρ equal to:

$$\rho = m_e/V_e$$
$$= m_e/[(4/3)\pi r_e^3],$$

where m_e is the earth's mass, r_e is the earth's radius, and $(4/3)\pi r_e^3$ is the earth's total volume.

b.) Consider the spherical shell of radius a and differential thickness da (see Figure 10.7). The differential volume dV of the shell equals the surface area of the shell ($4\pi a^2$) times the shell's thickness da. Written mathematically, this is:

$$dV = (4\pi a^2)da.$$

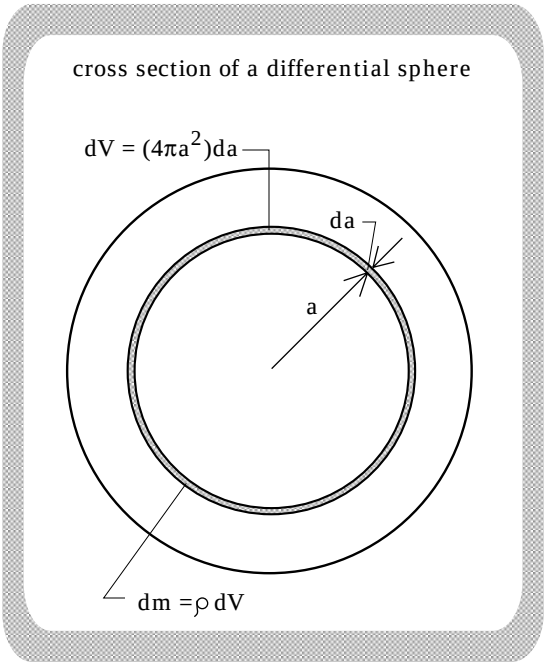


FIGURE 10.7

c.) The differential mass *dm* in the differential volume *dV* equals:

$$dm = \rho\, dV.$$

Substituting ρ and *dV* into the above expression yields:

$$\begin{aligned} dm &= [m_e/[(4/3)\pi r_e^{\,3}]]\ [(4\pi a^2)da] \\ &= (3m_e/r_e^{\,3})a^2 da. \end{aligned}$$

d.) Summing all the *dm's* between $a = 0$ to $a = r$ will give us the total mass inside the sphere upon which the body sits.

5.) Using all this in conjunction with Newton's universal gravitational equation yields:

$$\begin{aligned} F_{grav} &= -G\,\frac{m_1\left[m_{\text{inside sphere}}\right]}{r^2} \\ &= -G\,\frac{m_1\left[\int_{a=0}^{r} dm\right]}{r^2} \\ &= -G\,\frac{m_1\left[\left(\frac{3m_e}{r_e^{\,3}}\right)\int_{a=0}^{r} a^2 da\right]}{r^2} \\ &= -G\,\frac{m_1}{r^2}\left[\left(\frac{3m_e}{r_e^{\,3}}\right)\left[\frac{a^3}{3}\right]_{a=0}^{r}\right] \\ &= -\left(G\,\frac{m_1 m_e}{r_e^{\,3}}\right)r. \end{aligned}$$

6.) In other words, though the force between any two bits of the system is proportional to $1/r_{bit}^{\,2}$, where r_{bit} is the distance between the two bits, the expression that characterizes the overall gravitational force on a body situated somewhere *inside* a larger structure is a linear function of *r*, where *r* is the radius of the sphere upon which the body sits. Being linear, that function equals zero at $r = 0$, just as we would expect.

7.) This approach works even when a body is symmetric but inhomogeneous (the earth, for instance, is not a uniformly solid object). In such

cases, the approach is exactly as presented above with the variable dm equaling $\rho\, dV$. The only alteration is in the fact that ρ is not constant for such problems--it must be defined as a function of r.

E.) Potential Energy and Gravitational Fields:

1.) Until now, our energy dealings with gravity have been solely from the perspective of a body near the earth's surface. That is, we have assumed a constant gravitational force field and have defined an appropriate gravitational *potential energy function* for the situation (i.e., a potential energy function that is linear--*mgy*--and that has a floating *zero potential energy level*).

In actuality, the earth's gravitational force field varies, diminishing as we get farther and farther away from the planet. The field is still conservative, but a linear *potential energy function* is no longer appropriate. In short, a different *potential energy function* must be derived.

2.) To determine a force field's *potential energy function*, we need to determine the amount of work the field does as a body moves from the *zero potential energy point* to some arbitrary point in the field. In the case of gravity, the *zero point* must be at infinity (that is where the gravitational force due to the field producing body is itself zero).

To make the calculation simple, assume that the field producing mass m_1 is at the origin of our coordinate axis and that m_2 moves in from infinity along the *x-axis* to a position r units from m_1 (this means that r in this equation is the *magnitude* of the distance between m_1*'s* center of mass and m_2*'s* center of mass). Mathematically, this operation looks like:

$$\begin{aligned} U(r) - U(\infty) &= -\int F \bullet dr \\ &= -\int_{x=\infty}^{r} \left(-G\frac{m_1 m_2}{x^2}i\right) \bullet (dx\,i + dy\,j + dz\,k) \\ &= Gm_1m_2\int_{x=\infty}^{r}\left(\frac{1}{x^2}\right)dx \\ &= Gm_1m_2\left(\frac{-1}{x}\right)_{x=\infty}^{r} \\ &= -G\frac{m_1m_2}{r}. \end{aligned}$$

3.) The negative sign should not be upsetting if you remember that potential energy functions make sense only when used in the context of $W_F = -\Delta U_F$. That is, only *changes* of potential energy have any meaning. As long as the *potential energy change* between two points yields the net work done on a body by the field as the body moves from the one point to the other point in the field, all is well. If there happens to be a negative sign in the expression that defines the potential field, so be it.

Big Note: The potential energy defined by this potential energy function is *not* zero at the earth's surface--it is zero at infinity. As such, a body resting on the earth's surface *has potential energy*. Don't forget this when using the *conservation of energy* equation.

F.) Potential Energy Wrapped up in a System of Bodies:

1.) To determine the two-body, gravitational potential energy function in the section above, we used the definition of *potential energy* to determine the amount of work gravity did on a body that was brought in from infinity to some arbitrarily chosen point *r*.

In its most general form, that calculation is denoted as:

$$U(r) - U(\infty) = -\int \mathbf{F} \cdot d\mathbf{r}.$$

Using the above form, the scalar expression derived for gravity was:

$$U(r) = -Gm_1m_2/r,$$

where *r* was effectively the distance between the *centers of mass* of the two bodies.

2.) The question arises, "How much potential energy is wrapped up in a system of masses?" To answer this question, we must extend our thinking just a bit. Specifically:

a.) Assume we want to assemble a three-mass system, the first mass of which is m_1.

b.) We have already derived an expression for the amount of *potential energy* the system will have after m_2 moves from infinity to a position a distance $r_{1,2}$ units from m_1. That expression is:

$$U_1(r_{1,2}) = -Gm_1m_2/r_{1,2}.$$

c.) Continuing in this vein, if a mass m_3 is brought in from infinity to a distance $r_{1,3}$ from mass m_1 and a distance $r_{2,3}$ from mass m_2, the ADDITIONAL gravitational potential energy provided to the system due to the presence of this new mass will be:

$$U_2(r_{1,3}) + U_3(r_{2,3}) = (-Gm_1m_3/r_{1,3}) + (-Gm_2m_3/r_{2,3}).$$

d.) The *total potential energy* of the system will therefore be:

$$U_{tot} = -Gm_1m_2/r_{1,2} - Gm_1m_3/r_{1,3} - Gm_2m_3/r_{2,3}.$$

Note 1: It doesn't matter which mass is placed first or which is brought in last, this expression will always be the same.

Note 2: If there were a fourth mass brought in, the expression would have *six* terms in it.

3.) It is interesting to note that the potential energy of this system is negative. What significance has this? Negative potential energy means the system is bound. Put another way, if you want to disassemble the system, you will have to provide to the system energy in the amount of $Gm_1m_2/r_{1,2} + Gm_1m_3/r_{1,3} + Gm_2m_3/r_{2,3}$.

This is called the *binding energy* of the system. It is defined as the amount of energy required to disassemble the system so that the parts no longer interact with one another. In the case of gravitational systems, it is the amount of energy required to push the masses infinitely far apart.

Note: The idea of *binding energy* is especially important in nuclear physics where there is an interest in the amount of energy required to spring a subatomic particle loose from its atom.

G.) Orbital Motion and Energy Considerations:

1.) It is interesting to see how energy distributes itself within an orbit-related system. Specifically, consider a small mass m (a satellite) orbiting in an approximately circular path at a distance r_1 units from the center of a much larger mass M (a planet). How much potential and kinetic energy is in the system, assuming that M is stationary?

a.) Mass *m's* potential energy is $U(r) = -G\dfrac{mM}{r}$.

Note: The *r* term in the potential energy function is defined as the distance from the center of the field-producing body (mass *M* in this case).

b.) Mass *m's* kinetic energy is $KE_r = (1/2)mv^2$.

i.) To finish this calculation, we need *v*. Consider Newton's Second Law and the fact that the motion is centripetal. Taking the direction of the gravitational force to be positive, we can write:

$\underline{SF_c}$:

$$G\frac{mM}{\left(r_{\text{between masses}}\right)^2} = ma_c$$

$$= m\left(\frac{v^2}{\left(r_{\text{radius of circle}}\right)}\right)$$

ii.) Note that the *r* terms on either side of this expression are the same. Why? The *r* term on the left side of N.S.L. is defined as the distance between *m's* center of mass and *M's* center of mass (if the bodies are far apart and considered point masses, this is *the distance between the two bodies*). The *r* term on the right side of N.S.L. is, according to the definition of centripetal acceleration, the radius of the circle upon which *m* travels. This is always measured from the SYSTEM'S *center of mass* (see Figure 10.8). Because *M* is so much more massive than *m* in this case, the distance between *m* and the system's center of mass is essentially the same as the distance between the two bodies, and the *r's* are equal.

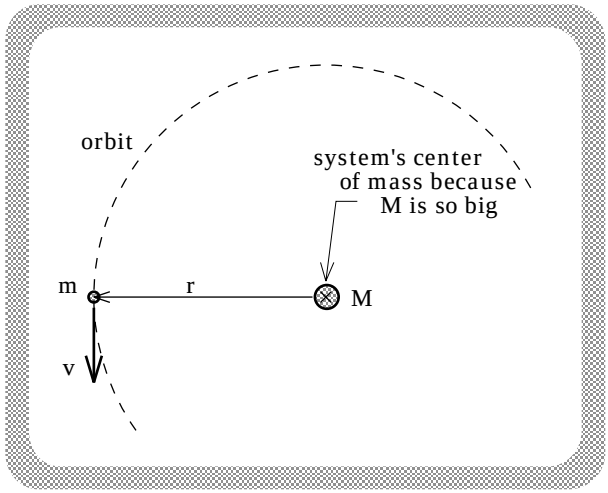


FIGURE 10.8

iii.) Solving for v, we get:

$$v^2 = \left(G\frac{M}{r} \right).$$

iv.) Finishing off the kinetic energy expression yields:

$$\begin{aligned} KE_r &= (1/2)mv^2 \\ &= \frac{1}{2}m\left(G\frac{M}{r} \right) \\ &= \frac{1}{2}G\frac{mM}{r}. \end{aligned}$$

c.) Putting it all together, we get the total energy (often referred to as *the mechanical energy*) E_{tot} of the system as:

$$\begin{aligned} E_{tot} &= \quad KE \quad + \quad U \\ &= \frac{1}{2}G\frac{mM}{r} - G\frac{mM}{r} \\ &= -\frac{1}{2}G\frac{mM}{r} \qquad \text{(this is usually written as } -G\frac{mM}{2r}\text{)}. \end{aligned}$$

2.) How much energy is required to move a satellite from a circular orbit of radius r_1 to a circular orbit of radius r_2, assuming the change occurs slowly (i.e., the satellite moves in a slowly changing series of circular paths).

a.) The temptation is to use the modified conservation of energy equation and write:

$$E_{tot,1} + W_{extraneous} = E_{tot,2}$$

$$\Rightarrow \left(\frac{1}{2}mv_1^2 - G\frac{mM}{r_1} \right) + W_{ext} = \left(\frac{1}{2}mv_2^2 - G\frac{mM}{r_2} \right)$$

Note that this expression needs both the radii *and* the velocities required for the satellite to sustain each orbit.

b.) BUT, because the sum of the potential and kinetic energy for a body moving in a CIRCULAR ORBIT is $-G\frac{mM}{2r}$, we can instead write:

$$E_{tot,1} + W_{ext} = E_{tot,2}$$

$$\Rightarrow \quad -G\frac{mM}{2r_1} + W_{ext} = -G\frac{mM}{2r_2}$$

c.) What's nice about this relationship is that you don't have to determine the velocity of the satellite in the final orbit.

Note: This amount of energy is *not* the same as the amount of energy required to take a satellite *sitting on the earth* and put it into orbit. In that case, you need to determine the required kinetic energy, given the orbit's radius, and add that to the *potential energy* <u>*difference*</u> (i.e., $\Delta U = (-Gm_s m_e/r_s) - (-Gm_s m_e/r_e)$) between the initial and final positions of the satellite. How so? $-\Delta U$ equals the amount of work *gravity* does on the body as it ascends; minus that amount (i.e., $+\Delta U$) equals the amount of work *you* must do to overcome gravity.

3.) In some cases, a more interesting way to look at energy considerations in the context of circular orbital motion is to see how the *kinetic energy*, *potential energy*, and *total energy* play off one another.

a.) To begin with, notice that for a particular circular orbit of radius r_1, there must be a given amount of potential energy ($-GmM/r_1$), kinetic energy ($GmM/2r_1$), and total energy ($-GmM/2r_1$) in the system if a body is to stay in that orbit.

Put another way, there is only one *kinetic energy/ potential energy/ total energy* combination, given *m* and *M*, that will allow a body to hold a particular orbit.

b.) To see how these energy quantities relate, consider the *Energy versus Orbital Radius* graph shown in Figure 10.9 to the right.

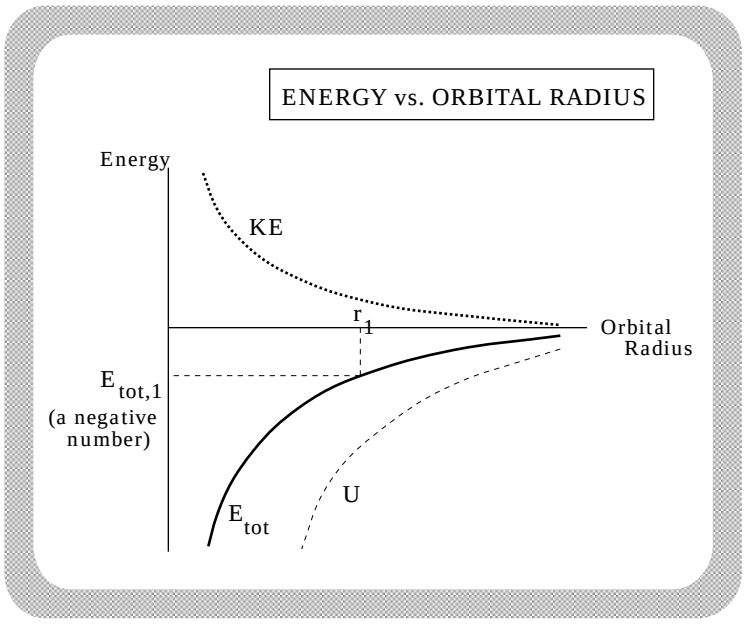


FIGURE 10.9

c.) Notice that as the radius of the motion gets larger (i.e., the bodies get further apart), the orbiting body's *kinetic energy* must

decrease (i.e., it slows down) and its *potential energy* must increase (i.e., becomes less negative).

d.) This means the binding energy, as measured by the potential energy in the system, is decreasing (getting closer to zero). Put another way, as a satellite gets farther away from its orbital center, it takes less energy to release it from its orbital bonds.

4.) One last observation: If we have a satellite moving in a circular orbit around the earth, and we want to put it into a higher velocity orbit, how can we use the satellite's thrusters to make the adjustment?

a.) The temptation is to shoot the thrusters *backwards*, thus applying a forward force on the satellite. The problem here is that in doing so, we will force the satellite into elliptical motion--something we do not want to do. But even if we apply the thrusters slowly and gently, we won't get the result we want. Doing so will result in forcing the satellite into an orbit that is *farther away from the earth*. With less gravitational force being applied to the satellite (i.e., less centripetal force acting on the satellite), the satellite's motion will require a lower velocity to hold its orbit. In short, the satellite will ultimately have to slow down to hold orbit.

From a little different perspective, firing the thrusters backwards will do *positive* work on the system. Putting energy *into* the system makes the *total energy* move closer to zero (remember, the total energy of an orbiting system is *negative*--adding energy moves the total energy from a large negative number to a smaller negative number). Looking at our graph, we see that when the total energy moves toward zero, the orbital radius increases (that is what we concluded above) and the kinetic energy (hence velocity) *decreases*.

b.) We know that higher velocity is associated with higher kinetic energy. That means we need to do whatever is required to increase the body's kinetic energy. Looking at our *energy versus orbital radius* graph (Figure 10.9), we see that putting the satellite into a smaller radius orbit will increase the satellite's kinetic energy and associated velocity.

c.) The trick, therefore, is to shoot the thrusters *very gently* in the *forward* direction, thereby doing negative work. As the satellite slows, gravity will pull the satellite in closer to the earth, but it will do it uniformly throughout the satellite's motion and as a consequence, the satellite will continue in nearly circular motion.

Following on, the satellite's gravitational potential energy will decrease (i.e., increase negatively) as the satellite gets closer to the earth. Although part of that potential energy loss will be caused by the satellite's absorption of

energy due to the negative work done by the thrusters, part of the potential energy loss will be due to a conversion of potential energy into kinetic energy. That is, the kinetic energy of the satellite will increase (and so will its velocity).

The observed increase of kinetic energy will *not* throw the satellite into a bigger orbit as would have been the case if we had fired the thrusters backwards. Why? Because firing backwards would have *increased* the total energy of the system whereas firing the thrusters forward *decreases* the total energy of the system. *The increase of velocity with diminished orbital radius is due to potential energy converting itself into kinetic energy. It is not due to extra energy being put into the system.*

From an *energy perspective*, firing the thruster forwards does negative work on the system. When added to the negative total energy, this *increases* the system's negative total energy. Looking at our graph to see what happens when the total energy becomes more negative, we find the orbital radius decreasing and the kinetic energy (and velocity) increasing.

H.) Energy Symmetry:

1.) In the section above, we observed a kind of symmetry between a body's orbital *kinetic* and *potential energy* due to the fact that the total energy of the motion had to be conserved. This idea of energy symmetry can be very useful in other ways.

2.) As an example, consider a stationary body poised a distance r_o units above the earth's center. Release the body and it begins to freefall (this is obviously *not* an orbital problem--the body is freefalling vertically). What does its *Kinetic Energy vs. Position* graph look like?

a.) We don't have a function for the kinetic energy of a freefalling body as its position changes, but we do have an expression for a body's *potential energy* as a function of position. That function was used to obtain the graph presented in Figure 10.10. This graph is *not* a function of *time*. It is designed to show how energy is distributed when the body is at various positions above the earth's surface. Do not be put off by the fact that the *beginning* position r_o (the word *beginning* being a time-related description) is found to the right of the so-called *final* position r_e.

b.) As there is no initial *kinetic energy*, the *total energy* must equal the initial *potential energy* in the system. That is, $E_{tot} = U_o = -GmM/r_o$.

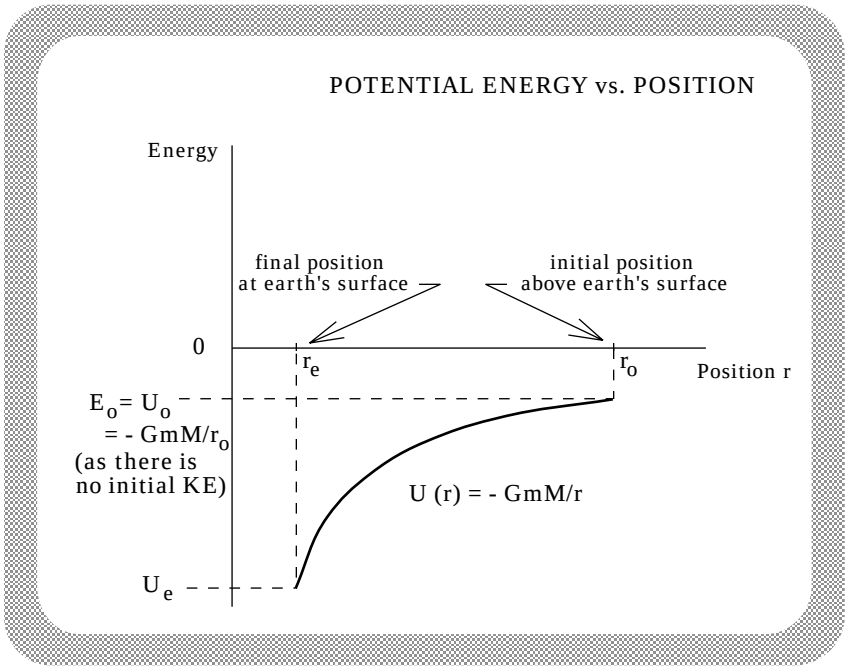


FIGURE 10.10

c.) This is where the symmetry comes in. We have a *potential energy versus position* graph, and we know that the system's *potential energy* and *kinetic energy* must always add to the same number--the *constant total energy*. From symmetry, then, the *kinetic energy versus position* graph must be constructed so that for every *potential energy* decrease, a corresponding *kinetic energy* increase exists.

In other words, the general outline of our desired kinetic energy graph must be a *mirror image* of our potential energy graph.

d.) In summary, even though *kinetic energy* is a function of v^2 and the potential energy is a function of $1/r$, the two functions are symmetric as plotted. Why? Because the sum of the two evaluated at a particular point must be a constant.

e.) A graph of the desired function is shown in Figure 10.11.

f.) Having said all that, what would the body's *Potential Energy vs. Velocity* graph look like? Think about it if you have the time!

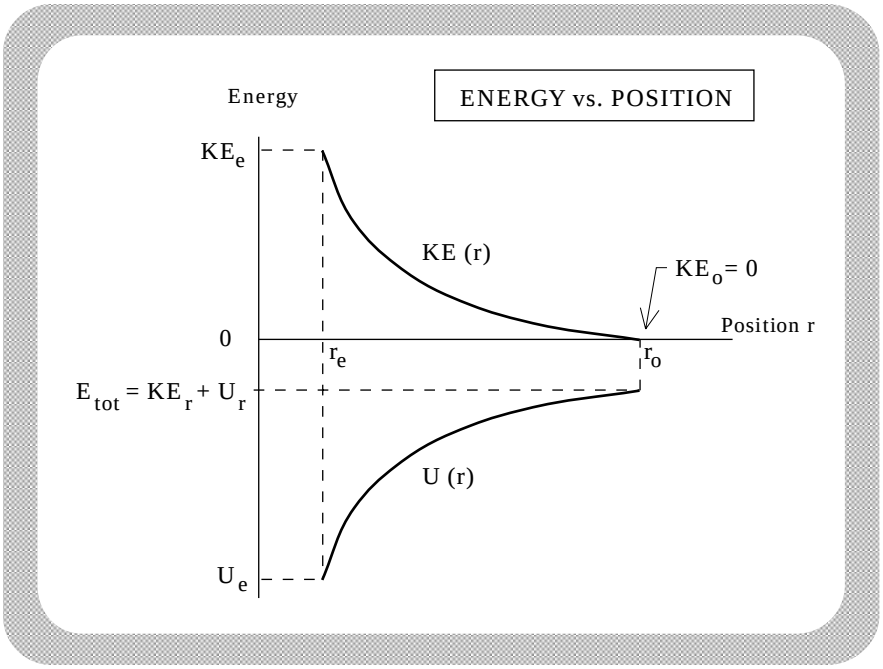


FIGURE 10.11

QUESTIONS

10.1) You and a friend sit 5 meters apart on an ice pond assumed to be a frictionless surface. Your mass is 80 kg and your friend's mass is 60 kg. If Newton was correct, you will exert a gravitational force on your friend, and vice versa (that's right, he maintained that gravitational forces act between *all* massive bodies, even little bodies like you and me). For the situation outlined above, determine:

a.) The gravitational force you exert on your friend; and

b.) The resulting acceleration of your friend.

10.2) Three bodies, each of mass m, orbit in such a way as to create an equilateral triangle (see Figure I to the right). If the distance between each body is d, determine the magnitude of the velocity of each body.

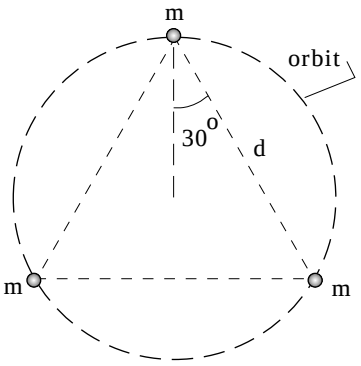


FIGURE I

10.3) The moon (mass 7.36×10^{22} kg and radius 1.74×10^{6} meters) is approximately 3.82×10^{8} meters from the earth (mass 5.98×10^{24} kg and radius of 6.37×10^{6} meters). If we could fix the earth, stop the moon, then allow the moon to freefall toward the earth:

a.) What would its initial acceleration be?

b.) What would its acceleration be just before striking the earth?

c.) How fast would it be moving when it reached the earth, assuming the earth remained stationary during the freefall?

d.) How would the problem have changed if the earth had not been assumed to be stationary?

10.4) A planet has a volume density distribution of $\rho = (m_p / r_p^{\,4})r$, where r_p and m_p are respectively the radius and mass of the planet and r is the distance from the planet's center to a point of interest. Assume a tunnel is dug through the planet.

a.) Derive an expression for the gravitational force applied to a mass m_1 when at a distance r units from the planet's center (remember, $r < r_p$).

b.) Derive a *potential energy function* for the force determined in *Part a*.

10.5) A satellite travels in a circular orbit a distance 1.3×10^6 meters above the earth's surface ($m_e = 5.98 \times 10^{24}$ *kg* and $r_e = 6.3 \times 10^6$ *m*). If the satellite's mass is 400 kg;

a.) How fast is it moving?

b.) What is its period?

c.) How much energy was required to put the satellite in orbit, assuming it started from rest at the earth's surface (that is, assuming we can ignore the earth's rotational speed).

d.) How would the calculation in *Part c* have changed if we hadn't ignored the earth's rotation?

e.) For whatever reason, the satellite loses energy at a rate of 2×10^5 joules per complete orbit. Assuming its radius of motion diminishes slowly, how far *from the surface* will it be after 1800 revolutions?

f.) For the situation outlined in *Part e* above, what is the magnitude of the average retarding force acting during the 1800 orbits?

g.) For the situation outlined in *Part e*, angular momentum is *not* conserved as there is an external torque acting on the system (it is due to the drag force acting on the satellite). Modify the *conservation of angular momentum* equation to determine the amount of *time* required for the 1800 revolutions (think about how the *conservation of linear momentum* equation was modified to accommodate external forces acting over a time interval Δt).

10.6) A planet moving in an elliptical orbit has 2.5×10^{33} joules of *kinetic energy* when at its closest point in its path around its star (this is approximately the kinetic energy of the earth as it orbits the sun). If we call this closest point r_{min}:

a.) What quantities are conserved for the planet's motion?

b.) The planet has a certain amount of *potential energy* due to its proximity to the star. Figure II on the next page shows the graph of the planet's *Potential Energy versus Distance from the star*. On the same axis, graph both the planet's *Total* and *Kinetic Energy versus Distance* functions.

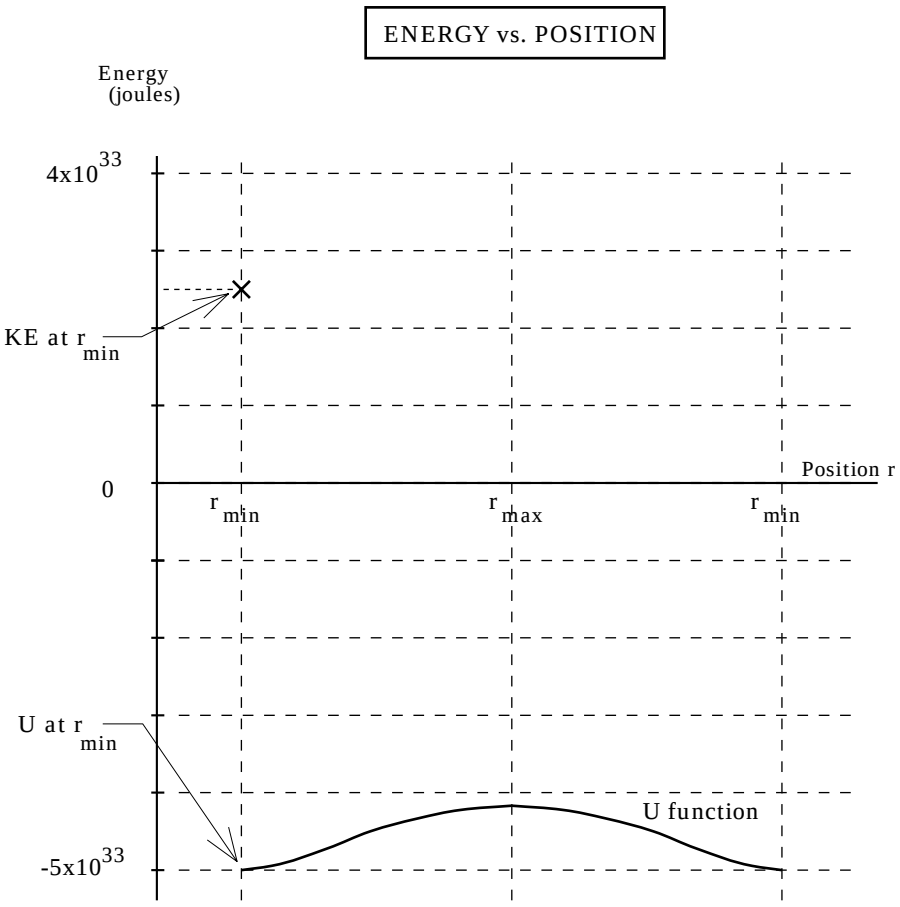

ENERGY vs. POSITION
Energy
(joules)
4x10^33
KE at r min
0
Position r
r min
r max
r min
U at r min
U function
-5x10^33

FIGURE II

Chapter 11

VIBRATORY MOTION

Note: There are two areas of interest when discussing oscillatory motion: the mathematical characterization of vibrating structures that generate waves and the interaction of waves with other waves and with their surroundings. We will examine the former topic in this chapter, the latter in the next chapter.

A.) Vibratory Motion–Basic Concepts:

1.) For any structure to vibrate periodically, there must be a restoring force on the body. A restoring force is a force that is constantly attempting to accelerate the object back toward its *equilibrium position*.

2.) The easiest way to examine vibratory motion is with an example. We will use a spring system:

a.) Consider the mass attached to the spring shown in Figure 11.1. When the spring is neither compressed nor elongated, the mass *feels no force* and is, hence, in a state of equilibrium.

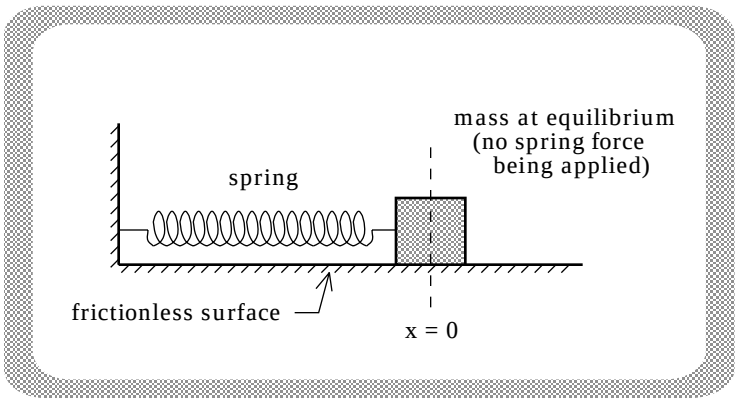


FIGURE 11.1

Note: For this and similar systems, the coordinate axis used to define *mass position* has its origin (i.e., $x = 0$) defined at the body's equilibrium position.

b.) It has been experimentally observed that if an "ideal" spring (i.e., one of those mythical types that loses no energy during oscillation) is displaced a distance Δx (see Figure 11.2), the force F required to displace the spring will be proportional to the displacement Δx. Put another way, if a mass is attached to the spring and the spring is displaced

a distance Δx, *the spring will exert a force F on the mass* when released. That force will be proportional to the spring's displacement from its equilibrium position.

Defining *k* as the proportionality constant (units: nt/m), this force is:

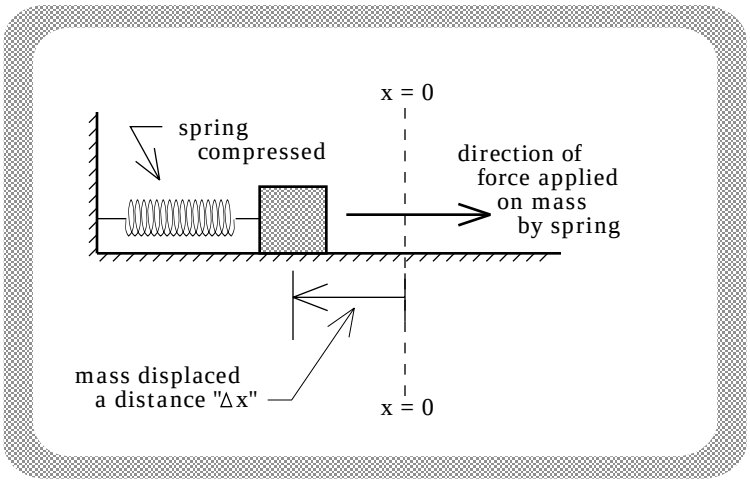


FIGURE 11.2

$$F = -k (\Delta x).$$

Called *Hooke's Law*, this relationship and the motion it describes is called "simple harmonic motion."

Note 1: Because Δx is measured *from equilibrium* (i.e., from $x = 0$), a displacement *x-units-long* will equal $\Delta x = x_{final} - x_{initial} = x - 0$. In other words, $\Delta x = x$. As such, HOOKE'S LAW IS ALWAYS WRITTEN:

$$F = - kx.$$

Note 2: Be sure you understand which force Hooke's Law alludes to: it is the force *the spring applies to the mass*, not the force the mass applies to the spring.

Note 3: The negative sign in front of the *kx* term insures that the force is always directed back toward the equilibrium position. To see this, assume the spring in Figure 11.2 has a spring constant of 2 nt/m and is displaced a distance $x = -.6$ *meters*. The force equation yields:

$$F = - [(2 \text{ nt/m}) (-.6 \text{ m})]$$
$$= +1.2 \text{ nts.}$$

The *direction* of the spring's force on the mass is *positive*, just as common sense would dictate. Without the *negative sign* on the right hand side of the force equation, the mathematics would not accurately model the situation.

3.) Here are some DEFINITIONS needed for the discussion of vibratory systems:

a.) Periodic motion: Any motion that *repeats itself* through time.

b.) Simple harmonic motion: Periodic motion whose force function is of the form $-kx$, where k is a constant and x is the displacement of the structure at some arbitrary point in time.

c.) Frequency (ν): The number of cycles swept through per-unit-time; the MKS units are *cycles per second* (i.e., *hertz*, abbreviated *Hz*). The symbol used for frequency-- ν--is the Greek letter *nu*.

Important Note: *Cycles* is not technically a unit. In many texts, *hertz* is defined as *inverse seconds* (i.e., *1/seconds*). We will use both, depending upon the situation.

d.) Period (T): The time required to sweep through one complete cycle. The units are *seconds per cycle* (or just *seconds*). Note that the *period* and *frequency* of a body's motion are inversely related. That is:

$$T = 1 / \nu.$$

e.) Displacement (x): The distance a vibrating object is from its equilibrium position at a given point in time. *Displacement* is a *time varying* quantity whose units are in *meters* or *centimeters* or whatever the distance units are for the system.

f.) Amplitude (A): The *maximum displacement* x_{max} of an oscillating body. *Assuming the vibratory motion does not lose energy*, the amplitude of the motion remains constant--it does *not* vary with time. *Amplitudes* are measured *from equilibrium* and have the same units as *displacement*.

Note: It is interesting to observe that because the *force function* for a spring is proportional to the spring's displacement ($F = -kx$), the *period* and, hence, *frequency* of a given spring/mass system will be a *constant*. Why?

An oscillation with a *very small displacement* will have a very small distance to travel during one period, but it will also have a *very small spring force* to motivate it. An oscillation with a *very large displacement* will have a very large distance to cover during one period, but it will have a *very large spring force* to help it along. The net result: whether you have big oscillations or small oscillations, it takes the same amount of *time* to oscillate through *one cycle*.

B.) The Mathematics of *Simple Harmonic Motion*:

1.) We would like to derive an expression that defines the *displacement* of a vibrating object from equilibrium as a function of time--i.e., *x(t)*. To generate the appropriate equation, we will examine the vibratory motion of a mass attached to a spring (see Figure 11.3), using Newton's Second Law to evaluate the motion.

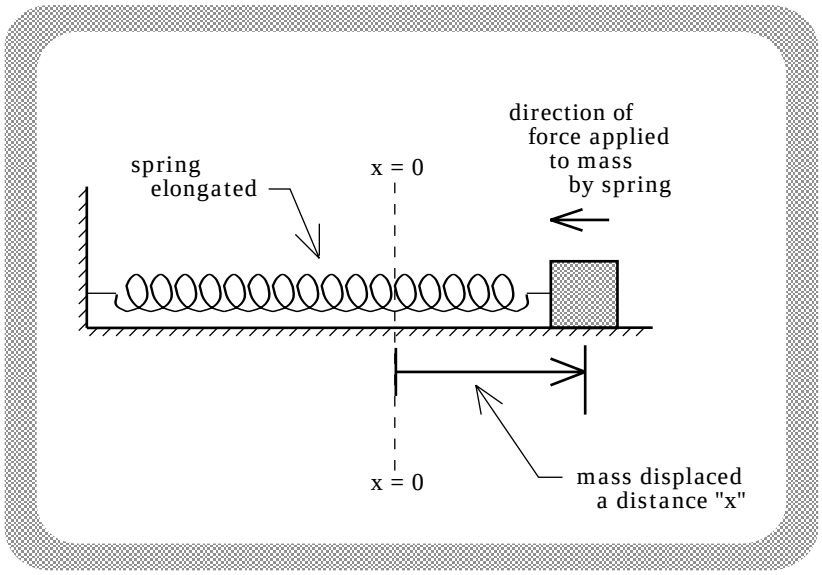


FIGURE 11.3

a.) A *free body diagram* is shown in Figure 11.4. Summing the forces in the horizontal, and leaving the *sign of the acceleration* embedded within the *ma*, we get:

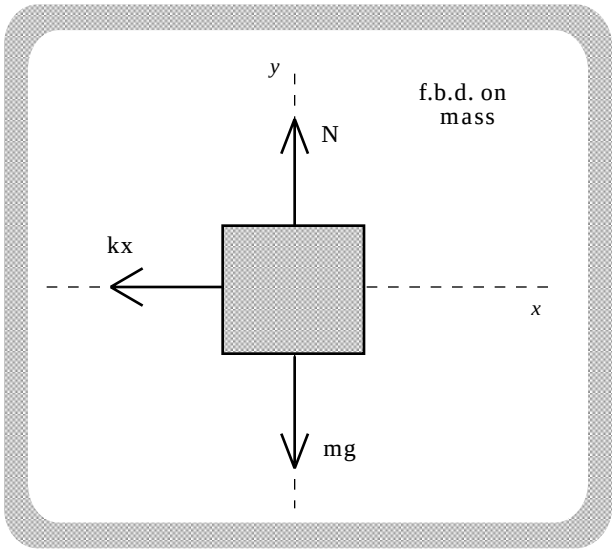


FIGURE 11.4

$\underline{\Sigma F_x}$:

$$-kx = ma$$

$$\Rightarrow \quad a + (k/m)x = 0.$$

b.) We know that the acceleration and velocity are related by $a = dv/dt$, and that the velocity and displacement are related by $v = dx/dt$. As such, it is true that $a = d^2x/dt^2$, where the notation used is meant to convey *the second derivative of the position with respect to time.*

c.) Substituting $a = d^2x/dt^2$ into the force expression yields:

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0.$$

d.) What does this equation really say? It suggests that there exists a function *x* such that when you add its *second derivative* to a *constant times itself*, you ALWAYS get zero.

The question is, "What function will do the job?"

The answer is, "A sine wave."

2.) The most general expression for a sine wave (see Figure 11.5a on the next page) is:

$$x(t) = A \sin(\omega t + \phi),$$

where *A* is the *amplitude* of the displacement (i.e., its maximum possible value); ω is a constant called the *angular frequency* whose units are *radians/second* and whose significance will become clear later; and ϕ is another constant called the *phase shift* whose units are in *radians* and whose significance will also be discussed later.

3.) Using the Calculus on our sine function, we find that if:

$$x(t) = A \sin(\omega t + \phi).$$

a.) The velocity of the motion will be:

$$\begin{aligned} v(t) &= dx/dt \\ &= \omega A \cos(\omega t + \phi). \end{aligned}$$

b.) The acceleration of the motion will be:

$$\begin{aligned} a(t) &= dv/dt \\ &= -\omega^2 A \sin(\omega t + \phi). \end{aligned}$$

c.) The graphs of all three of these functions are found in Figures 11.5a, b, and c.

Note: Notice that the *horizontal axis* is not labeled in time *t* but rather in ωt. Sine waves are functions of angles. Angles must have arguments in angular measure (radians in this case). That means the expression $x = A \sin \omega t$ makes no sense as written--you can't have a sine argument whose units are in *time*. To get around the problem, we modify the time variable by multiplying by ω *radians per second*.

d.) The *maximum value* for both a *sine* and a *cosine* function is *1*. This means:

$$v_{max} = \omega A$$

and

$$a_{max} = \omega^2 A$$

(where a_{max} is a magnitude).

4.) Analyzing the graphs:

a.) Look at the *first* long, vertical dotted line spanning Figures 11.5a, b, and c:

i.) The graphs suggest that when the displacement *x* is maximum-and-positive (i.e., as far to the right of equilibrium as it gets), the acceleration is maximum-and-negative.

Note: A *negative amplitude* value ($-\omega^2 A = -10\ m/s^2$, for instance) does not signify a minimum. The fact that -10 is smaller than +10 on a number line is not relevant here. The

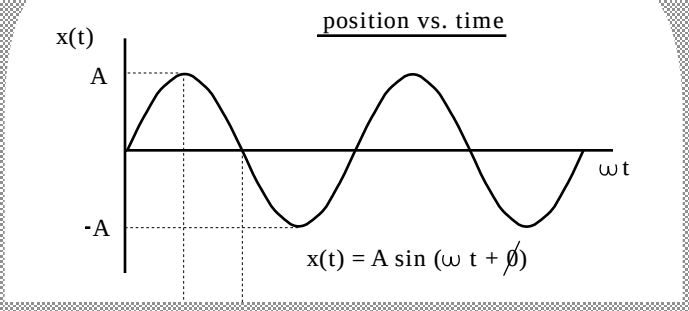


FIGURE 11.5a

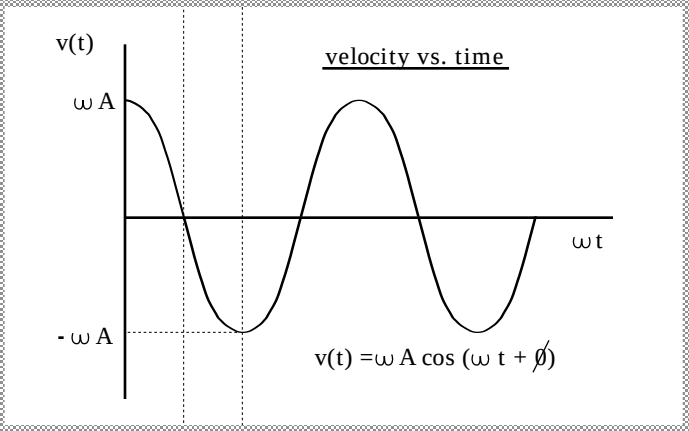


FIGURE 11.5b

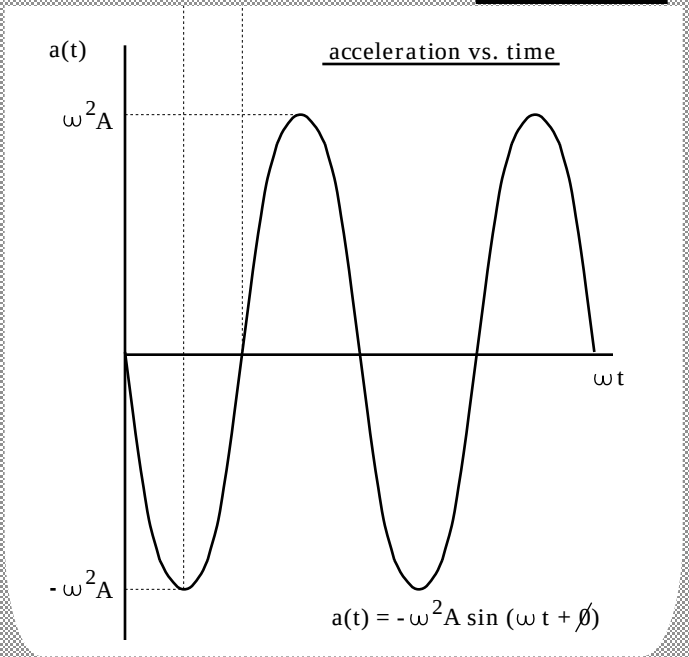


FIGURE 11.5c

value $-\omega^2 A$ is the *largest acceleration possible*; the negative sign simply tells you the *direction* in which it is a maximum.

ii.) Back to the "first vertical line:" When x is at its extremes, the *velocity* of the body is zero. This makes sense. At the extremes the body *stops* before beginning back in the opposite direction.

b.) Look at the *second* long, vertical dotted line spanning Figures 11.5a, b, and c:

i.) The graphs suggest that when x is zero (i.e., the body is at equilibrium), the acceleration is zero. This makes sense. At equilibrium the force applied to the body by the spring is zero, hence zero acceleration would be expected.

ii.) When x is at equilibrium, the *velocity* is a positive or negative maximum, depending upon which direction the body is moving. This also makes sense intuitively. Only when every bit of *acceleration* has been exhausted in motivating the mass back toward the equilibrium point will the velocity be at its maximum. That occurs at equilibrium.

5.) If we go back to Newton's Second Law with this information:

a.) Substituting our sine-related $x(t)$ and $a(t)$ functions back into our force expression (i.e., $a + (k/m)\, x = 0$), we get:

$$[-w^2 A \sin(\omega t + \phi)] + (k/m)\,[A \sin(\omega t + \phi)] = 0.$$

b.) Noting that the *A's* and the *sine functions* cancel, we end up with:

$$-\omega^2 + (k/m) = 0$$
$$\Rightarrow \quad \omega = (k/m)^{1/2}.$$

6.) Evidently, the function $x(t) = A \sin(\omega t + f)$ satisfies the equation $a + (k/m)\, x = 0$ as long as $\omega = (k/m)^{1/2}$.

a.) BIG GENERAL POINT: If you can manipulate a Newton's Second Law equation into the form:

acceleration + (some constant) (displacement) = 0,

you know for certain that:

i.) The motion will be *simple harmonic motion* (versus some other form of oscillatory motion); and

ii.) The *angular frequency* ω of the motion will be equal to the *square root* of the *constant in front of the displacement variable* in the manipulated N.S.L. equation. In the case of a spring, Newton's Second Law yielded:

$$a + (k/m)\, x = 0,$$

and the constant in front of the *position* variable k/m was found to be such that:

$$\omega = (k/m)^{1/2}.$$

b.) You might not think much of this revelation now, but it is going to be very useful later when we examine other kinds of vibrating systems.

Before we can look at these *other types of vibrating systems*, though, we need to make some sense out of the *angular frequency* ω and the *phase shift* ϕ terms.

C.) Angular Frequency (ω):

1.) Look at the POSITION VERSUS TIME graph of a vibrating body (Figure 11.6). How can I tell you where the body is in its motion at a given point in time? How, for instance, can I tell you that the body is at, say, *Point A* in Figure 11.6?

There are three ways to do the deed. Each is useful in its own way; each is listed below:

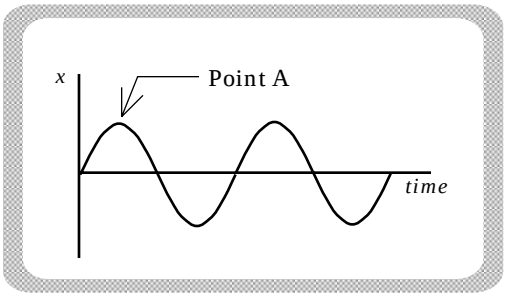


FIGURE 11.6

a.) The first way has already been discussed. I could simply say, "The body is at $x = A$." In that case, I am giving you the "linear displacement" of the body at the point-in-time of interest.

Though simple, it is not very useful if we want to know something about *how fast* the oscillations are taking place. That is, if I know the *time* it takes for the body to get from $x = 0$ to $x = A$, dividing the time into

the *displacement* will give only the *average velocity* over the motion--a none-too-useful commodity in most cases.

b.) Another possibility is to say, "The body is one quarter of a cycle through its motion." In that case, I am giving you the *cyclic displacement* of the body.

If I additionally tell you how long it takes to achieve that position within its cyclic motion, we can divide the time into that cyclic displacement and come up with an expression for the body's *frequency* in *cycles per second*.

For oscillatory motion, *frequency* measurements are very useful.

c.) Another more exotic possibility is to say, "The body is *π/2 radians* through its motion." In this case, I would be giving you the angular displacement of the body.

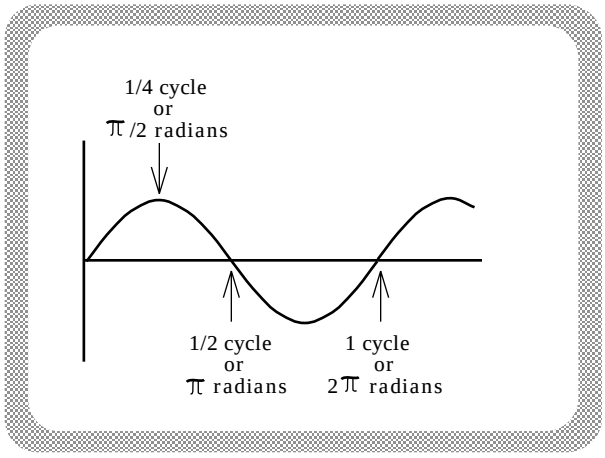


FIGURE 11.7

This very peculiar way of characterizing the position of a vibrating body is made simply by looking at the graph of a sine function (see Figure 11.7). Notice that a body having completed one full cycle has moved through an angular displacement of *2π radians*. Following logically, a body having moved through one-half cycle has displaced an angular measure of *π radians* and a body having moved through one-quarter cycle has displaced *π/2 radians*. In other words, if you understand the language, an angular displacement can tell you where a body is in its motion just as well as a cyclic displacement can.

If we further divide this radian-displacement by the time required to get to that position, we end up with an expression for the body's angular frequency ω in *radians per second*.

2.) *One cycle* is the equivalent of an angular measure of *2π radians*. That means that oscillatory motion whose frequency is *1 cycle/second* has an angular frequency of *2π radians/second*. Expanding this, it becomes obvious that the relationship between frequency ν and angular frequency ω is:

$$\omega = 2 \pi \nu.$$

3.) Reconsidering $x(t) = A \sin(\omega t + \phi)$, the angular frequency ω governs *how fast the function, hence body position, changes.* Large ω means it takes very little time for ωt to increment by 2π (i.e., move through one cycle), which means the *period* of the function and the body's motion is small. This corresponds to a *high frequency* oscillation. A small ω does just the opposite.

D.) Phase Shift (ϕ):

1.) A typical sine wave function is characterized by the graph shown in Figure 11.8 and is mathematically written as:

$$x(t) = A \sin(\omega t).$$

a.) This expression predicts that at $t = 0$, $x = 0$ (i.e., put in $t = 0$ and you get $x = 0$!). What's more:

b.) *Just after* $t = 0$, the value of $A \sin(\omega t)$ is *positive* and gets larger as time proceeds, just as the graph shows.

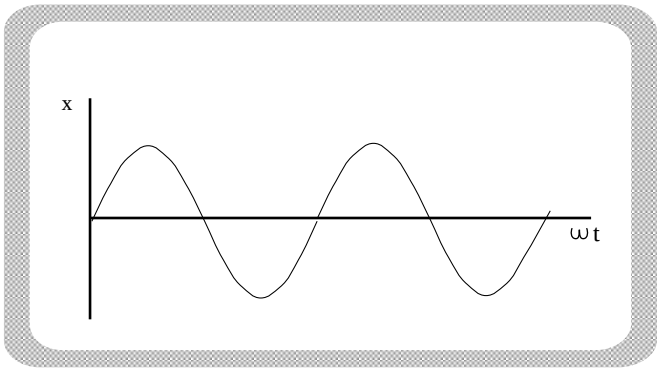


FIGURE 11.8

BIG NOTE: The displacement variable *x(t)* is measured from EQUILIBRIUM. That means that if you look at a graph of *x(t)* and see that the variable is getting larger over a particular time interval (either large in a positive sense or larger in a negative sense), it means that the function is modeling motion that is moving AWAY FROM equilibrium.

c.) The problem arises when we do not want the body to be at equilibrium ($x = 0$) at $t = 0$. For instance, what do we do if we want it to be at $x = A$ when we start the clock (i.e., at $t = 0$)? Dealing with such problems is exactly what the phase shift ϕ is designed to do. It allows one to make compensations in the math so that a sine function can be used to characterize oscillatory motion that doesn't assume $x = 0$ at $t = 0$.

2.) Easy Example: Assume we define the position of an oscillating body as $x = +A$ at $t = 0$. How can we use a *sine function* to characterize that motion?

a.) Notice that if we shift the vertical axis of the *sine wave* shown in Figure 11.9a (next page) by $\pi/2$ *radians* (see Figure 11.9b for the shifted

version), we end up with a graph that gives us $x = +A$ at $t = 0$ (Figure 11.9c). In other words, adding $\pi/2$ to the *sine's angle* will do for us exactly what we want.

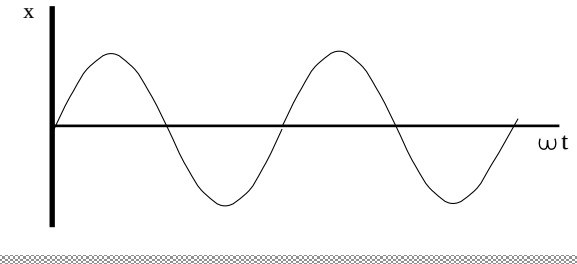


FIGURE 11.9a

b.) Mathematically, we are suggesting that the correct function is:

$$x(t) = A \sin (\omega t + \pi/2).$$

i.) To check, we know what the function's value should be at $t = 0$: it should be $x = +A$.

ii.) Plugging $t = 0$ into the function we are testing yields:

$$\begin{aligned} x(t=0) &= A \sin (\omega\ (0) + \pi/2) \\ &= A \sin (\pi/2) \\ &= A. \end{aligned}$$

Our function works!

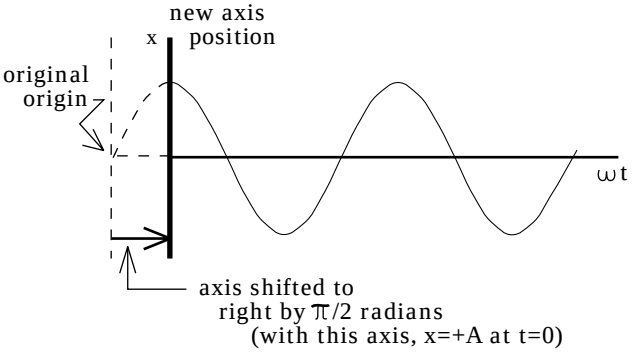


FIGURE 11.9b

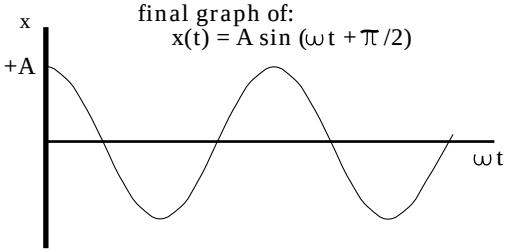


FIGURE 11.9c

c.) The moral: The *phase shift* tells us how much we have to translate (shift) the vertical axis to define the correct displacement x at $t = 0$.

Note: A "+" *phase shift* shifts the axis <u>to the right</u>; a "-" *phase shift* shifts the axis <u>to the left</u>.

3.) In the case above, it was obvious that the shift needed to be $\pi/2$ *radians*. Unfortunately, not all problems are this easy. How does one determine the *phase shift* for more complex situations?

a.) Assume you know where an oscillating body is supposed to be at $t = 0$. The key to determining the general *sine wave function* that will fit the situation lies in evaluating the displacement equation $x(t)$ at a known point in time (preferably at $t = 0$), then solving that expression for the appropriate ϕ. The process will be formally presented using the relatively easy case cited above. We will then try the approach on more difficult problems.

4.) Advanced Example #1: Assume that at $t = 0$, $x = +A$.

a.) Putting that information into our displacement expression

$$x(t) = A \sin(\omega t + \phi)$$

yields

$$A = A \sin(\omega(0) + \phi).$$

b.) Dividing by A and multiplying w by zero gives us:

$$1 = \sin(\phi)$$
$$\Rightarrow \quad \phi = \sin^{-1}(1)$$
$$= 1.57 \qquad \text{(i.e., } \pi/2\text{)}.$$

This is exactly what we expected.

c.) Knowing ϕ for one point in time means we know it for all points in time (ϕ is a constant for the motion). Putting it back into our general algebraic expression for the *displacement* gives us:

$$x(t) = A \sin(\omega t + 1.57).$$

Note: In most problems, you will have already determined both A and w. That is, both will have numeric values. As an example, if $A = 2$ *meters* and $\omega = 7.5$ *radians/second*, the finished expression will look like:

$$x(t) = 2 \sin(7.5t + 1.57).$$

5.) Example #2: Determine the general algebraic expression for the *displacement* of a spring-mass system whose *position* at $t = 0$ is $(3/4)A$ going *away from equilibrium* (see Figure 11.10a and 11.10b on the next page).

a.) In general:

$x(t) = A \sin\left(\omega t + \phi_1\right)$.

b.) Substituting *t = 0* and *x = (3/4)A* into our general equation yields:

$(3/4)A = A \sin\left(\omega(0) + \phi_1\right)$.

c.) Dividing by *A* and multiplying ω by zero gives us:

$3/4 = \sin(\phi_1)$,

which implies that ϕ_1 is the angle whose *sine* is *3/4*, or

$\phi_1 = \sin^{-1}(3/4)$
$= .848$ radians.

d.) Putting our value for ϕ back into our general algebraic expression for the *displacement* gives us:

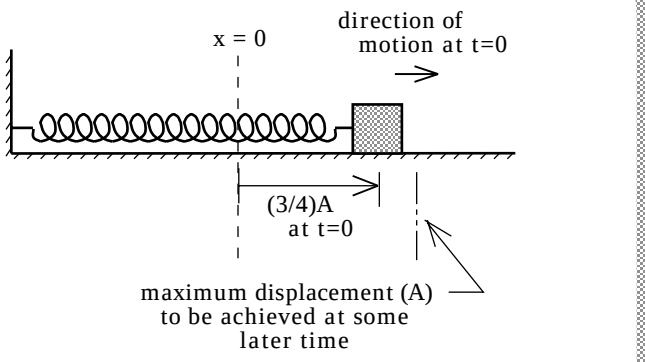


FIGURE 11.10a

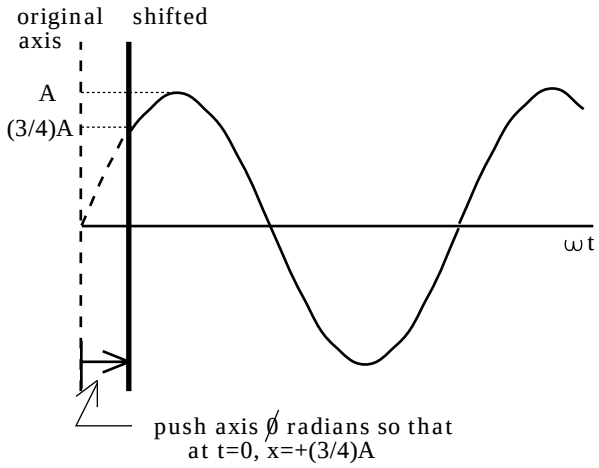


FIGURE 11.10b

$x(t) = A \sin(\omega t + .848)$.

e.) By shifting the axis of the sine wave by *.848 radians* (see Figure 11.10b), we get a graph that has the body's position equal to *.75A* at *t = 0* and that additionally has the displacement proceeding *away from equilibrium* just after *t = 0*.

6.) Example #3--a little different twist: Determine the general algebraic expression for the displacement of a spring/mass system whose position at *t = 0* is *(3/4)A going toward equilibrium*.

Note: This is almost exactly the same as the problem in #5. The only difference is in the direction of the motion just after *t = 0*. Proceeding through the steps:

a.) In general:

$x(t) = A \sin\left(\omega t + \phi_2\right)$.

b.) Substituting *t = 0* and *x = (3/4)A* into our general equation yields:

$(3/4)A = A \sin\left(\omega(0) + \phi_2\right)$.

c.) Dividing by *A* and multiplying ω by zero gives us:

$3/4 = \sin(\phi_2)$,

which implies that ϕ_2 is the angle whose *sine* is *3/4*, or

$\phi_2 = \sin^{-1}(3/4)$
$= .848$ radians.

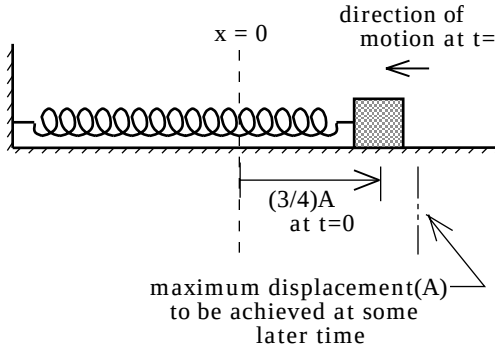


FIGURE 11.11a

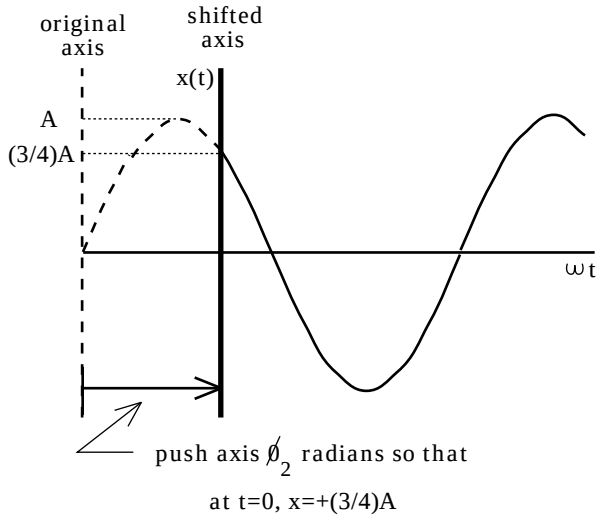


FIGURE 11.11b

d.) THE SNAG: This suggests that the angle ϕ_2 equals the angle ϕ_1, which clearly can't be the case (see Figure 11.12). What we need is an angle that predicts motion that proceeds *back toward equilibrium* just after *t = 0* . . . not an angle that predicts motion that proceeds *away from* equilibrium after *t = 0*.

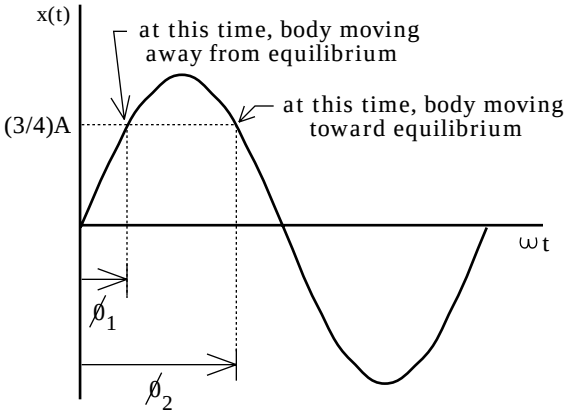


FIGURE 11.12

Put a little differently, it is clear from the sketch that there are *two* phase shifts that can put $x = (3/4)A$ at $t = 0$. The first (i.e., ϕ_1) is the one we used in #5. It corresponds to the situation when, just after $t = 0$, the motion proceeds *away from equilibrium* (look at the graph--the value for x gets *more* positive as time progresses).

The second phase shift (ϕ_2) is the one we want here. It makes $x = (3/4)A$ at $t = 0$, and it also has the displacement going *back toward* equilibrium as time progresses.

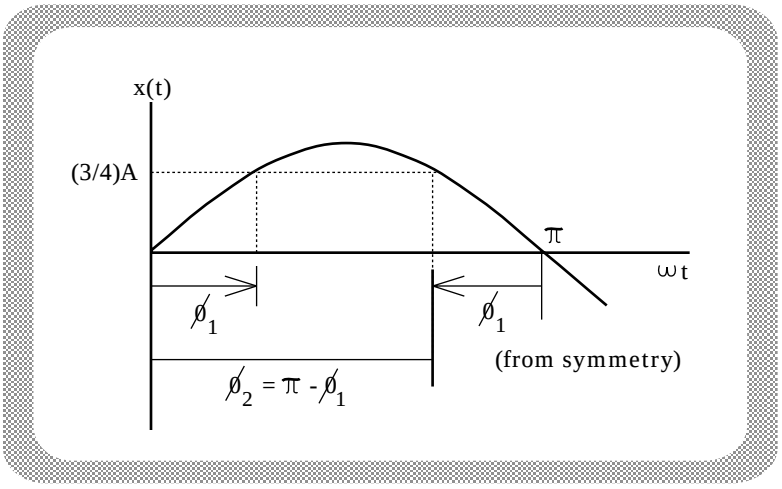


FIGURE 11.13

e.) To determine ϕ_2, we need to use the symmetry of the *sine function* (see Figure 11.13). Notice from the figure that the phase shift (ϕ_2) is equal to π - *.848 radians*, or *2.29 radians*.

Using this, the final expression becomes:

$$x(t) = A \sin(\omega t + 2.29).$$

f.) Bottom line: Before deciding if the angle your calculator produces is correct, *make a sketch of a sine wave and decide whether you need* ϕ_1 *or* ϕ_2.

7.) Example #4: Determine a general algebraic expression for the displacement of an oscillating body whose position at $t = 0$ is $(-3/4)A$ going *away from* equilibrium (see Figures 11.14a). Assume also that $A = .6$ *meters* and $\omega = 12$ *rad/sec*.

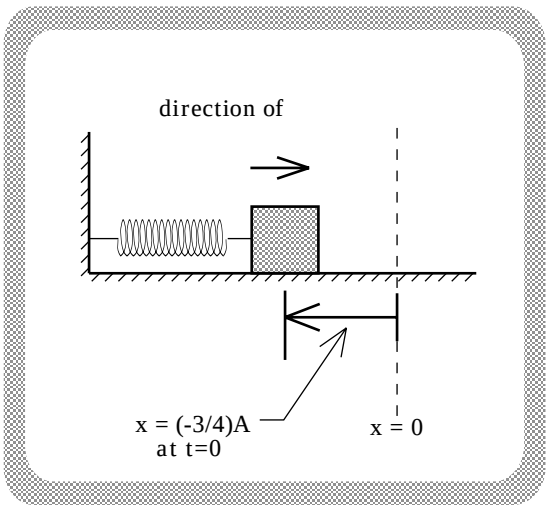


FIGURE 11.14a

a.) Using the same approach as before:

$$x(t) = A \sin\left(\omega t + \phi_3\right)$$

$$\Rightarrow \quad (-3/4)(.6) = (.6) \sin\left(\omega(0) + \phi_3\right)$$

$$\Rightarrow \quad -3/4 = \sin(\phi_3)$$

$$\Rightarrow \quad \phi_3 = \sin^{-1}(-3/4)$$

$$= -.848 \text{ radians.}$$

b.) A <u>*negative*</u> *phase shift* moves the axis to the *left*. Again, there are *two* positions where an axis can be placed so that at $t = 0$, $x = (-3/4)A$ (see Figure 11.14b). The first, corresponding to an *angular shift* of the axis of ϕ_3, has the body moving *toward equilibrium* just after $t = 0$; the second, corresponding to an angular shift of the axis of ϕ_4, has the body moving *away from* equilibrium just after $t = 0$.

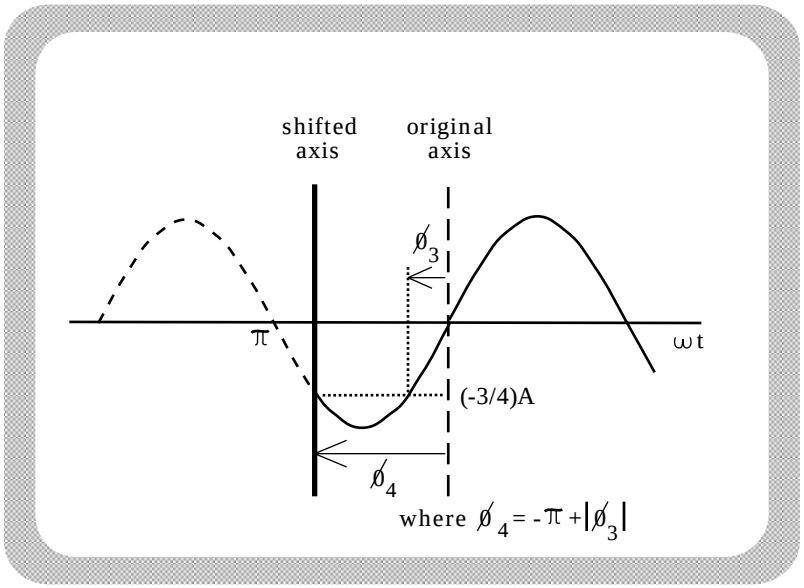


FIGURE 11.14b

In our example, the appropriate angular shift is ϕ_4. The displacement expression is, therefore:

$$x(t) = (.6) \sin\left(12t + 2.29\right).$$

8.) The technique for determining *phase shifts* is simple. Put the $t = 0$ value for displacement into $x(t) = A \sin(\omega t + \phi)$, solve algebraically for ϕ, and your calculator will crank out a number for you.

a.) If the calculator's number is *positive*, shift the axis to the right. If the number is *negative*, shift the axis to the left.

b.) The only thing tricky about the operation: in almost all cases there will be two possible axes (i.e., shift angles) that will correspond to the required $t = 0$ displacement. Determine which is appropriate by noting whether the motion is proceeding *away from* equilibrium or *toward* equilibrium. That information will dictate whether you can use

your calculator-provided *phase shift* value or whether you will have to add or subtract π.

c.) Whatever the case, you should end up with an expression that looks something like *x(t) = 2 sin (7.5t + 1.57).*

E.) Energy in a Vibrating System:

1.) Consider the motion of a mass attached to a vibrating spring:

a.) At the extremes, the body's *velocity* is zero (it's at a turn-around point), its *position* is a maximum (i.e., $x = A$), and all the *energy* in the system is *potential energy*.

That is, at the extremes:

$$E_{total} = U\,(x_{max}).$$

b.) The *potential energy* function for a spring system is $(1/2)kx^2$. This means:

$$\begin{aligned} E_{total} &= U\,(x_{max}) \\ &= (1/2)kx_{max}^{\ 2} \\ &= (1/2)kA^2. \end{aligned}$$

c.) *Assuming there is no energy loss* during the motion, the *amplitude* of the motion remains constant and the total energy of the system is conserved. The energy flows back and forth between being *potential* and *kinetic*, but the sum of the two is always equal to $(1/2)kA^2$.

F.) A summary example:

1.) You have a spring hanging from the ceiling. You know that if you elongate the spring by 3 meters, it will take 330 nts of force to hold it at that elongated position.

The spring is hung and a 5 kg mass is attached. The system is allowed to reach equilibrium; then is displaced an additional 1.5 meters and released. For this system, what is the:

a.) *Spring constant*?

b.) *Angular frequency*?

c.) *Amplitude*?

d.) *Frequency*?

e.) *Period*?

f.) *Total energy*?

g.) *Maximum velocity* of the mass?

h.) *Position* of the mass at maximum velocity?

i.) *Maximum acceleration* of the mass?

j.) *Position* of the mass at maximum acceleration?

k.) General *algebraic expression* for the position of the mass as a function of time, assuming that at $t = 0$ the body's position is located at $y = -A/4$ going away from equilibrium?

2.) Solutions:

a.) F/x = 110 nt/m; **b.)** $(k/m)^{1/2} = 4.7$ rad/sec; **c.)** 1.5 m (from observation); **d.)** $\omega / 2\pi = .75$ hz; **e.)** $1 / \nu = 1.33$ sec/cycle; **f.)** $(1/2)kA^2 = 123.75$ joules; **g.)** $\omega A = 7.05$ m/s; **h.)** at equilibrium position; **i.)** $\omega^2 A = 33.135$ m/s^2; **j.)** at the extremes; **k.)** either $x(t) = 1.5 \sin (4.7t + 3.39)$ or $x(t) = 1.5 \sin (4.7t - 2.89)$.

G.) Another Kind of Vibratory Motion–The Pendulum:

1.) Consider a swinging pendulum bob of mass m at the end of a string of length L positioned at an arbitrary angle θ as shown in Figure 11.15. What is the system's *frequency*, *period* of oscillation, *angular frequency*, etc.?

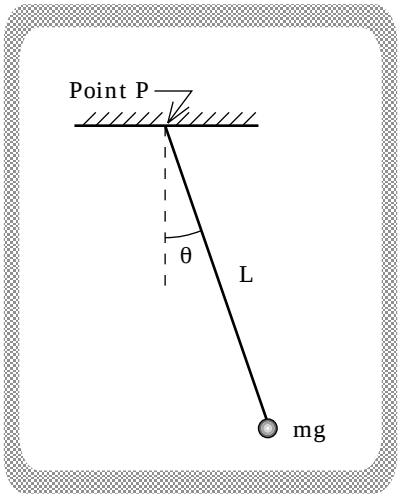


FIGURE 11.15

a.) We will begin the same way we did with the spring. If the Newton's Second Law equation for this situation matches the form:

acc. + (constant) disp. = 0,

we know the motion will be *simple harmonic* and we know that the *constant* will numerically equal the *angular frequency squared.*

b.) The only difference between this situation and the spring situation is that in this case, the pendulum bob is moving in a *rotational* sense around the string's point of attachment *P.* The version of N.S.L. that is applicable here, therefore, is the rotational version.

c.) Figure 11.16a shows the *free body diagram* for the set-up. Figure 11.16b shows that the *torque* about *Point P* due to the tension *T* is zero (the tension force passes through *Point P*), and the *torque* due to gravity is *mg (L sin θ)* (in this case, $r_\perp$ is *L sin θ*). Remembering that the *moment of inertia* for a point mass is $I_{ptmass} = mL^2$, the rotational counterpart to Newton's Second Law yields:

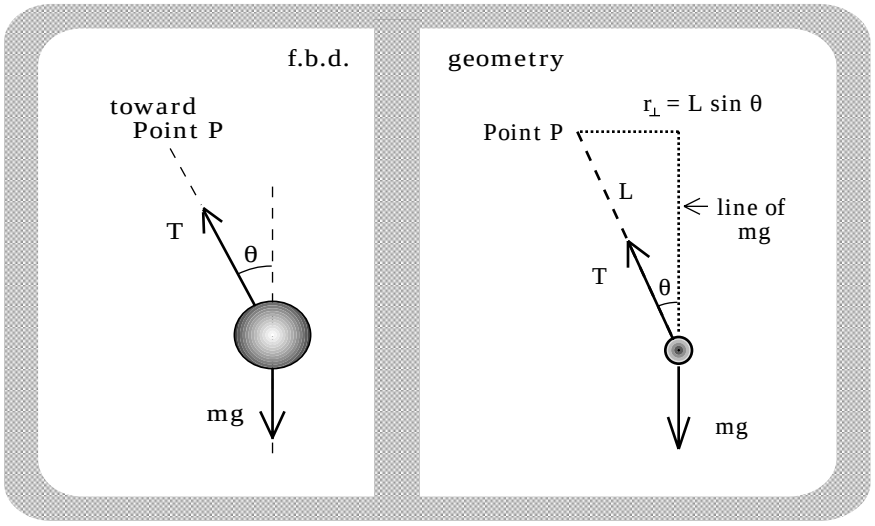


FIGURE 11.16a FIGURE 11.16b

$\underline{\Sigma \Gamma_p}$:

$$-mg\,(L \sin\theta) = I\alpha$$
$$= (mL^2)\,a$$

which implies:

$$\alpha + (g/L) \sin\theta = 0.$$

d.) This is not the form for which we were hoping. Fortunately, if θ is <u>small</u> and measured in *radians*, $\sin\theta = \theta$ (put your calculator in *radian* mode and see what *sin (.02)* is--you should find that it is .01999999--.02 to a good approximation).

e.) Making the *small angle approximation*, we get:

for $\theta <<$:

$$\alpha + (g/L)\ \theta = 0.$$

f.) Running a parallel from our spring experience, we know that the oscillation's *angular frequency* must be:

$$\omega = (g/L)^{1/2}.$$

g.) With the angular frequency ω, we can determine general algebraic expressions for the motion's frequency ($\omega/2\pi$) and period ($1/\nu$).

2.) Reiteration: If you are ever asked to determine either the *period* or *frequency* of an exotic oscillatory system, use N.S.L. and see if you can put the resulting *equation of motion* into the form:

$$\text{acc.} + (\text{constant})(\text{displ.}) = 0.$$

If you can do so, the motion will be simple harmonic in nature and the *angular frequency* will equal *the square root of the constant*. From there you can easily determine the motion's *frequency* and/or *period*.

QUESTIONS

11.1) A spring/mass set-up oscillating in the vertical is found to vibrate with an amplitude of .5 meters and a period of .3 seconds per cycle. If the mass is 1.2 kg, determine:

a.) The frequency of oscillation;
b.) The angular frequency;
c.) The spring constant;
d.) The maximum velocity (in general, where does this happen);
e.) The maximum acceleration (in general, where does this happen);
f.) How much energy is wrapped up in the system?

11.2) A .25 kg mass sliding over a frictionless horizontal surface is attached to a spring whose spring constant is 500 nt/m. If the spring's maximum velocity is 3 m/s, determine the motion's:

a.) Angular frequency;
b.) Frequency;
c.) Period;
d.) Amplitude;
e.) Total energy;
f.) Maximum force applied to the mass.

11.3) A body's motion is characterized by the expression:

$$x(t) = .7 \sin (14t - .35).$$

Determine the motion's:

a.) Amplitude?
b.) Angular frequency?
c.) Frequency?
d.) Position at t = 3 seconds?
e.) Position at t = 3.4 seconds?
f.) Velocity at t = 0?
g.) Acceleration at t = 0?

11.4) A pendulum consists of a small, 2 kg weight attached to a light string of length 1.75 meters. The pendulum is set up on a distant planet and set in motion. Doing so, it is observed that its period is 2 seconds per cycle. What is the acceleration of gravity on the planet?

11.5) The Newton's Second Law equation shown below came from the analysis of an exotic pendulum system oscillating with a small angular displacement. It is:

$$\alpha + (12g/7L)\ \theta = 0$$

a.) Given the information provided above, how can you tell that the system oscillates with *simple harmonic motion*?

b.) What is the system's theoretical *frequency of oscillation* if the pendulum length is assumed to be 1.3 meters?

11.6) A 3 kg block is attached to a vertical spring. The spring and mass are allowed to gently elongate until they reach equilibrium a distance .7 meters below their initial position. Once at equilibrium, the system is displaced an additional .4 meters. A stopwatch is then used to track the position of the mass as a function of time. The clock is started when the mass is at $y = -.15$ *meters* (relative to equilibrium) moving *away from* equilibrium. Knowing all this, what is:

a.) The *spring constant*?

b.) The oscillation's *angular frequency*?

c.) The oscillation's *amplitude*?

d.) The oscillation's *frequency*?

e.) The *period*?

f.) The *energy* of the system?

g.) The *maximum velocity* of the mass?

h.) The *position* when at the maximum velocity?

i.) The *maximum acceleration* of the mass?

j.) The *position* when at the maximum acceleration?

k.) A general *algebraic expression* for the position of the mass as a function of time?

11.7) A tunnel is dug through the earth from the North Pole to the South Pole. When done, Jack (the idiot) goes for the thrill of his life and jumps into the hole. The gravitational force on him is always directed toward the earth's center, so Jack ends up oscillating back and forth between the two poles.

In the chapter on Gravitation, we derived an expression for the magnitude of the gravitational force acting on a mass a distance r units from the earth's center, where $r < r_e$ with r_e being the earth's radius.

Tailored to our situation, that expression is:

$$F_J = -\left[\frac{G m_e m_J}{r_e^3}\right] r$$

where m_e and r_e are the mass and radius of the earth, respectively, m_J is Jack's mass, and r is Jack's position along the *y-axis* (we are assuming the tunnel is in the vertical).

Jack's father misses him. As such, Papa has hired a surveillance satellite whose orbit is such that every time Jack's *head* emerges momentarily from the hole, the satellite and its cameras are directly above to snap photos.

a.) For this to work, what must the satellite's period be?

b.) Given the satellite's period, what must its orbital radius and velocity be? (This part is more a *gravitation* problem than a *vibratory motion* problem, but it's good review.)

Chapter 12

WAVE MOTION

A.) Characteristics of Waves:

1.) A wave is *a disturbance that moves through a medium*.

a.) Example: A pebble is dropped into a quiet pond. The DISTURBANCE made by the pebble is what moves outward over the water's once-still surface. Water molecules are certainly jostled by the wave, but after the wave passes by, each molecule finds itself back in its original, pre-disturbance position (at least to a good approximation).

In other words, water waves are *not* made up of lumps of water that move across the water's surface. They are *disturbances* that move through the water that only temporarily displace water molecules in the process.

Note 1: A group of waves is called a *wave train*.

Note 2: Looking down from above, pebble-produced water waves will look like a series of crests and troughs moving in ever-expanding circles outward away from the pebble's point-of-entry into the water. If, at a given instant, lines are drawn on the crests, we find a visual presentation of the waves as shown in Figure 12.1b. Figure 12.1a shows a side-view of this same situation.

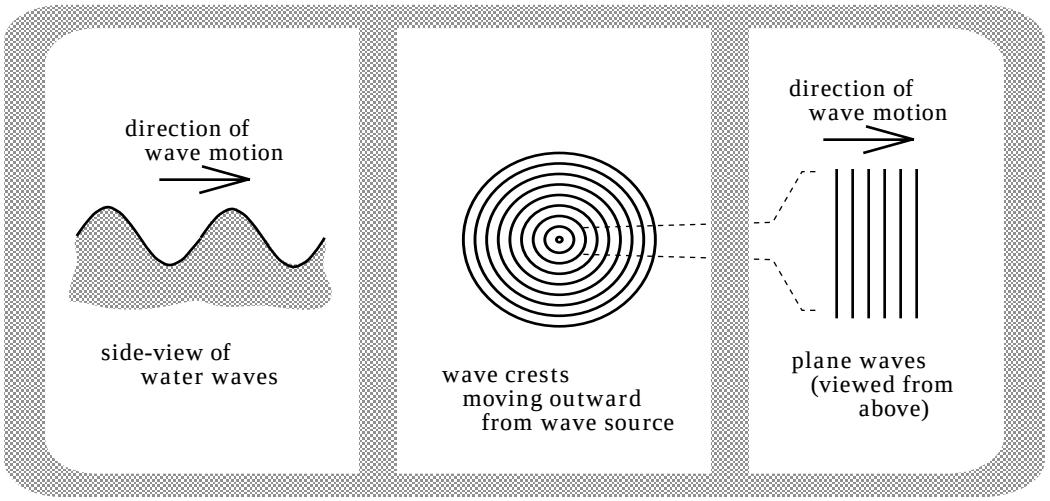


FIGURE 12.1a FIGURE 12.1b FIGURE 12.1c

Note 3: Once a *wave train* has moved far enough away from its source, crests in the immediate vicinity of one another are to a very good approximation parallel to one another (look at the outer sections in Figure 12.1a). Waves in this situation are called *plane waves* and are shown on the previous page in Figure 12.1c. It is not uncommon for *plane waves* to be assumed when wave-phenomena are being discussed.

b.) A *sound wave* is a <u>*pressure disturbance*</u> that moves through air or water, or whatever the host medium happens to be.

Human vocal sound is generated by the back and forth vibration of the *vocal cords*. When these cords are extended, they momentarily compress air molecules together creating a high pressure region that is accelerated outward. As the cords pull back, they generate a momentary vacuum--a low pressure region (in between these two situations, the air pressure obviously passes through a "normal" pressure circumstance). In other words, the vibration of the vocal cords creates regions of high pressure, then normal pressure, then low pressure, then normal pressure, then high pressure, etc., as they vibrate back and forth (Figure 12.2a presents a representation of what

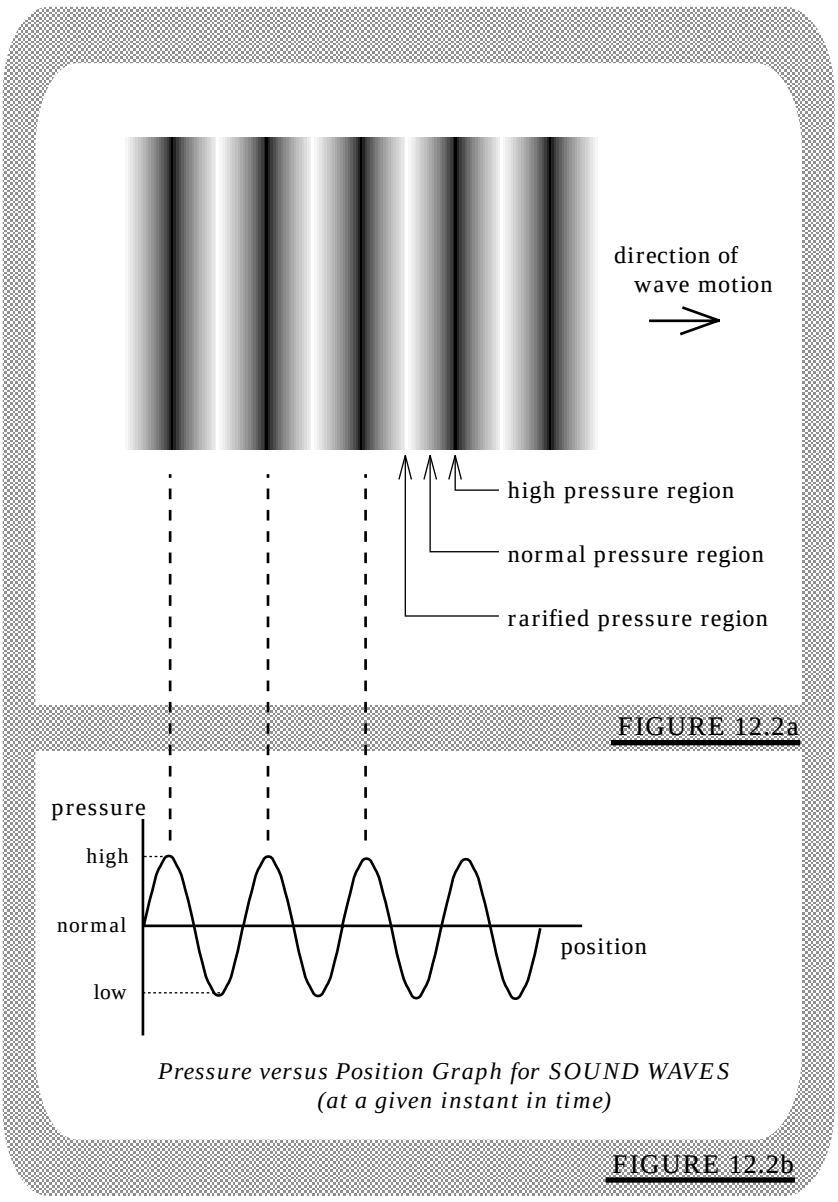


FIGURE 12.2a

FIGURE 12.2b

sound waves would look like if our eyes were sensitive to very subtle pressure variations--Figure 12.2b graphs *pressure variation* versus *position* for sound at a *given point in time*).

These *pressure disturbances* move out into the surrounding air at approximately 330 meters per second (i.e., the *speed of sound*). As they pass a hearing person, the pressure variations motivate tiny hairs in the listener's ears to vibrate generating electrical signals which, upon reaching the brain, are translated into incoming sound.

Again, sound waves are a *disturbance moving through a medium*. Without the medium, there can be no sound.

Note: That's right, the next time you see *Star Wars* and they show a big battle scene viewed from space, you have every right to stand up in the middle of the movie theater and shout at the top of your lungs, "WAIT, WAIT, THIS CAN'T BE. THERE IS NO SOUND IN SPACE!!" They'll probably throw you out of the theater for causing a disturbance (i.e., for making waves--a little physics humor), but you will be correct in exposing one of Hollywood's greatest displays of scientific misinformation ever.

2.) Waves are important because they carry *energy*.

Note: If you think about it, this should be obvious. If waves didn't carry energy, *sound waves* wouldn't have the wherewithal to wiggle those little ear-hairs that allow you to hear, and *tidal waves* would not have the ability to blow away whole island-populations with a single achoo.

3.) There are two kinds of waves, both of which are identified by how *the disturbance-producing force* is applied:

a.) Transverse waves: These are waves that are created by a force that is applied to a medium *perpendicular* to the direction of the wave's motion in the medium.

i.) An example: When a *pebble* enters a pond, it applies a *force* to the water that is *perpendicular* to the water's surface, hence *perpendicular* to the wave's direction, as it moves out over the water's surface. As such, this is a *transverse wave*.

b.) Longitudinal waves: These are waves created by a force applied to a medium *in the same direction* as the wave's motion in the medium.

i.) An example: When sound from a loud-speaker is produced, the speaker cone applies a force to air molecules that is *in the same direction* as the subsequent pressure-waves that move out from the device.

4.) Wave reflection:

a.) Consider a taut rope fixed to a door. Flicking the rope at the unattached end will produce a single wave that will travel down the rope (see Figure 12.3a). When it gets to the door, the wave will bounce off the fixed end, flip 180^o (π radians; one-half a cycle; whatever--see Figure 12.3b) and proceed back down the line. This half-wave inversion is typical of wave-reflection off *fixed ends*.

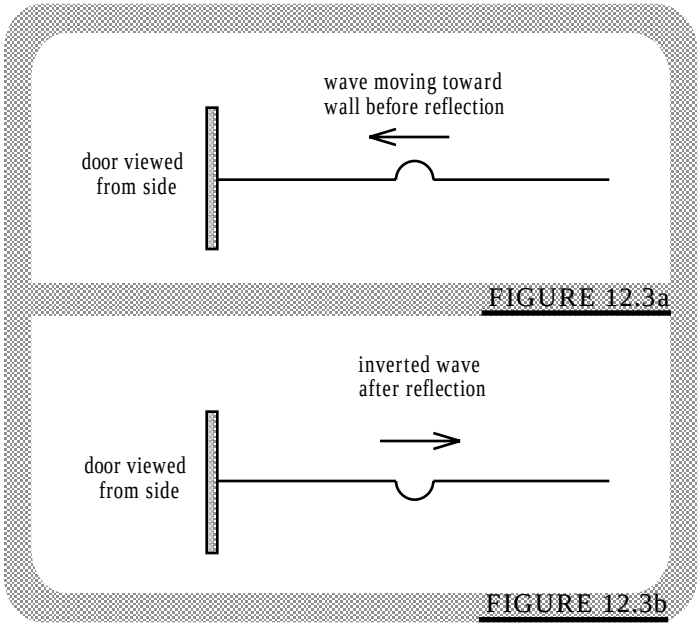


FIGURE 12.3a

FIGURE 12.3b

b.) Consider a rope hanging freely from a ceiling. A single wave moving downward (see Figure 12.4a) will bounce off the free bottom and proceed back up toward its origin.

The bounce-back flips the wave 360^o (i.e., it comes back to its original position) before the wave proceeds back up the rope (see Figure 12.4a). This full-wave inversion (net effect--no inversion at all) is typical of wave reflection off *free ends*.

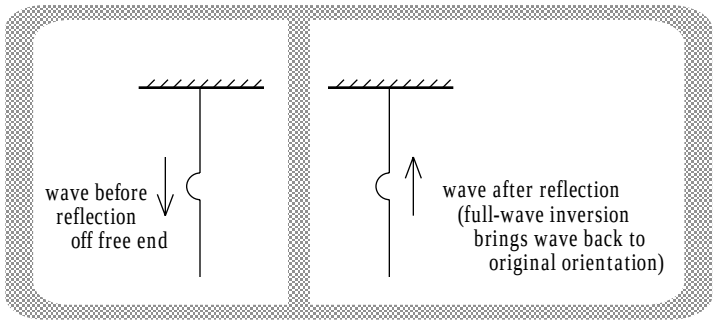


FIGURE 12.4a

FIGURE 12.4b

5.) Some DEFINITIONS:

a.) Wavelength (λ in meters): the distance between two successive crests, or two successive troughs, or between two successive positions along the wave that are exact duplicates of one another (see Figure 12.5).

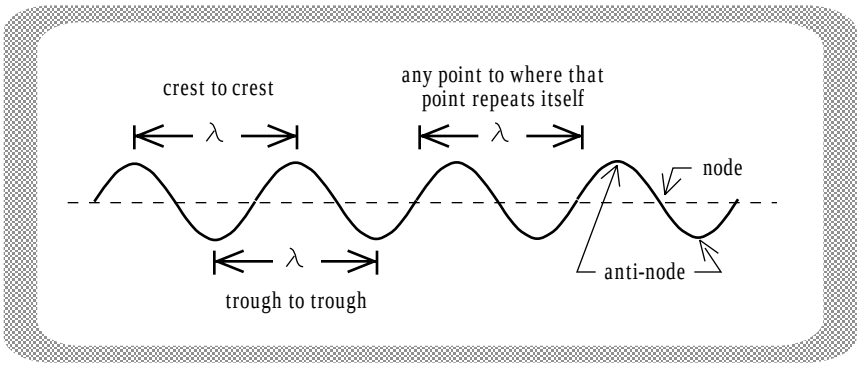


FIGURE 12.5

b.) Frequency ("ν" in cycles/second--this symbol is a Greek "nu"): the *number of wavelengths* that pass a fixed observer *per second*.

c.) Period ("T" in seconds/cycle): the *time* required for one full wavelength to pass a fixed observer. As in vibratory motion, $T = 1/\nu$.

d.) Wave velocity ("v" in meters/second): the velocity of a wave disturbance as it moves through its medium. Mathematically:

$$v = \lambda \nu.$$

(Don't believe me? Check the units.)

A consequence of this relationship: for a given wave, *high frequency* corresponds to *short wavelength* and vice versa.

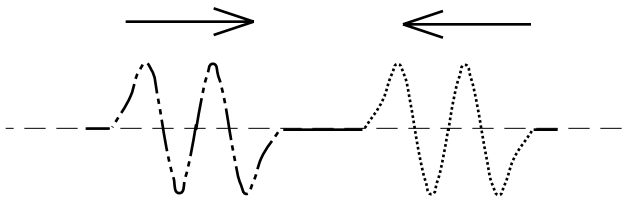
e.) Nodes and anti-nodes: a *node* is a null spot on the wave. It corresponds to a place where the displacement of the wave is zero (see Figure 12.5). An *anti-node* is a spot where the displacement is a maximum. It corresponds to a crest or trough (see Figure 12.5).

f.) Superposition of waves: when two waves in the same medium run into one another, the two disturbances will add to one another in a linear way. Given such a situation, there are a number of outcomes:

i.) Constructive superposition: a situation in which the two waves momentarily produce a single wave that is *larger* than the original two. For two waves with the same amplitude *A*, completely *constructive superposition* will yield a displacement of *2A*.

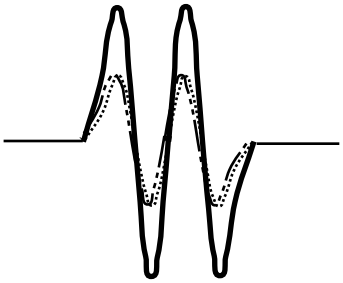
ii.) Destructive superposition: a situation in which the two waves produce a single wave that is *smaller* than the largest of the original two. For two waves with the same amplitude *A*, completely *destructive superposition* will produce a net displacement of zero.

iii.) Figure 12.6a shows two waves (one denoted with dots, one denoted with dots and dashes) moving in opposite directions in the same medium. Figures 12.6b and 12.6c show the waves at various stages of superposition.



original waves moving in opposite directions in same medium

FIGURE 12.6a



completely constructive superposition
(the dark line shows the net effect)

FIGURE 12.6b

partially destructive superposition
(central superposition is totally destructive)

FIGURE 12.6c

B.) Mathematics of Traveling Waves:

1.) The displacement of a traveling sine wave is a function of both *time* and *position*. Its displacement will vary *at a given time* from place to place in addition to varying *at a given place* as time proceeds.

2.) The function that characterizes this situation is:

$$y(x,t) = A \sin (kx \pm \omega t),$$

where *A* is the amplitude of the wave, *k* (the *wave number*) is defined as $2\pi/\lambda$ (just as ω--the angular frequency--governs how fast the function changes relative to time, *k* governs how fast the function changes relative to position--it is like a *positional-frequency* function), and *x* and *t* are the two variable-parameters that allow one to zero in on a *particular place* at a *particular time of interest* (for more, see *Problem 12.3* at the chapter's end).

C.) Resonance:

1.) Vibrating systems usually have at least one *frequency* at which the system will naturally vibrate. A swing, for instance, acts like a pendulum. In the last chapter, we found that for small angles the *frequency* of a pendulum is dependent upon the *acceleration of gravity* and the *length of the pendulum arm*. Assuming neither of those changes in a given situation, a pendulum has only *one frequency* at which it will freely, naturally oscillate (in fact, this is approximately true even for moderately large amplitude oscillations). Although you haven't run into any yet, there are systems that have more than one *natural frequency*.

2.) *Resonance* is a situation in which the *frequency* of an *applied force* matches one of the *natural frequencies* of the system to which the force is applied. The consequence of such a condition is an increase in the system's *energy* and *amplitude of vibration*.

3.) Example: Consider a mass attached to the spring shown in Figure 12.7. Every such spring/mass combination will have one natural frequency at which the system will oscillate (remember $\nu = [(k/m)^{1/2}]/2\pi$?). For this example, assume that the *natural frequency* is *1/2 cycle per second*. As the period is $T = 1/\nu$, the *period* of this oscillation will be *2 seconds per cycle*. With the mass at rest:

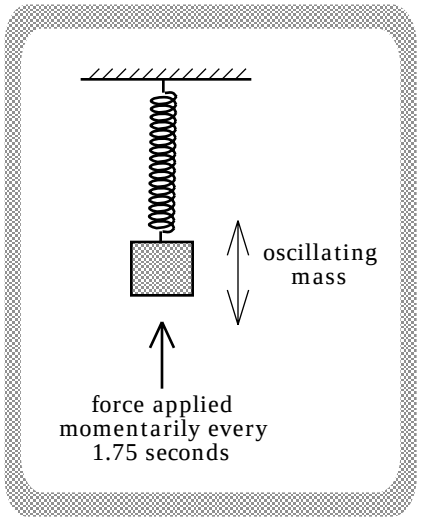


FIGURE 12.7

a.) A force is briefly applied to the mass, then applied periodically for just a moment every 1.75 seconds (see Figure 12.7). Will the force help or hinder the oscillatory motion?

i.) As the *force's frequency* is out of sync with the *natural frequency* of the system, it doesn't take a genius to see that the force will fight the natural motion of the system. The consequence: the mass's motion will be disorderly and chaotic.

b.) Another possibility: the *frequency* of the *applied force* is now *2 seconds/cycle*. Under this condition, the push comes each time the mass is at its lowest point--each time the mass is naturally ready to start upwards again. The force helps the motion because the *frequency of the applied force* matches the *natural frequency* of the oscillating system. As

such, the *amplitude* of the motion gets bigger and bigger with each cycle, and the *energy* of the vibrating system increases.

Note: You might at first think that because the *amplitude* of the oscillation is changing, the frequency must change. Not so for ideal springs. The *frequency* of an oscillating spring/mass system is *not* a function of *amplitude*; the *frequency* will remain the same no matter how large the *amplitude*.

D.) Resonance and String Systems:

1.) A more complex *resonance* situation is found in what are called *standing waves*.

Note: Although a *spring* has only one natural frequency, a *string* system has an infinite number of possible natural frequencies. Why this is so will become evident shortly.

a.) Consider a rope of length L and mass density h (h is the rope's mass/length) stretched taut between two fixed points. A small *transverse force* is periodically applied at one end creating a series of waves moving down the string (this is called a *wave train*). As the waves move, each will sooner or later bounce off the fixed end, flipping a half-cycle before proceeding back toward its origin. These reflected waves will run into newly formed waves coming in the other direction and a *superposition of waves* will take place along the entire string.

b.) Just as was the case with the mass/spring system, the *frequency of the applied force* has a lot to do with how this superposition proceeds.

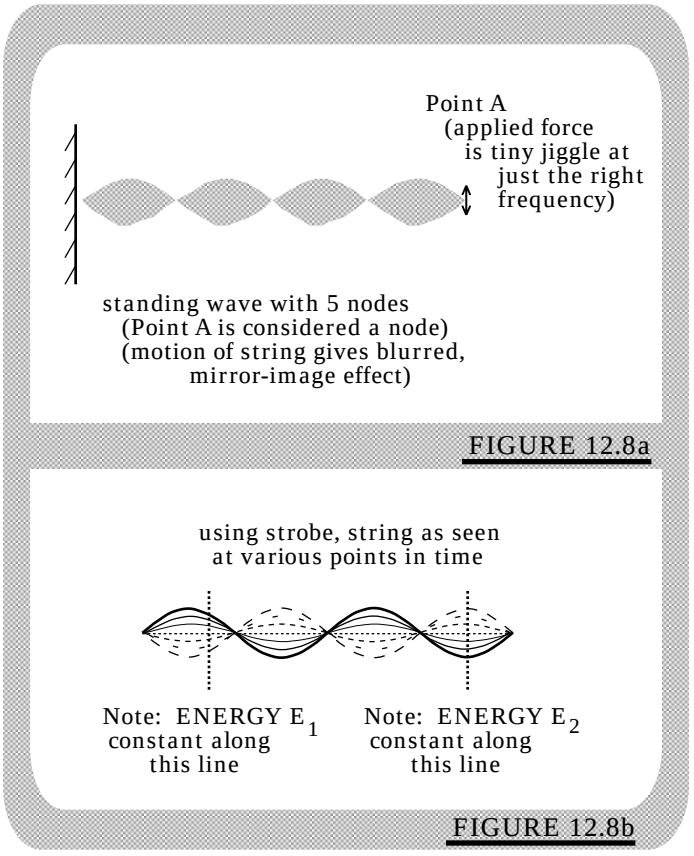
i.) If there is <u>no</u> *applied force* once the first few waves are set in motion, the waves will travel up and back and up and back until frictional effects dampen them out (this usually happens quickly).

ii.) What happens if the *applied force* at *Point A* continues after the first few waves are formed? In that case, it depends upon the frequency of the *applied force*. If the *frequency* is <u>out of sync</u> with the *natural frequency* of oscillation on the string, the returning waves will be met by an *applied force* that fights their natural motion. The net result will be a *superposition* of incoming and outgoing waves along the string that is disorganized and the string will jerk around chaotically.

Note: This is just like applying a force whose frequency is *1.75 cycles per second* to a swing whose natural frequency is *2 cycles per second*.

iii.) If, on the other hand, the *frequency of the applied force* matches one of the *natural frequencies* of the system, the *applied force* will reinforce the waves bouncing off *Point A* and the superposition that subsequently occurs on the string will be orderly and well defined. When such a situation arises, the *net wave* formed on the string is the consequence of superposition and is called a *standing wave*.

iv.) At a given instant, the waveform for the standing wave described above will look like a *sine wave*. Because the wave's net displacement is constantly changing quickly, such standing waves usually look like the blurred form shown in Figure 12.8a. Figure 12.8b uses a strobe effect to show a snapshot view of the string at different points in time.



Note: A given point on the string will move up and down with simple harmonic motion. In terms of *energy*, that means *each point* will have energy $(1/2)kA_p^2$ associated with it, where k is some constant and A_p is the amplitude--the maximum displacement--of that point. As the amplitude of each point on the string varies, the amount of energy associated with each point on the string will be different.

This fact is where the name *standing wave* came from. True, the point displacements vary so fast that the net wave looks like a single, blurred, two-dimensional standing form, but in fact the name was originated to honor the fact that the *energy of a particular point stands constant throughout the motion* (see Figure 12.8b).

v.) Bottom line: When an *applied force* is in sync with one of the *natural frequencies* of a string system, resonance occurs and the displacement of each point on the string is found to move in a very precise, harmonious way. The net effect is to produce string motion that outlines a mirror-image sine wave.

2.) In terms of the system's parameters (i.e., wave velocity v, string length L, etc.), how can we determine the *natural frequencies* of a string system (i.e., what is the *lowest* natural frequency, the *second lowest* natural frequency, etc.) and, hence, the *applied-force frequencies* that will allow the system to *resonate*?

The answer to that question is outlined below.

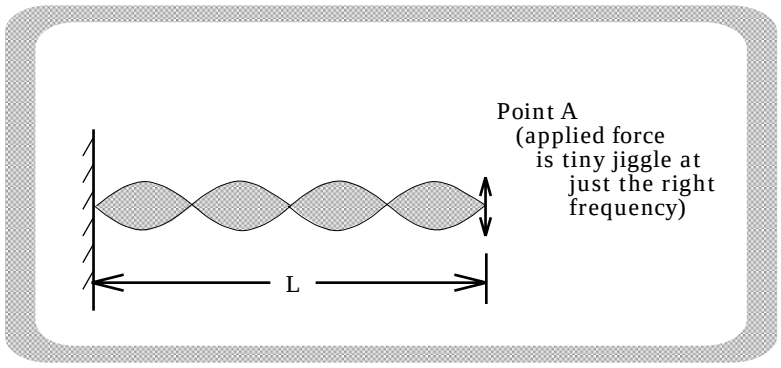


FIGURE 12.9

a.) Observation #1: Begin by considering the standing wave shown in Figure 12.9.

b.) Observation #2: We know that there exists a relationship between a wave's velocity v, wavelength λ_a, and wave frequency ν_a. It is: $v = \lambda_a \nu_a$.

Note: If we can determine both the wave velocity v and the wave's wavelength λ_a when our standing wave exists on the string, we can use the above relationship to determine the natural frequency ν_a that corresponds to that standing wave.

c.) Observation #3: There is a known relationship between the velocity v of a wave moving on a string of tension T and mass density h. That relationship is: $v = (T/h)^{1/2}$.

d.) Observation #4: By inspection, we can determine a standing wave's wavelength λ_a in terms of the known string length L. Looking at Figure 12.9 we ask, "How many wavelengths of length λ_a are there in the length L?" Mathematically, this is like querying, "Find the N in the equation $N\lambda_a = L$." Examining Figure 12.9, it is obvious that there are two λ_a's in the length L, or $2\lambda_a = L$. From this, we get $\lambda_a = L/2$.

e.) Observation #5: With the information found in Observations 2, 3, and 4, we can determine the *natural frequency* of the wave shown in Figure 12.9. It is:

$$v = \lambda_a \, \nu_a.$$
$$\Rightarrow \quad \nu_a = v/\lambda_a$$
$$= v/(L/2).$$

f.) We have just found one *natural frequency* at which a system will oscillate; we now know one *applied-force frequency* at which that system will *resonate*.

3.) This approach works fine as long as you have the sketch. What happens when that isn't the case? What happens when the question simply states, "Given a string length L and wave velocity v, determine the *three lowest frequencies* of wave that can stand on this system?" Using an "observation" approach:

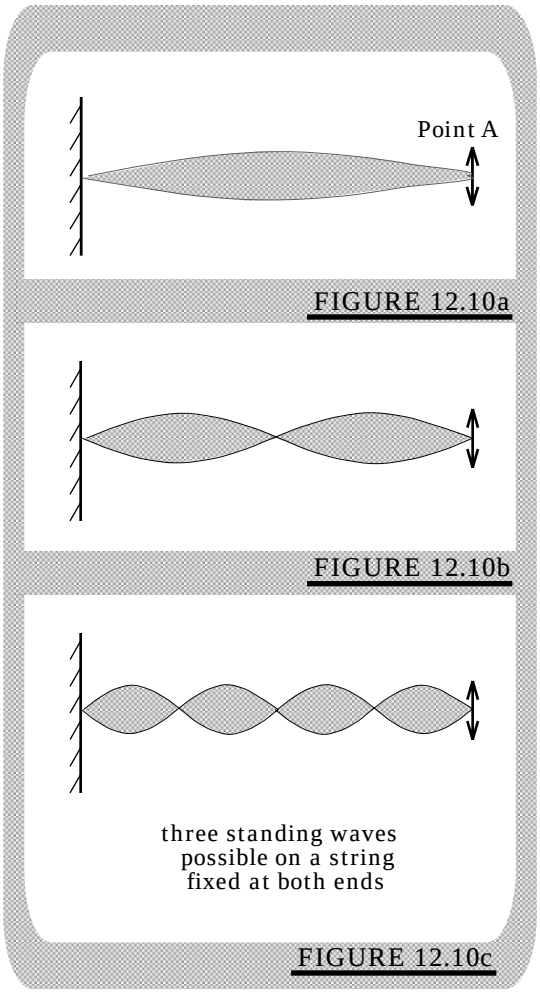


FIGURE 12.10a

FIGURE 12.10b

FIGURE 12.10c

a.) Observation #1: Figures 12.10 a, b, and c show *different frequency* standing waves standing on the same system (i.e., the same length of string). What do they have in common?

i.) In all cases, there are *nodes at each end of the string*. Each wave has a wavelength that fits the constraint of having nodes a distance L meters apart.

ii.) Does this make sense? Certainly! The string is *fixed* at each end. That means there can be no motion at those points (as the force at *Point A* has a tiny amplitude--it is legitimate to ignore it, calling *Point A* a node).

b.) Observation #2: If you didn't *have* the sketches to look at, could you have created them? Yes!

i.) Draw a sine wave (see Figure 12.11 on the next page).

ii.) Determine the left-end constraint for your system (the left-end constraint in our example is a node). Proceed toward the right along the *sine wave* until you come to a point that matches the *right-end constraint* (in our example, this is another node). If there are no other constraints between the endpoints (i.e., there are no forced nodes in between due to somebody *holding the string*), *the waveform between the endpoints* will be the *longest wavelength waveform* (i.e., *lowest frequency waveform*) THAT FITS THE CONSTRAINTS OF THE SYSTEM.

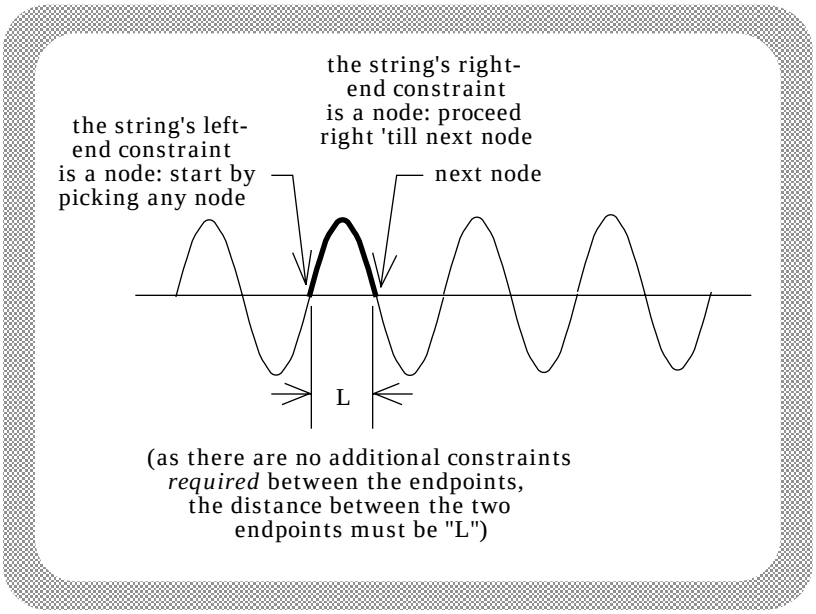

the string's right-end constraint is a node: proceed right 'till next node
the string's left-end constraint is a node: start by picking any node
next node
L
(as there are no additional constraints *required* between the endpoints, the distance between the two endpoints must be "L")

FIGURE 12.11

iii.) From here, you can either re-draw the section-of-sine-wave you are interested in, or simply use the original drawing to put in the length *L* (Figure 12.11 shows *L* placed directly onto the working *sine wave*).

iv.) Having the sketch, you can then proceed to ask, "How many wavelengths are there in the length *L*?" Mathematically, this can be written:

$$N\lambda_n = L.$$

Once answered, proceed from there.

Note: If you want the *second lowest frequency* for the example, the process is exactly the same. Draw a sine wave; match the string's *left-end constraint* (a node in this case) with a point on the sketch; proceed to the right until you find a point that matches the string's *right-end constraint*. If the waveform encompassed between those two points has already been used (in this case, the *two-consecutive-nodes* waveform was used for the lowest frequency waveform), continue to the right until you find the *next* point that satisfies the right-end constraint (see Figure 12.12). If there are no additional constraints between the two ends (nobody is holding the string in the middle somewhere), *the waveform*

you see in between your two chosen points (both nodes in this case) will correspond to the *second longest wavelength* waveform (i.e., the *second lowest frequency* waveform). Define the distance between the two endpoints as L and proceed as before.

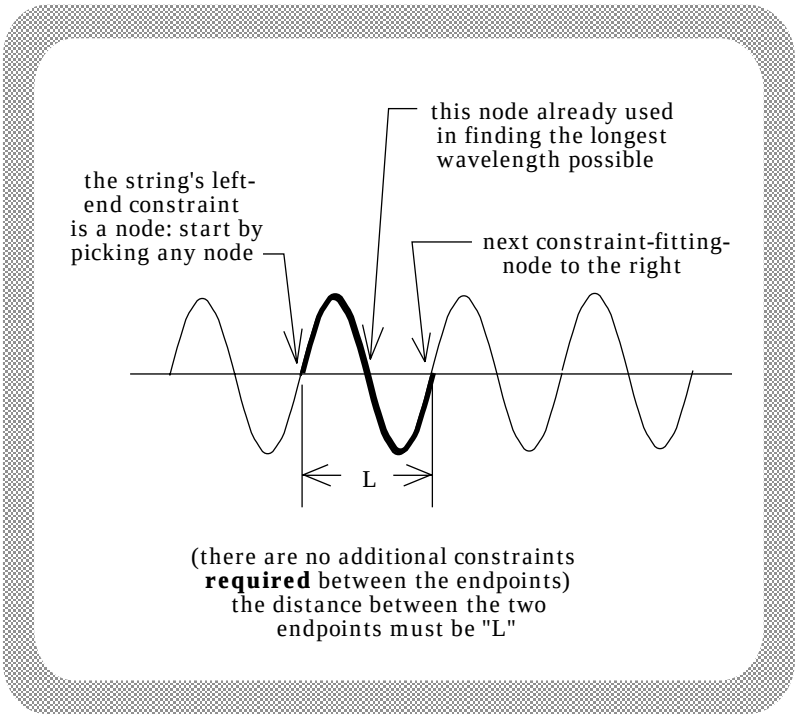

this node already used in finding the longest wavelength possible
the string's left-end constraint is a node: start by picking any node
next constraint-fitting-node to the right
L
(there are no additional constraints **required** between the endpoints) the distance between the two endpoints must be "L"

FIGURE 12.12

4.) Consider a string of length L constrained to hang from the ceiling as shown in Figure 12.13. Assume the velocity of a wave moving on the string is a known v. If a *small, periodic, transverse force* is applied to the string at the ceiling, what are the *two lowest frequencies* at which the string will *resonate*? That is, what are the two lowest frequencies that can *stand* on the string?

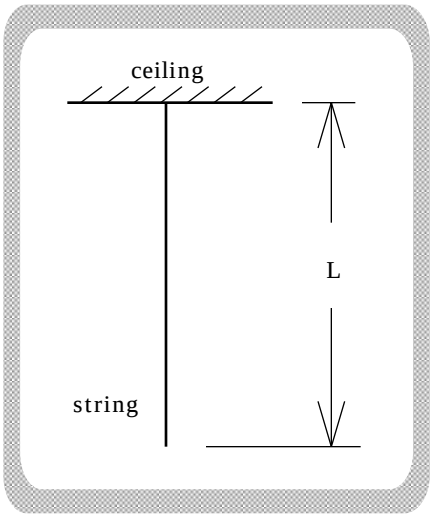


FIGURE 12.13

a.) First, we need to identify the constraints. They are:

i.) A node at the ceiling (ignoring the applied jiggle the string is fixed there);

ii.) An anti-node at the free end; and

iii.) There are no additional constraints between the two ends.

b.) With the constraints in mind, consider the vertical sine wave shown in Figure 12.14 (a vertical *sine wave* has been used to better reflect the situation--a horizontal sine wave would have worked just as well). Beginning at <u>any convenient node</u> (there must be a node at the ceiling), proceed down until you run into the first anti-node (this is the constraint at the free end). The *longest wavelength* waveform that can fit the system's constraints is the smallest section of sine wave to fit the constraints (see Figure 12.14).

i.) You can re-draw this waveform or put L on the original waveform as is done in Figure 12.14.

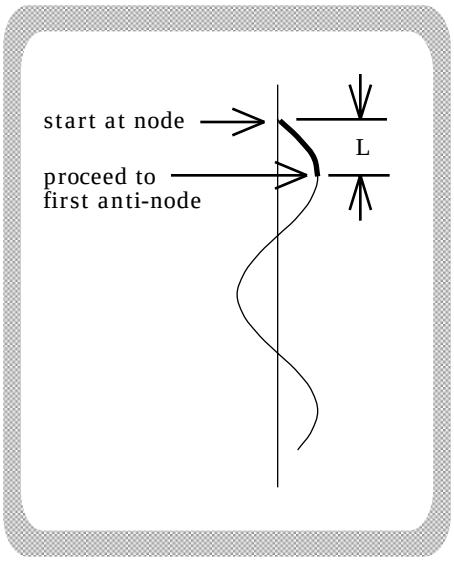


FIGURE 12.14

c.) Asking the question: How many of these wavelengths are there in the length L, or $N\lambda_1 = L$, we get:

$$(1/4)\ \lambda_1 = \mathrm{L}$$
$$\Rightarrow\quad \lambda_1 = 4\mathrm{L}.$$

d.) Putting it all together, we end up with:

$$\mathrm{v} = \lambda_1 \nu_1$$
$$\Rightarrow\quad \nu_1 = \mathrm{v}/\lambda_1$$
$$= \mathrm{v}/(4\mathrm{L}).$$

e.) Following a similar series of steps, the *second longest wavelength* waveform to satisfy the constraints is shown in Figure 12.15. The math yields:

$$(3/4)\ \lambda_2 = \mathrm{L}$$
$$\Rightarrow\quad \lambda_2 = (4/3)\mathrm{L}.$$

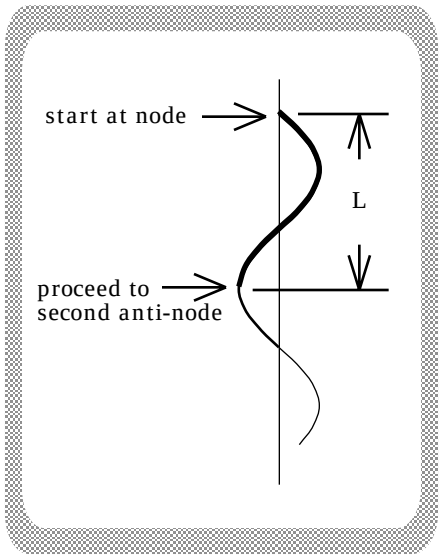


FIGURE 12.15

As $v = \lambda_2 \nu_2$, we can write:

$$\nu_2 = \mathrm{v} / \lambda_2$$
$$= (3\mathrm{v})/(4\mathrm{L}).$$

Note: Although you can have either a node or anti-node at a given end, you can only have nodes *in between* endpoints as there is no way to force a system to exhibit *anti-nodes* in its mid-section.

E.) Resonance and a Steel Bar:

1.) Consider the bar of length L shown in Figure 12.16. It is constrained by a clamp located half way down the bar's length. If the bar is tapped once at the end, a series of waves will be found to "stand" on the bar. If the velocity of the waveform on the bar is a known value v, what are the *two lowest frequencies* at which the bar will ring?

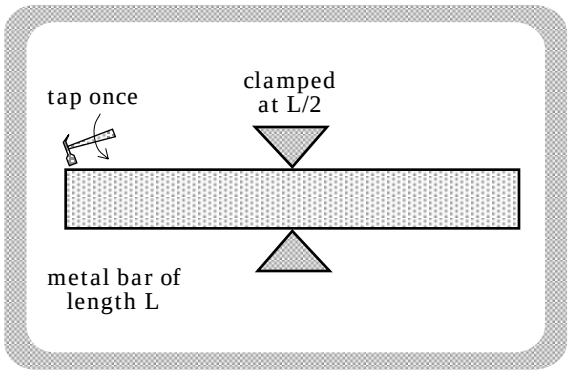


FIGURE 12.16

a.) The constraints: *anti-nodes at both ends* and *a node in the middle.*

b.) Consider the *sine wave* drawn in Figure 12.17.

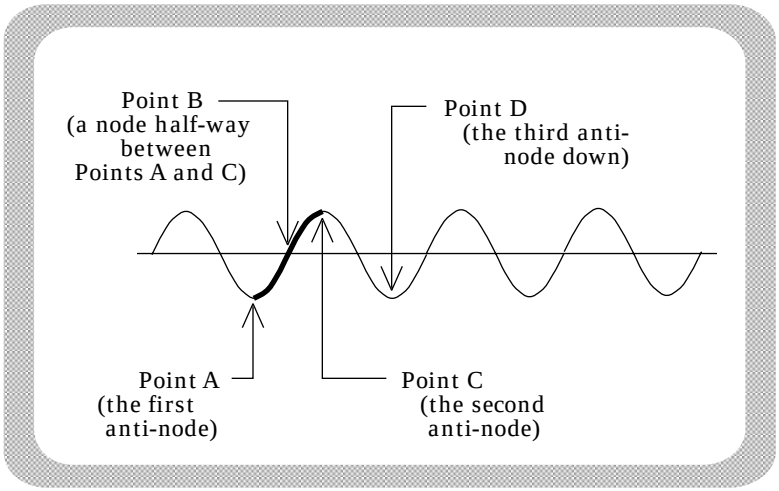


FIGURE 12.17

i.) If we start at the *anti-node* at *Point A* and proceed to the right, the next *anti-node* we come to is located at *Point C* on the sketch. That takes care of the end constraint. But . . .

ii.) The waveform we are looking for must additionally have a *node* in the middle. Fortunately for us, this waveform *has* a node halfway between the endpoints, which means we have found the longest wavelength (lowest frequency) wave form that conforms to the constraints of the problem.

c.) The math proceeds just as it did with the *string systems*: We begin by asking, "How many λ *'s* in L?" Proceeding yields:

$$(1/2)\lambda_1 = L$$
$$\Rightarrow \quad \lambda_1 = 2L.$$

As $v = \lambda_1 \nu_1$, we can write:

$$\nu_1 = v / \lambda_1$$
$$= v/(2L).$$

d.) For the *second longest wavelength* waveform, it might be tempting to assume the third anti-node (the one located at *Point D* in Figure 12.17 on the previous page) is the appropriate choice. A closer look shows that the waveform extending from *Point A* to *Point D* has an *anti-node* at its center. Our constraints require a node.

The waveform that matches our constraints extends from *Point A* to *Point E* in Figure 12.18. The math yields:

$$(3/2)\ \lambda_2 = L$$
$$\Rightarrow \quad \lambda_2 = (2/3)\ L.$$

As $v = \lambda_2 \nu_2$,

$$\nu_2 = v / \lambda_2$$
$$= (3v)/(2L).$$

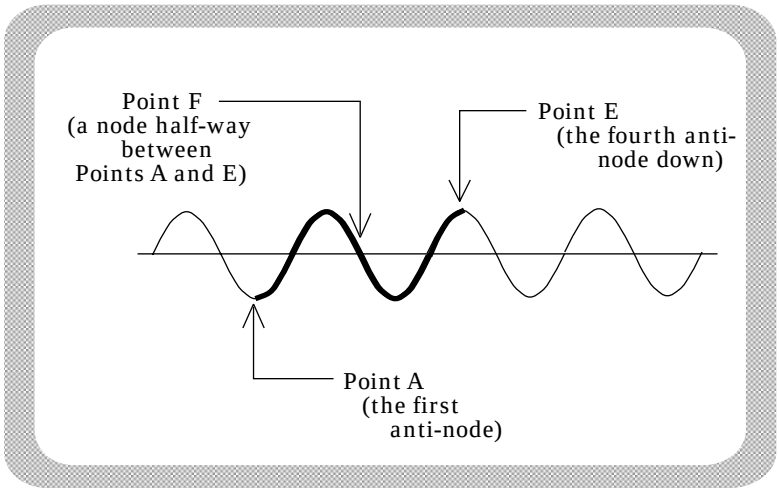


FIGURE 12.18

2.) Consider the bar shown in Figure 12.19. It is constrained by a clamp *at one end* and by a second clamp located *two-thirds of the way down the bar's length*. What are the *two lowest frequencies* at which this bar system will ring?

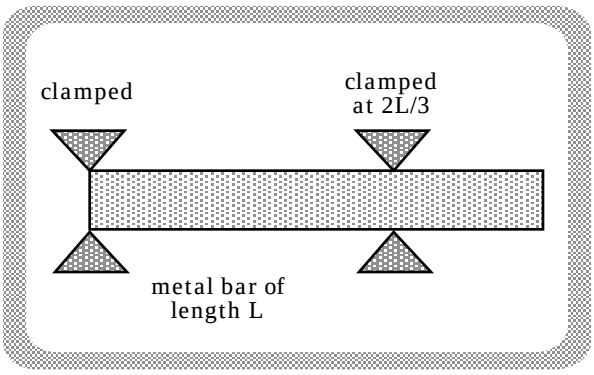


FIGURE 12.19

a.) The constraints: a node at one end, a node two-thirds of the way down the bar, and an *anti-node* at the other end.

b.) The longest waveform to conform to our constraints is shown in Figure 12.20a on the next page.

c.) With a sketch of the appropriate waveform, we write:

(3/4) $\lambda_1 = L$

$\Rightarrow \quad \lambda_1 = (4/3)L.$

As $v = \lambda_1 \nu_1$,

$\nu_1 = v / \lambda_1$

$= (3v)/(4L).$

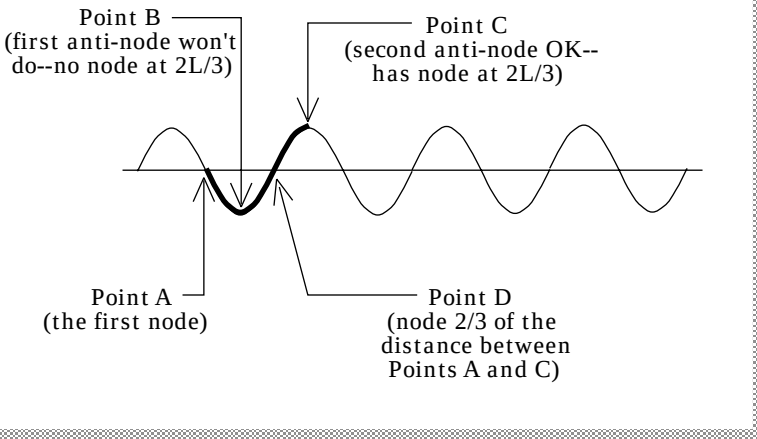

Point B
(first anti-node won't do--no node at 2L/3)
Point C
(second anti-node OK--has node at 2L/3)
Point A
(the first node)
Point D
(node 2/3 of the distance between Points A and C)

FIGURE 12.20a

d.) Following a similar series of steps, the *second longest wavelength* waveform to satisfy the constraints is shown in Figure 12.20b. The math yields:

(9/4) $\lambda_2 = L$

$\Rightarrow \quad \lambda_2 = (4/9)L.$

As $v = \lambda_2 \nu_2$,

$\nu_2 = v/\lambda_2$

$= (9v)/(4L).$

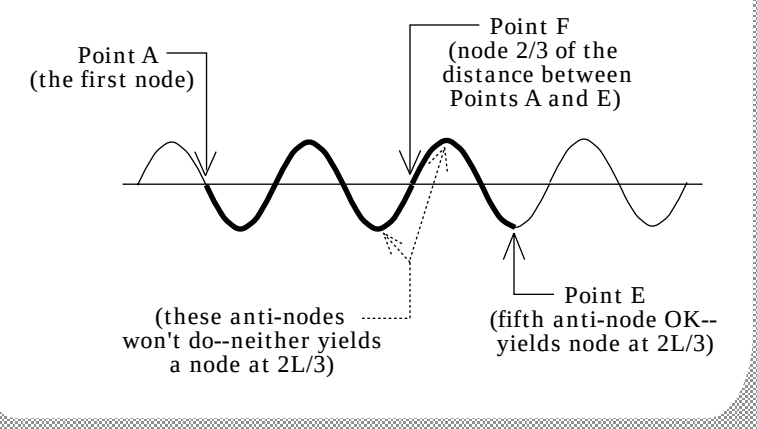

Point A
(the first node)
Point F
(node 2/3 of the distance between Points A and E)
(these anti-nodes won't do--neither yields a node at 2L/3)
Point E
(fifth anti-node OK--yields node at 2L/3)

FIGURE 12.20b

Note: The temptation might have been to go to the node just after the anti-node used in the "longest wavelength" part of the problem, or for that matter the anti-node after that one (see "won't do" note on Figure 12.20b). The problem with both of these choices is that there is no node *two-thirds of the way* between either of them and the node at *Point A*. All the constraints must be met before we have our waveform.

3.) Understand this approach to the point where you can find the two or three *lowest frequencies* for any combination of clamps.

4.) Although we will use a *simple* situation to deduce the relevant points, the following observations should be helpful when dealing with *complex* standing wave problems. Consider: A system has a node at its left-end, an *anti-node* at its right-end, and an additional node-constraint 2/3 of the way from the left-end. Calling this 2/3 ratio the *fractional distance* between the mid-range node and the left-end constraint, what is the lowest frequency that will stand on this system?

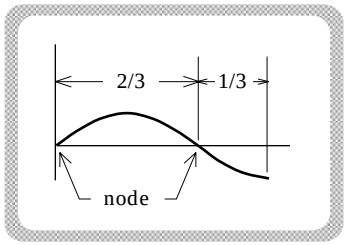


FIGURE 12.20c

a.) By observation, the waveform that fits the constraints is shown to the right in Figure 12.20c.

b.) The question? How can we be sure that the waveform satisfies the constraints of a system? That is, if we have a waveform that we think fits the bill, how can we check it?

c.) The approach:

i.) Count the number of *QUARTER WAVELENGTHS* in the waveform. In our example, there are *3 quarter wavelengths*.

ii.) Note the *fractional distance* between one of the end-constraints (we have used the left-end NODE in our example) and the mid-range constraint (mid-range constraints are always nodes). In our example, this is *2/3*.

iii.) Determine the product of the *fractional distance* and the *number of quarter wavelengths in the waveform*. In our case, this equals *(2/3)(3)= 2*.

Note: If we had started at the other end (i.e., at the end constrained to act as an anti-node), the product of the *fractional distance* and the *number of quarter wavelengths* would have been *(1/3)(3) = 1*.

v.) Conclusion (not a proof, but it happens to be always true)

1.) If the product of the *fractional distance* and *the number of quarter wavelengths* equals a WHOLE NUMBER, then the waveform you are examining satisfies the mid-range constraint in question.

2.) If the product does *not* equal a whole number, the waveform does NOT satisfy the mid-range constraint.

vi.) For a waveform to stand (resonate) on a particular system, ALL of the mid-range constraints must satisfy the *fractional-distance/quarter-wavelength* test.

F.) Resonance of Sound in Columns of Air:

1.) Sound is a longitudinal wave. It is made up of successive high, then low, then high pressure zones moving out through a medium.

a.) Figure 12.21a is the sketch of a sound wave as it would appear if your eyes were sensitive to pressure waves.

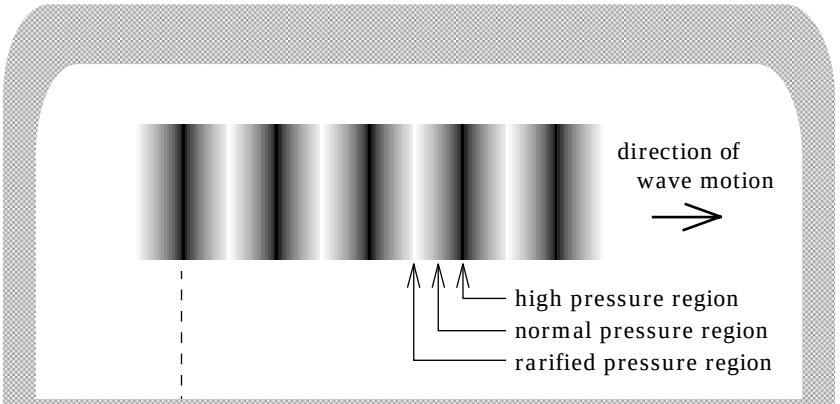


FIGURE 12.21a

b.) Figure 12.21b graphs *pressure* as a function of *position in space* at a particular point in time.

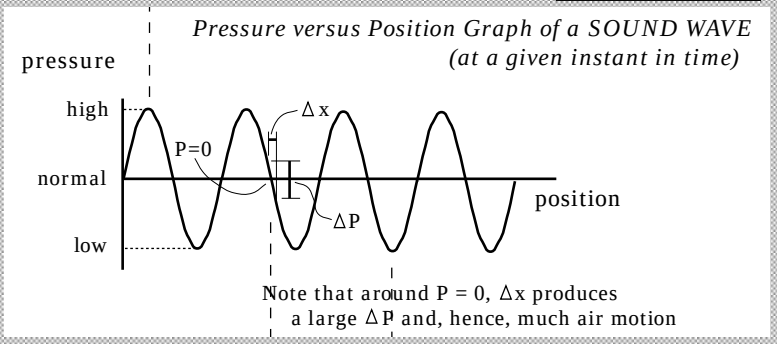


FIGURE 12.21b

c.) A pressure *difference* motivates air particles to move from higher to lower pressure.

d.) When standing waves are set up in air, the motion of the air medium is what dictates whether a particular point is a node or an anti-node.

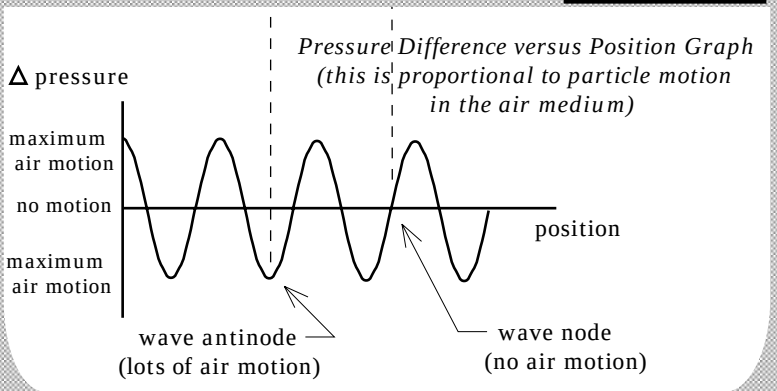


FIGURE 12.21c

e.) On either side of a pressure *maximum*, the pressure difference is very small, there is very little air motion, and with little motion the medium (the air) acts like a node. On either side of a pressure *zero*, the pressure difference is very large, there is great air motion, and with much motion the medium acts like anti-nodes. In short, the graph used to analyze sound standing waves is really a *pressure difference* graph (see Figure 12.21c).

2.) When sound moving down a wave-guide bounces off a wall, it acts like a wave on a string bouncing off a fixed end. For sound-generated standing waves, a *node* will exist wherever the sound bounces off a *solid surface* (this is like saying that there is *no <u>net</u> air motion* at the wall). An *open surface* like the end of an open pipe (see Figure 12.22a, for example) acts like an *anti-node*. With this in mind, consider the following situation.

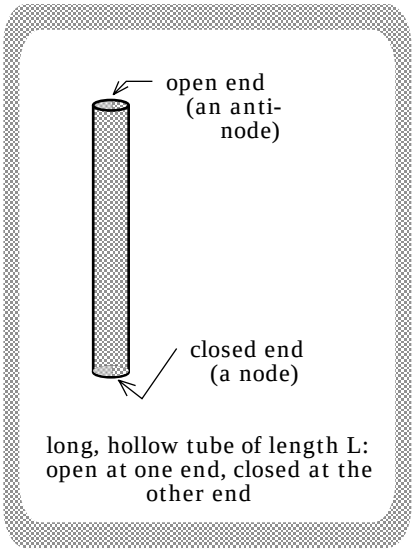


FIGURE 12.22a

3.) A tube of known length L is open at one end and closed at the other (see Figure 12.22a). A series of sound waves is projected into the tube. What are the *three lowest frequencies* that will *resonate* in the air column?

a.) The constraints are: an *anti-node* at the top and a *node* at the bottom. There are no additional constraints in-between the ends (i.e., no baffles anywhere).

b.) Consider the sine wave drawn in Figure 12.22b (this *sine wave* is presented in both a vertical and horizontal setting: there is no difference--use the one that seems the least confusing). There are no *additional constraints* between the endpoints, so we will start at an *anti-node* (*Point A*) and proceed until we hit the first node (*Point B*). Making this distance equal to the tube's length L, we do the process.

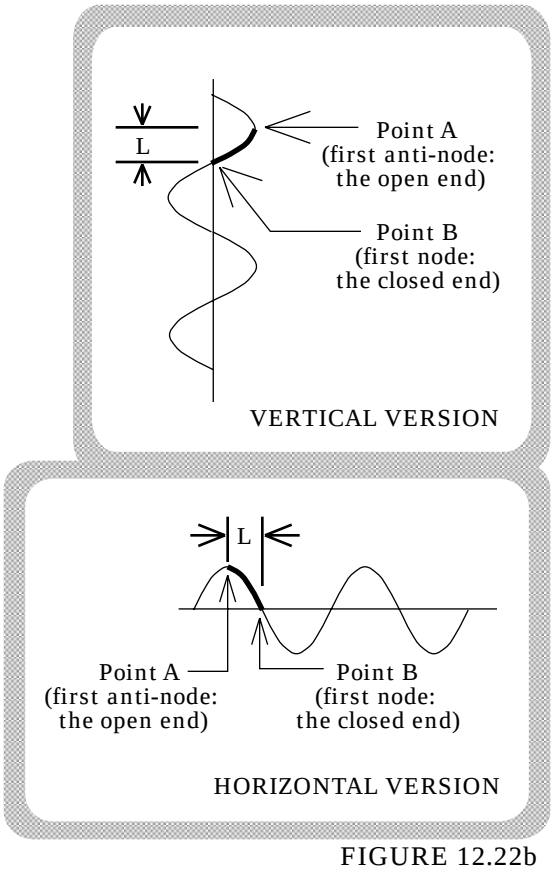


FIGURE 12.22b

c.) The math dictates:

$$(1/4)\ \lambda_1 = L$$
$$\Rightarrow \quad \lambda_1 = 4L.$$

As $v = \lambda_1 \nu_1$.

$$\nu_1 = v/\lambda_1$$
$$= v/(4L).$$

d.) $L = 1.5\ m$, the speed of sound in air is approximately 330 m/s, and remembering that THE UNIT FOR FREQUENCY is technically *1/seconds*:

$$\nu_1 = v/\lambda_1$$
$$= (330 \text{ m/s})/[(4)(1.5 \text{ m})]$$
$$= 55 \text{ Hz.}$$

e.) The *second longest wavelength* is found in Figure 12.23 (we will use a vertical *sine wave* only).
The math yields:

$$(3/4)\ \lambda_2 = L$$
$$\Rightarrow \quad \lambda_2 = (4/3)L.$$

As $v = \lambda_2 \nu_2$,

$$\nu_2 = v/\lambda_2$$
$$= (3v)/(4L).$$
$$= 3(330 \text{ m/s})/[(4)(1.5 \text{ m})]$$
$$= 165 \text{ Hz.}$$

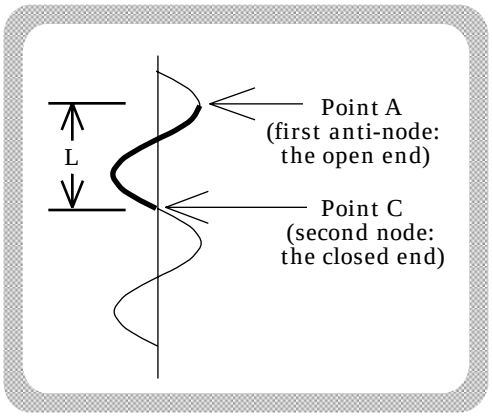


FIGURE 12.23

f.) The *third longest wavelength* is found in Figure 12.24. The math yields:

$$(5/4)\ \lambda_2 = L$$
$$\Rightarrow \quad \lambda_2 = (4/5)L.$$

As $v = \lambda_2 \nu_2$,

$$\nu_2 = v/\lambda_2$$
$$= (5v)/(4L)$$
$$= 5(330 \text{ m/s})/[(4)(1.5 \text{ m})]$$
$$= 275 \text{ Hz.}$$

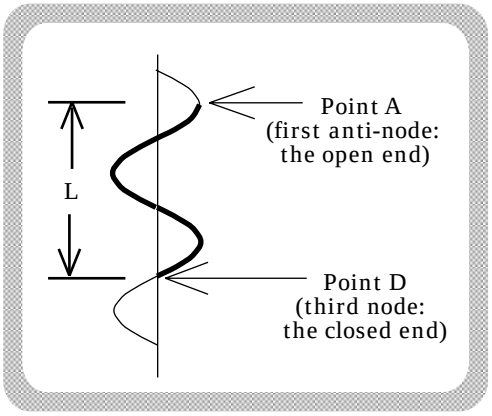


FIGURE 12.24

4.) Now consider the same pipe open at *both* ends. What is the <u>*third*</u> *lowest frequency* at which this system will resonate?

a.) The constraints: anti-nodes at both ends.

b.) The waveform that conforms to our constraints and specifications (i.e., the *third* lowest frequency) is shown in Figure 12.25 to the right.

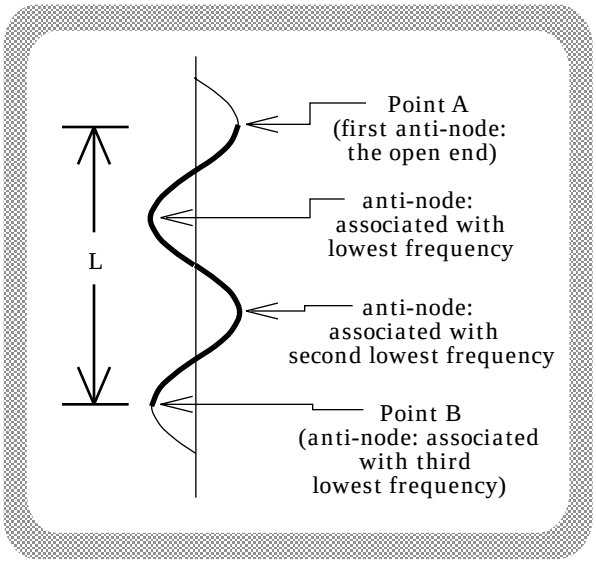


FIGURE 12.25

c.) From the sketch, we write:

$$(3/2)\ \lambda_3 = L$$
$$\Rightarrow \quad \lambda_3 = (2/3)\ L.$$

As $v = \lambda_3 \nu_3$,

$$\nu_3 = v/\lambda_3$$
$$= (3v)/(2L)$$
$$= 3(330\ \text{m/s})/[(2)(1.5\ \text{m})]$$
$$= 330\ \text{Hz}.$$

5.) Parting shot: Look at the constraints; draw your sketch; find l in terms of L; then use $\nu = v/\lambda$ to determine the resonance frequency.

G.) Odds and Ends--Beats and the Doppler Effect:

1.) There are two other areas of wave phenomena that need to be briefly examined. The first is *beats*, the second is *the Doppler Effect.*

2.) BEATS:

a.) Figure 12.26 on the next page shows two equal-amplitude sound waves of slightly different frequency mingling with one another. The same figure also shows the superposition of those two waves.

b.) Because one waveform has a slightly higher frequency, it cycles a little bit faster than the other. That means that if both start out in phase (i.e., if both have their peaks and troughs initially aligned), the two will sooner or later go out of phase.

c.) When *in phase*, the superposition of the waves is constructive and the amplitude of the net wave is large (this corresponds to loud sound). When *out of phase*, the superposition of waves is destructive and there is no net wave (this corresponds to no sound).

d.) The frequency of this sound variation (sometimes called *a warble*) is called the *beat frequency* of the superimposing waves.

e.) Numerically, the *beat frequency* equals the *difference* between the two superimposing frequencies.

i.) Example: A 220 Hz wave coexists with a 218 Hz wave. The beat frequency heard as a consequence of this super-position will be 2 cycles per second. If the frequencies had been 220 Hz and 222 Hz, the beat frequency would again have been 2 Hz.

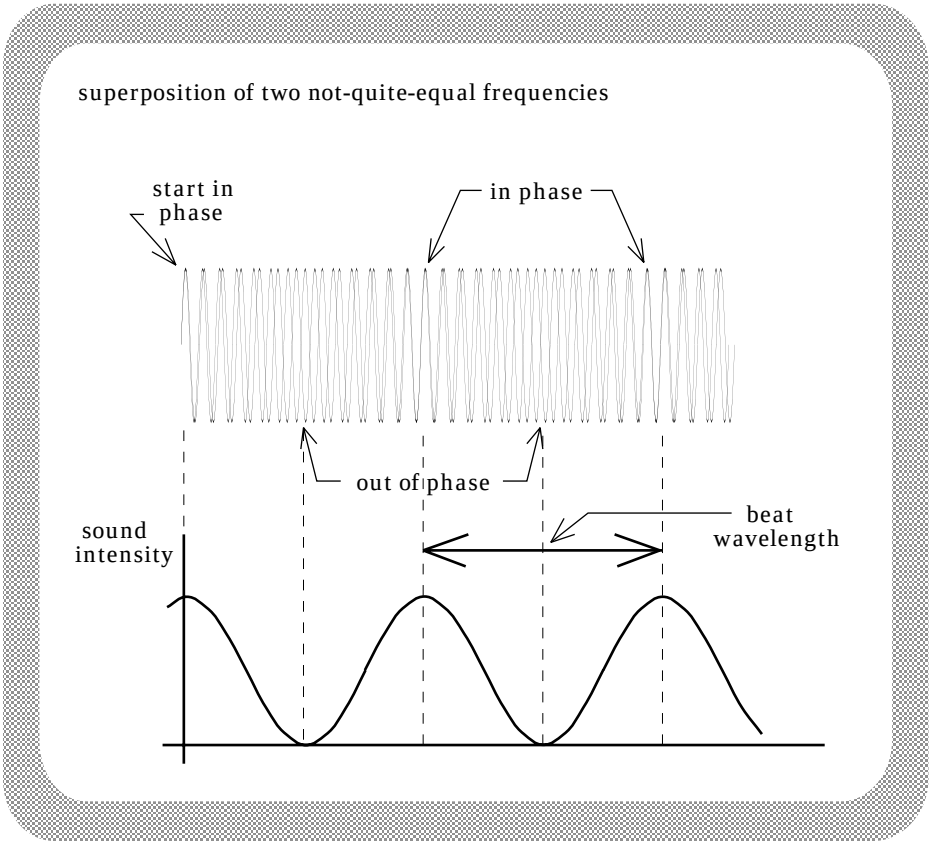


FIGURE 12.26

f.) Guitarists use beats in tuning their instruments. They do so by plucking two strings that, if tuned, will have the same frequency. If beats are heard, the strings are out of tune.

3.) The DOPPLER EFFECT:

a.) Have you ever had a train pass you while its whistle was being blown? The sound has a relatively high frequency as the train approaches, then abruptly drops in frequency after having gone by. This is the consequence of what is called *the Doppler Effect*.

b.) Consider a single sound source putting out sound of wavelength λ_1. At regular intervals, a sound crest (i.e., a high pressure ridge) is emitted from the sound source and moves out at approximately 330 m/s (i.e., the speed of sound

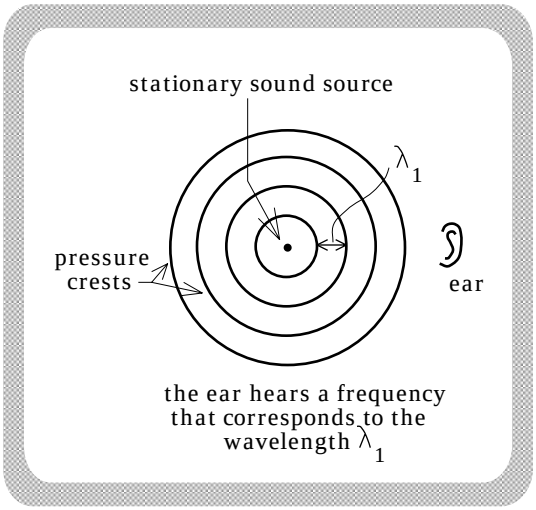


FIGURE 12.27

in air). Figure 12.27 on the next page shows a progression of such crests. Note that the listener in the sketch hears sound whose wavelength is λ_1 and whose frequency is $\nu_1 = v_{sound}/\lambda_1$.

c.) What happens when the sound source is moving toward the listener (or vice versa)? Figure 12.28 below shows the situation. Assume that at $t = 0$, the sound source puts out its first crest. After a time interval equal to the *period T* of the wave (remember, the period is the time required for one cycle to pass by . . . it is also equal to $1/\nu_{source}$), a second crest is emitted. After another period's worth of time, another crest is emitted, etc.

The distance between successive crests should be λ_1, but that is not what the listener perceives. Why? Because the sound source is moving toward the listener. As such, the distance between crests is smaller than λ_1.

d.) In fact, that wavelength will be the source's wavelength λ_1 minus the distance the source moved in time T (i.e., $v_{source}T$). Mathematically, this is:

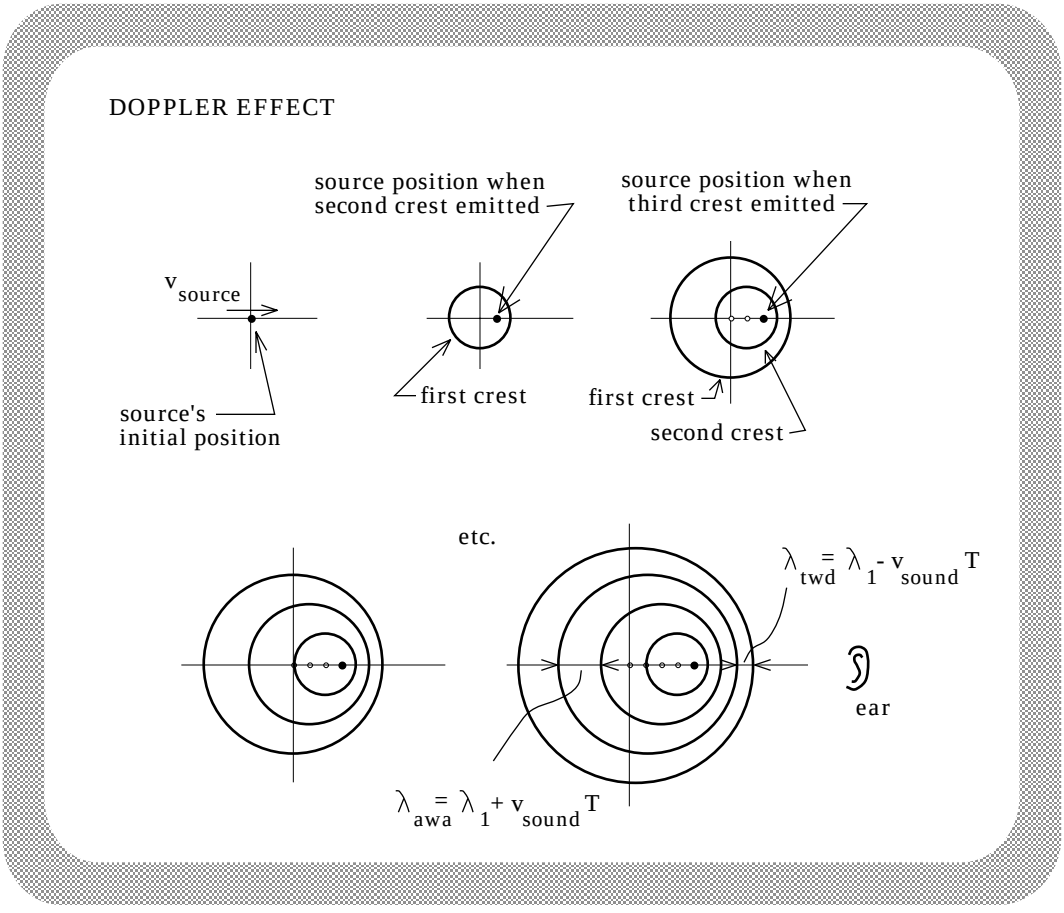


FIGURE 12.28

$$\lambda_{new} = \lambda_1 - v_{source}T \qquad \text{(Equ. A).}$$

e.) As $T = 1/\nu_{source}$, $\lambda_{new} = v_{sound}/\nu_{new}$, and $\lambda_1 = v_{sound}/\nu_1$, we can substitute everything into *Equation A* and come up with the expression:

$$\lambda_{new} = \lambda_1 - v_{source}T$$
$$v_{sound}/\nu_{new} = v_{sound}/\nu_{source} - v_{source}/\nu_{source}.$$

Rearranging yields:

$$\nu_{new} = v_{sound}\,\nu_{source}/[v_{sound} - v_{source}] \qquad \text{(Equ. B).}$$

Note 1: If the source had been moving *away* from the listener, the expression would have had a *positive* sign in place of the *negative* sign.

Note 2: Looking at the Doppler Shift sketches in Figure 12.28, what would have happened if the velocity of the sound source had been *greater* than the velocity of sound itself? The situation is pictured in Figure 12.29. In that case, a series of high pressure ridges would superimpose in such a way as to create a single, super-high intensity pressure ridge. This shock wave is what causes *sonic booms* when jets exceed Mach 1 (i.e., the speed of sound).

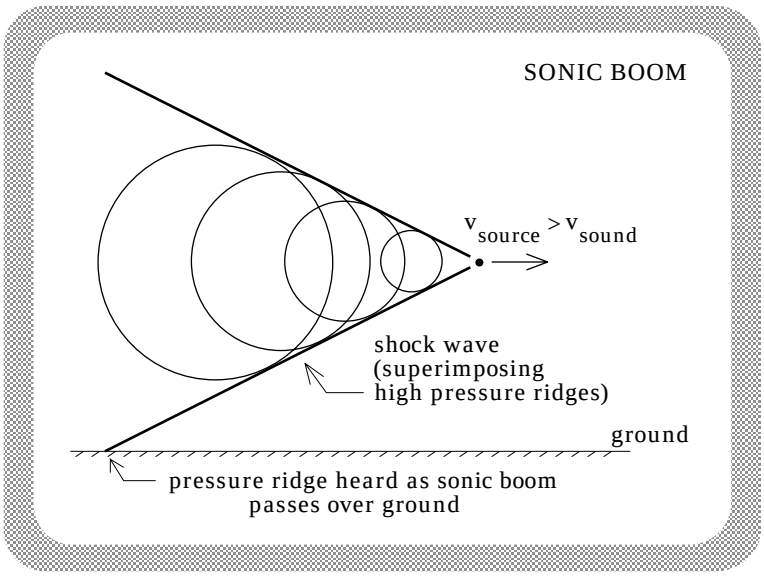


FIGURE 12.29

f.) Going back to the train whistle: When the train approaches, a listener will perceive crests coming in at a shorter wavelength than the actual wavelength of the whistle (again, see Figure 12.28). Shorter wavelengths correspond to higher frequencies, which means the sound heard by the listener will be of a *higher* frequency than the whistle's actual frequency. By the same token, after the train passes the listener will hear a longer wavelength (i.e., lower frequency) sound.

As such, the train whistle seems to drop from high to low frequency as the train transits from *approaching* to *retreating*.

g.) The Doppler Effect is particularly useful in astronomy. White light is the superposition of all of the frequencies of electromagnetic radiation to which our eyes are sensitive. A prism (or, for that matter, a diffraction grating) can spread these superimposed frequencies out. That is why white light passed through a prism yields the colors of the rainbow.

Stars give off white light, but gases in the star's atmosphere absorb out certain frequencies. That means that when star light is passed through a prism or diffraction grating, frequency gaps called spectral lines are observed.

One particular gas, hydrogen, is found in the atmosphere of all stars. Its absorption spectra (i.e., the series of spectral lines that are absorbed out when white light passes through hydrogen) is well known. What is interesting is that when we look for the pattern of hydrogen spectral lines in light from stars, we find the lines, but we find them shifted toward the red end of the spectrum. That is, their calculated frequencies are *lower* than they should be.

This is a Doppler shift caused by the motion of the star relative to Earth. The shift is associated with light waves instead of sound waves, but the principles are the same. From the observed shift, we can deduce two things. First, as all star light seems to be red-shifted, all stars must be moving away from us (a shift toward lower frequency is observed when a wave source recedes from an observer). Second, by measuring how large the red shift is, we can determine the speed of the star relative to the earth.

i.) Example: A particular spectral line in the hydrogen spectrum should have a frequency of $6\text{x}10^{14}$ Hz. When observed from a star, the line seems to have a frequency of $5\text{x}10^{14}$ Hz. How fast is the star receding from the earth?

Solution: Using our frequency expression (Equ. B) and substituting the speed of light in for the speed of sound, we can write:

$$\nu_{new} = v_{light}\ \nu_{source}\ /[\ v_{light} - v_{source}]$$

$$(6\text{x}10^{14}\text{ Hz}) = (3\text{x}10^{8}\text{ m/s})(5\text{x}10^{14}\text{ Hz})/[3\text{x}10^{8}\text{ m/s} - v_{source}].$$

Solving for the velocity of the source, we get $v_{source} = .5\text{x}10^{8}$ *m/s*.

QUESTIONS

12.1) On the assumption that one's hearing has not been ruined by loud music or contact with intense noise, good ears can hear sounds between 20 Hz and 20,000 Hz.

a.) What is the wavelength associated with a 20 Hz sound wave? How does this compare with everyday size?

b.) What is the wavelength associated with a 20,000 Hz sound wave? How does this compare with everyday size?

12.2) Three different waves are generated so as to coexist in the same medium as they move to the right. Figure I shows the outline of the three waves as they would appear if they were to exist alone.

a.) Make an approximate sketch of the wave that would actually exist in the medium--i.e., the superimposed wave (do it lightly in pencil directly on Figure I).

b.) All the waves are moving with the same speed but their *amplitudes* and *angular frequencies* are different. As can be seen, each wave is a sine wave which means each can be characterized using $x = A \sin \omega t$ (the *amplitudes* and *angular frequencies* are different but the general form of each wave's expression is the same). Using the information available on the graph to determine the appropriate

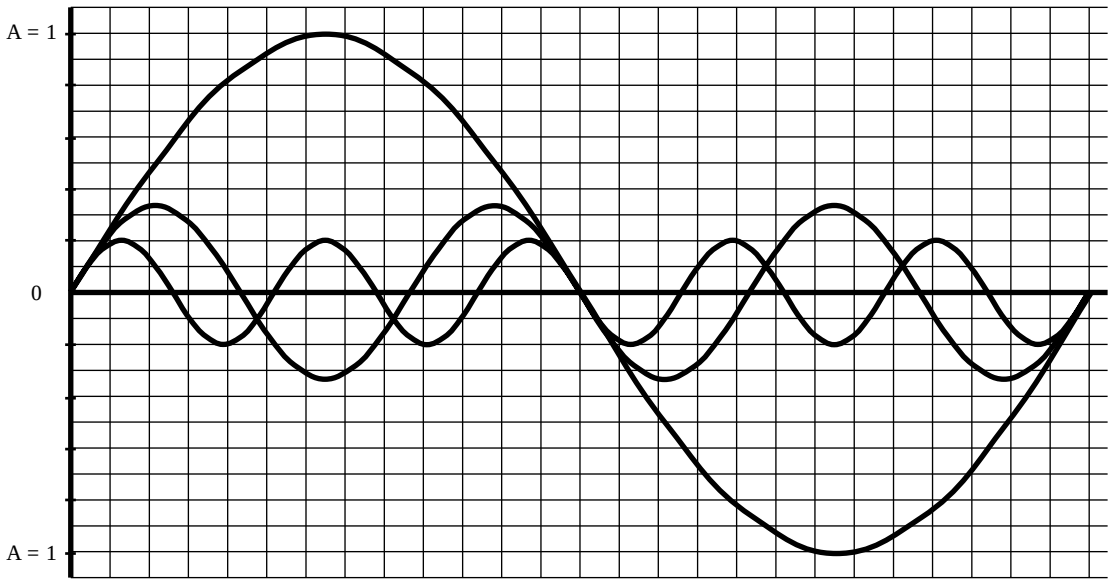


FIGURE I

parameters, write out the *algebraic expression* that characterizes the superposition of these three waves. That is: take the largest wave's *amplitude* and *angular frequency* and assume each is numerically equal to 1; determine the *amplitude* and *angular frequency* of the other two waves RELATIVE TO THE LARGEST (use the sketch and your head to get the required information); write the $x = A \sin \omega t$ expression for all three waves; then add them.

c.) In looking at the expression determined in *Part b*, you should see a pattern. Following the pattern, write out the first six terms of the series that seems to be emerging. (Note: If you look at the series from *Part b* in *fraction form*--that is, for instance, using 1/4 instead of .25--the pattern should become obvious.)

d.) By looking at the superimposed wave you drew in *Part a,* you should get a feel for the general wave form to which the series is converging. Draw a sketch of that wave form.

12.3) For the *traveling wave* characterized below:

$$y(x,t) = 12 \sin (25x - .67t).$$

a.) Sketch *y* as a function of *x* from *t = 0* to *t = 1 second*.

b.) Is this *traveling wave* moving to the right or the left?

c.) What can you conclude about the *negative sign* between the *x* term and the *t* term in the function? That is, in what direction would the wave have traveled if that sign had been *positive*?

d.) Determine the wave's *frequency*;

e.) Determine the wave's *period*;

f.) Determine the wave's *wavelength*;

g.) Determine the wave's *velocity*;

h.) Determine the wave's *amplitude*.

12.4) A *traveling wave* moving left has a *frequency* of 225 Hz, an *amplitude* of .7 meters, and a *wave velocity* of 140 m/s. Characterize the wave mathematically.

12.5) A damaged meterstick (length $L = .8$ *meters*) having a mass of $m = .4$ *kg* is pinned through one end and made to swing back and forth executing *small-angle* oscillations. Newton's Second Law for this situation (i.e., after a *small angle approximation* is made) yields the equation:

$$(2/3)\,\alpha + (g/L)\,\theta = 0.$$

a.) Determine the *resonant frequency* of the swinging meter stick.

b.) If small, periodic, tangential forces (i.e., a sequence of pushes designed to make it swing) are applied to the bottom of the stick over time, will its amplitude of motion become large or stay small if the *period* of the applied pushes is:

i.) 1.31 seconds/cycle (approximately);

ii.) 1.47 seconds/cycle (approximately)?

12.6) A violin string vibrating at 800 Hz has five nodes along its length of .3 meters (this includes two nodes at the ends). What is the *velocity* of the wave on the string?

12.7) Determine the *third lowest resonant frequency* for each of the systems shown in Figure II. Assume you know L and the wave velocity v.

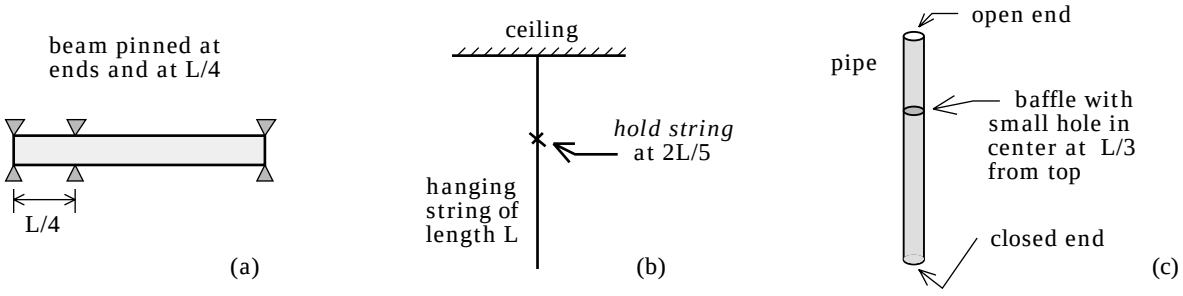


FIGURE II

Chapter Reviews

Prologue

THESE SUMMARIES SHOULD NOT BE USED AS YOUR PRIMARY LEARNING EXPERIENCE. THEY ARE FOR REVIEW ONLY.

Explanation: One of the things that you as a student will have to deal with before a final exam and/or AP test is the hassle of mentally dredging up all of the information you have dealt with over the course of the year. As a compilation of concepts and formulas, the following material is meant to help with that process. It should be noted, though, that simply knowing formulas will not effect success on such tests. You actually need to understand what each relationship does for you, what the variables inside it mean, when it IS and IS NOT applicable and, above all else, the conceptual consequences any given relationship brings. In other words, keep in mind when looking over the following material that this is a bare-bones REVIEW ONLY.

Chapter 1
Math Review Chapter Review

EQUATIONS of which to be aware: (No, knowing them won't guarantee success on your next test. You actually need to understand what each relationship *means*, when it is and is not *applicable*, and the conceptual consequences the relationship brings. In other words, as was said in the Prologue, this is a REVIEW, not a primary learning experience.)

- $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ [In *unit vector notation*, vector $\boldsymbol{A}$, denoted in bold face, is equal to the vector sum of its *x component* A_x times a unit vector $\boldsymbol{i}$ in the *+x direction*, plus its *y component* A_y times a unit vector $\boldsymbol{j}$ in the *+y direction*, plus its *z component* A_z times a unit vector $\boldsymbol{k}$ in the *+z direction*. An example is $\boldsymbol{A} = -3\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}$.]

- $|A| = (A_x^{\,2} + A_y^{\,2} + A_z^{\,2})^{1/2}$ [If a vector is defined in unit vector notation, the magnitude of the vector is equal to the square root of the sum of the squares of its components.]

- $\theta = tan^{-1}\left(\frac{A_y}{A_x} \right)$ [If a vector is defined in unit vector notation, the angle between the vector and the *+x axis* is equal to the inverse tangent of the ratio of the vector's *y component* and its *x component*.]

- $\boldsymbol{A} = |A| \angle \theta$ [In *polar notation*, vector $\boldsymbol{A}$ is equal to the magnitude of $\boldsymbol{A}$ at an angle θ.]

- $A_x = |A| cos\,\theta$ [If a vector is defined in polar notation, its *x component* can be determined by multiplying the vector's magnitude by the cosine of the angle θ, where θ is measured from the *+x axis* counterclockwise. IN SOME CASES, it is easier to generate your own right triangle and use whatever trig function is appropriate to determine that component, especially when you are dealing with vectors in the second and third quadrants.]

- $A_y = |A| sin\,\theta$ [If a vector is defined in polar notation, its *y component* can be determined by multiplying the vector's magnitude by the sine of the angle θ, where θ is measured from the *+x axis*. Again, in some cases it is easier to generate your own right triangle and use whatever trig function is appropriate to determine that component.]

- $\boldsymbol{A} \bullet \boldsymbol{B} = |A||B| cos\,\phi$ [If two vectors are defined in polar notation, the *dot product* between the two is defined as the product of the magnitudes of the vectors times the cosine of the angle between the vectors. If *the angle between the two vectors* isn't obvious (that is, if the two aren't drawn so as to produce a common vertex), the line of each vector must be extended until they cross. Extending each vector a little beyond that crossing exposes the appropriate angle.]

- $\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$ [If two vectors are defined in unit vector notation, the *dot product* between the two is defined as the sum of the products of the like components.]

- $|\boldsymbol{A}x\boldsymbol{B}| = |A||B|\sin\phi$ [If two vectors are defined in polar notation, the MAGNITUDE of the *cross product vector* is equal to the product of the magnitudes of the vectors times the sine of the angle between the vectors. The direction must be determined using the *right hand rule*.]

- $\boldsymbol{B}x\boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$ [If two vectors are defined in unit vector notation, the *cross product* AS A VECTOR is equal to the evaluation of the three by three matrix shown. Note that the second row is made up of the components of the *first* vector denoted in the cross product. Note also that you DON'T have to use the *right hand rule* to determine the direction as the direction is determined directly through the matrix evaluation.]

<u>COMMENTS, HINTS, and THINGS to be aware of</u>:

- The example given above for a vector written in **unit vector notation** was $\boldsymbol{A} = -3\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}$. This is a perfectly good notation, specifically because it allows one to see whether a particular component is in the positive or negative direction. *Technically*, though, the pluses and minuses are associated with the **direction** of the vector and should be attached to the unit vector itself. That is, a technical rendering of the vector should be $\boldsymbol{A} = 3(-\boldsymbol{i}) + 4\boldsymbol{j} + 6(-\boldsymbol{k})$. NOBODY WRITES VECTORS THIS WAY. Nevertheless, there will be times when knowing this picky little point will come in handy.

- When dealing with **polar notation**:
  --The ***magnitude*** (ALWAYS POSITIVE) is presented first.
  --The ***angle*** takes care of the sign. That is, if the vector is in the *-x direction*, you don't put a negative sign in front of the magnitude. Instead, you make the angle 180^o.
  --**When converting** from *unit vector notation* to *polar notation*, remember that your calculator will give you first and fourth quadrant angles ONLY. When you take the inverse tangent to determine an angle, make a little graph to the side to confirm which quadrant your vector should be in. If the vector turns out to be in the second or third quadrant, you will have to mess with the angle your calculator gave you to get the correct angle for the situation.

- When dealing with **unit vector notation**:
  --Don't be bashful about **creating your own right triangle**, then determining the vector components relative to the known angle in that triangle. IF YOU DO, remember that you will have to manually place negative signs into the component representations when appropriate.

- A **dot product** is a *scalar*. That is, you *can't* do a dot product between two vectors, then dot that result into a third vector. Also, dot products can be either positive or negative. The significance of the sign depends upon the situation in which the dot product is used. As an example, when we get to the energy chapter, the *work* done by a force acting on a body over a distance *d* is defined as $\boldsymbol{F}\cdot\boldsymbol{d}$. From that definition, *positive* work is associated with forces that put energy *into* a system whereas *negative* work is associated with forces that

take energy *out of* a system. In any case, what is important to understand is that because dot products aren't vectors, the sign (plus or minus) of a dot product is *not* associated with a direction.

- A **dot product** yields a number that is equivalent to the product of *the magnitude of one vector* and *the magnitude of the parallel component of the second vector*. What is *the parallel component of the second vector*? It is defined as the component whose direction is *along the line of the first vector*.

- A **cross product** is a *vector*. That is, you can do a cross product between two vectors, then cross that result into a third vector.

- The **magnitude** of a **cross product** yields a number that is equivalent to the product of *the magnitude of one vector* and *the magnitude of the perpendicular component of the second vector*. What is *the perpendicular component of the second vector*? It is defined as the component whose direction is *perpendicular to the line of the first vector*.

- The **direction** of a **cross product** is ALWAYS *perpendicular to the plane* defined by the two vectors being operated upon.
  --For vectors denoted in *polar notation*, the direction must be determined using *the right hand rule*. This is weird, but we teachers don't get paid much, and making our students do apparently undignified contortions in this fashion is one of our few sources of amusement.
  --For vectors denoted in *unit vector notation*, the direction will become apparent in the evaluation of the matrix.

- There are two ways to execute **the right hand rule**.
  --The first way requires that the length of your *right hand* be extended along the line of the *first* vector denoted in the cross product. With the thumb extended out perpendicular to that direction in an open handed hitch hiking pose, the hand must be rotated until it can *wave* in the direction of the second vector (once the wave happens, the hand will be left in a closed handed hitch hiking position). Upon completion of the maneuver, the direction of the thumb will denote the direction of the cross product (remember, that direction must be perpendicular to the two vectors involved in the cross product operation).
  --The second way simply requires that the right thumb be directed along the line of the *first* vector denoted in the cross product, and the length of the fingers of the right hand be directed along the *second* vector denoted in the cross product. Once done, the direction of the cross product will be the same as the direction of a line drawn perpendicularly outward from the palm.

Chapter 4
Kinematics Chapter Review

EQUATIONS:

- $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ [This is the formal presentation of a position vector in unit vector coordinates. It identifies the position of an object at a given instant. Example: $-3\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$, where *-3* is the *x component* of the object's position, *+4* is the *y component* of the object's position, and *-5* is the *z component* of the object's position. Note that the units of a position vector are *meters*.]

- $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$ [This is the formal presentation of a velocity vector given in unit vector notation. It identifies the rate of position change and the direction in which an object is traveling at a given instant. Example: $-3\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$, where *-3* is the *x component* of the rate of position change with time, *+4* is the *y component* of the rate of position change with time, and *-5* is the *z component* of the rate of position change with time. Note that the units of velocity are *meters per second*.]

- $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$ [This is the formal presentation of an acceleration vector, given in unit vector notation. It identifies the rate at which an object's velocity changes at a given instant. Example: $-3\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$, where *-3* is the *x component* of the rate of velocity change with time, *+4* is the *y component* of the rate of velocity change with time, and *-5* is the *z component* of the rate of velocity change with time. Note that the units of acceleration are *meters per second per second*.]

- $v_x = \frac{dx}{dt}$ [This is the relationship between the velocity in the *x direction* and the rate of position change in the *x direction*.]

- $v_x\,dt = dx$ [An extension of the expression just above, this is the relationship between a differential (read this *tiny, tiny*) position change *dx*, the instantaneous velocity in the *x direction* during the change, and the differential time interval *dt* over which the change occurs. Note that this *appears* to follow in an algebraic sense from $v_x = \frac{dx}{dt}$, though mathematicians curl up their toes and spout steam from their ears when physicists take such shortcuts. Note also that from $v_x\,dt = dx$, it follows that $\int_{t_1}^{t_2} v_x\,dt = \int_{x_1}^{x_2} dx$. For constant velocity, this becomes $v_x(t_2 - t_1) = (x_2 - x_1)$, or $x_2 = x_1 + v_x\Delta t$, or $v_x\Delta t = \Delta x$, depending upon how you want to present the evaluation.]

- $a_x = \frac{dv_x}{dt}$ [This is the relationship between the instantaneous acceleration in the *x direction* and the rate of velocity change in the *x direction*.]

- $a_x\,dt = dv_x$ [An extension of the expression just above, this is the relationship between a differential velocity change *dv*, the instantaneous acceleration in the *x* direction during the

change, and the differential time interval *dt* over which the change occurred. It also follows that $\int_{t_1}^{t_2} a_x dt = \int_{v_1}^{v_2} dv_x$ . . .]

- $a_x = \frac{d^2x}{dt^2}$ [This is the relationship between the instantaneous acceleration in the *x direction* and the *rate* of the rate of position change in the *x direction*.]

- $v_{avg} = \frac{v_1 + v_2}{2}$ [FOR A CONSTANT ACCELERATION SITUATION, this relates the *average* velocity over a given time interval Δt to the initial and final velocities associated with that interval. This expression is RARELY USED.]

- $v_{avg} = \frac{x_2 - x_1}{\Delta t}$ ***or*** $x_2 = x_1 + v_{avg}\Delta t$ [FOR A CONSTANT ACCELERATION SITUATION, this relates the *average* velocity over a given time interval Δt to the initial and final positions associated with that interval. This expression is RARELY USED.]

- $a_{avg} = a_{instantaneous} = a$ [FOR A CONSTANT ACCELERATION SITUATION, the instantaneous acceleration and the average acceleration are the same, often denoted simply by *a*.]

- $a = \frac{v_2 - v_1}{\Delta t}$ or $v_2 = v_1 + a\Delta t$ [FOR A CONSTANT ACCELERATION SITUATION, this relates the instantaneous acceleration and the velocity change $(v_2 - v_1)$ over a given time interval Δt. This expression is USED A LOT.)]

- $x_2 = x_1 + v_1(\Delta t) + \frac{1}{2}a(\Delta t)^2$ or $\Delta x = v_1 t + \frac{1}{2}at^2$ [FOR A CONSTANT ACCELERATION SITUATION, this relates the position change over a time interval Δt, the interval's initial velocity, and the acceleration. This expression is USED A LOT. Also, note the somewhat standard notation wherein the Δt terms are simply presented as *t* terms. This is sloppy but somewhat conventional. My apologies, but get used to it.]

- $v_2^2 = v_1^2 + 2a(x_2 - x_1)$ or $v_2^2 = v_1^2 + 2a\Delta x$ [FOR A CONSTANT ACCELERATION SITUATION, this relates the velocities at the extremes of an interval, the position change over that interval, and the system's acceleration. This expression is USED A LOT.]

<u>COMMENTS, HINTS, and THINGS to be aware of</u>:

- **Kinematics** is essentially idiot physics, not because any idiot can do it (in fact, very intelligent people have trouble with it), but because all that is needed to use it successfully is the ability to identify what is given in a problem, to identify what is asked for, and to pick the kinematic equation that incorporates both what is known and what is required. Yes, there may be problems in which you have to solve *two* kinematic equations

simultaneously. But on the whole, your ability or inability to use kinematics intelligently will have absolutely no bearing on how well you will do when we begin to look at more formal, advanced approaches to physics.

- The **direction of a velocity vector** at a given instant is the same as the direction of the body's motion at that instant.

- The *net force* on a body and the body's *acceleration* go hand in hand. That is, if the net force on a body is in the negative direction, the acceleration will be negative. With that in mind, consider a body moving in one dimensional motion. If the net force (and, by extension, acceleration) is in the positive direction while the body's velocity is in the negative direction, what will the body do? It will slow down! And if the net force (and, by extension, acceleration) is in the negative direction and the body's velocity is still in the negative direction, what will the body do? It will speed up! So what does the **sign of the acceleration** tell you? Nothing, unless you additionally know the sign of the velocity. If the two are the same (both positive or both negative), the body will speed up. If the two are different, the body will slow down.

- It is not unusual to see the symbol ***t*** in kinematic equations. This stands for a TIME INTERVAL. It should be written as Δt. However, convention and sloppiness have prevailed and the symbol is usually shortened to t.

- You should be able to take a ***velocity versus time*** and corresponding ***position versus time*** **graph** and pick off all of the information needed to determine, say, the system's acceleration. You should be able to do this TWO or THREE DIFFERENT WAYS using the kinematic relationships as a guide (that is, using $v_2^2 = v_1^2 + 2a\Delta x$, or $x_2 = x_1 + v_1 t + .5at^2$, or whatever). In other words, you need to understand what the variables in the kinematic relationships REALLY MEAN.

- You should be able to make sense out of a ***velocity versus time*** **graph**. That is, you should know that the area under the graph over a particular time interval equals the distance traveled during that interval, and that the slope of the tangent to the curve, evaluated at a particular instant, equals the instantaneous acceleration at that point in time. Both of these observations are true whether the acceleration is constant or not, but when used in conjunction with a constant acceleration, the kinematic equations follow naturally.

- You should be able to make sense out of a ***position versus time*** **graph**. That is, you should know that the slope of the tangent to the curve, evaluated at a particular instant, equals the instantaneous velocity at that point in time.

- You should be able to make sense out of an ***acceleration versus time*** **graph**. That is, you should know that the area under the graph over a particular time interval equals the velocity change Δv during that interval.

Chapter 5
Newton's Laws Chapter Review

EQUATIONS:

- $f_k = \mu_k N$ [This is the *kinetic friction* expression, where μ_k is the coefficient of kinetic friction and N is the normal force applied to the body in question. Note that there is only one kinetic frictional force between two surfaces. It is *not* a velocity dependent quantity, assuming you don't change the characteristics of the surfaces through, say, heating during the slide.]

- $f_s = \mu_s N$ [This expression, which is associated with *static friction*, looks very much like the expression for kinetic friction. In it, μ_s is the coefficient of static friction and N is the normal force applied to the body in question. What makes the two expressions different is that f_s is defined as the MAXIMUM possible static friction force between two surfaces (whereas there is only *one* possible kinetic friction force). Also, f_s will always be larger than f_k.]

- $F_{net,x} = ma_x$ [This is Newton's Second Law, where $F_{net,x}$ is the net force on a body in the x *direction*, m is the body's mass, and a_x is the body's acceleration in the *x direction*.]

- $a_c = v^2/R$ [This is the centripetal acceleration expression, where v is the magnitude of the body's velocity and R is the radius of the circle upon which the body moves. The center seeking direction is ALWAYS perpendicular to the motion and always toward the CENTER of the path upon which the body moves.]

- $a = dv/dt$ [This is the acceleration relationship needed whenever a time-dependent velocity is requested. Remember that in problems like this, the acceleration direction (assumed to be along the line of motion) will be positive or negative *depending upon the sign of the velocity vector*. If this isn't clear, review the section associated with this kind of problem.]

- $F_{grav} = mg$ [This is the gravitational force exerted by the earth on mass m (it is also called *the mass's weight*). The constant g is the magnitude of the acceleration of gravity close to the surface of the earth, or 9.8 m/s^2 in the MKS system.]

COMMENTS, HINTS, and THINGS to be aware of:

- **Newton's three laws** are:
  --1.) In an *inertial frame of reference*, bodies in motion tend to stay in motion with a constant velocity, and bodies at rest tend to stay at rest, unless impinged upon by an external force.
  --2.) The sum of the forces acting in a particular direction on a body will equal the product of the body's mass and its acceleration in that direction. As a vector, this is written as $\boldsymbol{F} = m\boldsymbol{a}$.

--3.) For every action (i.e., for every *force*), there must be an equal and opposite reaction somewhere in the universe.

- The **five kinds of forces**:
  --Normal force: Forces that are provided by some kind of support. They are always perpendicular to the surface providing the support, and are usually symbolized by an *N*.
  --Frictional force: Kinetic friction is always opposite the direction of motion. Static friction is always opposite the direction the body *would* move should it break loose.
  --Gravitational force: Near the earth's surface, this is always down and equal to *mg*, where *m* is the body's mass and *g* is defined as the MAGNITUDE of the acceleration of gravity near the earth's surface. Note that the magnitude of this force is always symbolized by the product of *m* and *g* in free body diagrams, and the direction is denoted by a downward arrow.
  --Tension force: Always directed *away from* the body to which the rope or string is attached, tension forces are usually symbolized by a *T*, often subscripted.
  --Push-me pull-you force: Any force that doesn't fit into the other four categories. They are usually characterized by an *F*, often subscripted.

- A **free body diagram** is a sketch designed to allow you to identify all of the forces acting on a single body. Force directions are depicted with appropriately oriented arrows (gravity, for instance, is characterized by an arrow pointing down), and force magnitudes are denoted algebraically next to the arrow (i.e., gravity is *mg*, tension is *T*, etc.). When doing an f.b.d., mentally run through the *kinds* of forces that could exist within the system (i.e., is there gravity?--if so, put it in, etc.). Forget one force and you're dead.

- As is the case with all approaches in physics, Newton's Second Law is designed to allow you to write down mathematical relationships that are *true*. **Newton's Second Law** says that for a single body, if you pick ANY direction (*x*, *y*, or *z*) and sum up all of the positive and negative forces that act on the body along that line, the sum will equal the product of the body's mass *m* and its acceleration *a* along that line.

- **The Approach** for using N.S.L. is:
  --Look at the problem and think to yourself, *I couldn't possibly do this*. Then, start *The Approach*.
  --Pick ONE body in the system and draw a free body diagram for it. Make sure you don't leave any forces out, and be sure you accurately show the forces that exist (i.e., a *normal force* should be directed *perpendicularly out* relative to the support that provides it--DON'T draw an arrow that makes it look like it has a component that is *parallel* to the surface).
  --Through observation, identify the body's *line of acceleration*. Place one coordinate axis along that line. Place a second axis perpendicular to that line. For both axes, it is *your choice* which direction you will define as *positive* (axes *vertically up* and *horizontally to the right* are usually defined as positive, though they certainly don't have to be).
  --Break all forces that are not along either of the axes into their components along each axis.
  --Being careful about signs, sum the forces directed along one axis. Make that sum equal to the product of the body's mass and acceleration (i.e., ma_x), subscripting the acceleration term to match the axis (do this only if needed for clarity).
  --There are instances in which dealing with an acceleration *magnitude* is clearly easier than dealing with an acceleration term that has a possible negative sign embedded within it. If you prefer to work with a magnitude instead of an acceleration vector (embedded sign and all), you can make the acceleration into a magnitude by unembedding the sign. That is, if

you think that the body is accelerating in the negative direction, relative to the positive axis direction as you've defined it, make the right side of N.S.L. equal to **-***ma*.
--If you can, use the resulting expression to solve for the unknown you are seeking. If not, and you find more unknowns in the relationship than you have equations, either repeat the process along the other coordinate axis or, if that doesn't help, pick another body in the system and start the whole process over again.
--By the time you are done, you should have enough equations to eliminate all of the unwanted unknowns and solve for the desired quantity.

- N.S.L. and **center seeking (centripetal) forces and acceleration**:
  --According to Newton's First Law, bodies in motion continue to move in a straight line unless impinged upon by a force. Forces that change the <u>*direction*</u> of a body's velocity vector (versus changing the *magnitude* of the body's velocity vector) are called *center seeking forces*, or *centripetal forces*. A centripetal force is NOT a new *kind* of force. The phrase *centripetal force* is a label that is used to identify any naturally occurring force in a system that changes the *direction* of a body's motion.

- Centripetal forces produce a centripetal acceleration. The **centripetal acceleration** a body experiences as it moves in a circular path is related to two quantities: the radius R of the body's path and the magnitude of the body's velocity v. The actual expression is $a_c = v^2/R$.

- One of the steps in the N.S.L. approach is to determine the **directions of the coordinate axes**. If you will remember, one axis is always directed along the line of the acceleration (this is important because the sum of the forces along that direction will equal *ma*, where *a* is often the parameter you are trying to determine). When a body is being pushed or pulled out of straight line motion, there must exist a single force, the component of a single force, or, possibly, a combination of forces and components, that act centripetally. The net **centripetal acceleration** generated by those forces will **ALWAYS** be **aimed** toward *the center of the circle* upon which the body is being forced to travel, and the magnitude will always be v^2/R.

- One of the hardest things for students to do is to **determine *when*** it is appropriate **to use v^2/R** (i.e., to identify when a centripetal situation exists). The best way to tell is to *look to see if the body is following a curved path* (i.e., is the body's direction changing?). If it is, there must be a naturally occurring force (gravity, tension, a component of a normal force, whatever) that is directed *toward the center of the circle* upon which the body is moving. Pick that direction for one axis, then use the N.S.L. approach as outlined above.

- A corollary to the above observations is the following: The most confusing characteristic of centripetal situations is that **centripetal acceleration** is always **PERPENDICULAR** to the **line of the motion** (i.e., to the direction of the velocity vector). Students become comfortable with inclined plane problems in which the acceleration in the *normal* direction (i.e., perpendicular to the direction of motion) is equal to *zero*. If you get in the habit of mindlessly zeroing the acceleration variable directed perpendicular to the motion, you will be lost when you run into a centripetal force problem. You need to LOOK AT THE PROBLEM, then decide what should and shouldn't be equal to zero.

- One bit of nastiness that sometimes arises is found in situations in which you are asked to determine the **velocity** of a body's motion **as a function of time** knowing nothing more

than the forces acting on the body. *The Approach* works just fine with this kind of problem, but there is one major difference in how you proceed. The acceleration, assumed to be along the line of motion, must be written as dv/dt. ALSO, the *sign* of that dv/dt term should *not* be determined by assuming an acceleration direction, then unembedding the acceleration sign as is done in, say, an inclined plane problem. The sign that goes in front of the dv/dt term must be determined by taking the derivative of the velocity vector as written in the form $\boldsymbol{v} = v(\pm\boldsymbol{i})$. Although you will most probably drop the unit vector, the appropriate sign will fall out with the derivative.

Chapter 6
Energy Chapter Review

EQUATIONS:

- $W_F = \boldsymbol{F} \cdot \boldsymbol{d}$ [This is the amount of work a force $\boldsymbol{F}$ does on a body as the body moves over a distance $\boldsymbol{d}$. This definition is applicable as long as both the force and the angle between the force and displacement are constant.]

- $W_F = \int \boldsymbol{F} \bullet d\boldsymbol{r}$ [This is the sum of the differential bits of work force $\boldsymbol{F}$ does on a body moving between two points over successive, differential path lengths each of which are characterized by $d\boldsymbol{r}$. This definition is always applicable, but is usually used only when the force is variable, the displacement is along an oddball path, and/or the angle between the force and displacement varies.]

- $d\boldsymbol{r}$ [This is a differential displacement in Cartesian coordinates, or $d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit vectors. In polar notation, the radial coordinate is usually the only one of interest, so the relationship is written as $d\boldsymbol{r} = dr\boldsymbol{r}$, where $\boldsymbol{r}$ is a unit vector in the radial direction. There is an additional note at the end of this summary about radial coordinates.]

- $KE = (1/2)mv^2$ [This is the amount of kinetic energy an object of mass m has while moving with velocity magnitude v.]

- $W_{net} = \Delta KE$ [This is the Work/Energy theorem. It states that the *total work* done on an object is always equal to the change of the object's kinetic energy.]

- $W_{\boldsymbol{F}} = -\Delta U$ [The amount of work a *conservative force field* does on an object moving from *point 1* to *point 2* in the field is equal to the change of the potential energy function associated with the force field, as evaluated at *points 1* and *2*.]

- $U(x) - U(\text{where force is zero}) = -\int_{\text{where force is zero}}^{x} \boldsymbol{F} \bullet d\boldsymbol{r}$ [This is the relationship between a force and its potential energy function. The evaluation of this integral is the way potential energy functions are derived.]

- $\boldsymbol{F} = -\nabla U$ [This is the relationship between a potential energy function and the force that generates it. The evaluation of the *del operator* is the way a force function is derived from its potential energy function.]

- $U_{\text{grav. near earth}} = mgy$ [This is the gravitational potential energy function near the earth's surface, where y is the distance above an arbitrarily set *zero potential energy level*. Note that this is arbitrary because there is no place on or near the earth's surface where the gravitational force is zero, hence no preferred spot for the potential energy to be zero.]

- $U_{grav.\ anywhere} = -G\frac{m_1 m_2}{r}$ [This is Newton's most general gravitational potential energy function. It is especially useful when far from the earth's surface (i.e., where the gravitational force is no longer equal to *mg*, but rather the more precise $G\frac{m_1 m_2}{r^2}(-\boldsymbol{r})$). Note that *U* is zero at *infinity*, which means that the potential energy function is *derived* assuming the zero potential energy position is at infinity. The kinds of gravitational situations in which this force and potential energy function would be useful will be dealt with later in the course. They have been introduced here primarily because they are force and potential energy functions that aren't equal to zero at the coordinate origin.]

- $\boldsymbol{F}_{spring} = -kx\boldsymbol{i}$ [This is the force function for a spring, where *k* is a spring constant and $\boldsymbol{F}_{spring}$ equals zero at $x = 0$ where the spring is at equilibrium.]

- $U_{spring} = (1/2)kx^2$ [This is the potential energy function for a spring, where the zero potential energy position is at equilibrium at $x = 0$.]

- $\Sigma KE_1 + \Sigma U_1 + \Sigma W_{extraneous} = \Sigma KE_2 + \Sigma U_2$ [This is the modified conservation of energy theorem, where the sum of KE_1 and U_1 is called the total mechanical energy at *time 1*, the sum of KE_2 and U_2 is the total mechanical energy existing within the system at *time 2*, and $W_{extraneous}$ is the total work done BETWEEN *times 1 and 2* by any force whose work hasn't been taken into account by the potential energy functions (remember, potential energy functions are useful only in that they allow you to determine how much work a conservative force field does as a body moves from one point to another in the field).]

- $P = W/t$ [This is the definition of *power*, being the amount of work *per unit time* that a force provides to a system. Its units in the MKS system are joules/second, or watts.]

- $W_{normal} = 0$ [The amount of work a *normal force* does is always zero because the normal direction is always perpendicular to the line of motion.]

- $W_{friction} = -fD$ [As frictional forces are always directed opposite the direction of motion, the work friction does is always negative and always equals the product of the frictional force and the TOTAL DISTANCE *D* over which the force is applied.]

COMMENTS, HINTS, and THINGS to be aware of:

- **Work**--what is it?
  --Newton observed that a net force acting on a body over a given distance would change the body's motion. The two quantities that affect the magnitude of the change are *the magnitude of the force component in the direction of motion* and *the distance over which the force acts*. Multiplying those two quantities yields a number that gives us a relative feel for how much motion change we can expect. Called *work*, it is this defined quantity from which the entire theory of *energy* begins.

- **Energy**--what is it?
  --Objects are said to have energy if they have the ability to do work, either due to the fact that they are moving (quantified by *kinetic energy*--the energy of motion), or due to the fact that their position in a force field will allow them to pick up motion at some time in the future (quantified by *potential energy*--the energy of position--this *potential to pick up kinetic energy* function can be defined only for so-called *conservative force fields*).

- **Conservative forces** are forces that do work that is path independent. That is, the amount of work a conservative force does on a body moving from one point to another in the field is determined solely by where the endpoints are, not by how far or in what direction(s) the body traveled in traversing between the points.

- If you are **given** a **potential energy function** and asked to **derive** its **force function**, don't let the derivation scare you. The partial derivative in a **del operator** is nothing more than a derivative taken with respect to *one variable* at a time.

- If you are **given** a **force function** and asked to **derive** its **potential energy function**, don't let the derivation scare you. **The integral** simply determines how much *work* the force field does as a body moves from the *zero potential energy point* (this will normally be where the force is zero) to an arbitrary point located somewhere out from the origin.

- The **energy approach** is a book keeping technique. What you are doing is keeping track of the amount of energy of motion (kinetic energy) and energy of position (potential energy) that a body or system of bodies has at various points in time.

- **Conserved quantities** are quantities that *do not change with time*. If the mechanical energy of a system is said to be *conserved*, that means that the sum of the system's kinetic and potential energies at any given instant will always add to the same number.

- Think about what the ***conservation of energy theorem*** says in its strictest form. It says: Determine the total mechanical energy in a system (i.e., the sum of the kinetic and potential energies of all the bodies in the system) as evaluated *at a particular point in time*. Do the same at a second point in time. The individual kinetic and potential energy quantities may change over the time interval, but those two SUMS will be the same UNLESS there is work being done by forces that you can't take into account using potential energy functions (i.e., forces that are non-conservative--friction, for instance--or forces that are conservative but for which you don't have potential energy functions). If there are no extraneous forces doing work, $\sum KE_1 + \sum U_1 = \sum KE_2 + \sum U_2$ and the *total mechanical energy* within the system is said to be *conserved*.

- If **non-conservative forces** exist within a system, or if there exist forces that are conservative but for which you don't have a potential energy function, you can still use the ***modified conservation of energy*** **expression**. It says: The total mechanical energy in a system evaluated at a particular point in time will be different than the total mechanical energy evaluated at a later point in time. The variation will be due to work done by extraneous forces for which you don't have a potential energy function. You must either use $\boldsymbol{F \cdot d}$ or $\int F \bullet \boldsymbol{d}r$ for each of those forces to determine that extraneous work. This is denoted in the

modified conservation of energy theorem as $W_{extraneous}$. Symbolically, this can be written as $\sum KE_1 + \sum U_1 + \sum W_{extraneous} = \sum KE_2 + \sum U_2$.

- **Gravitational potential energy** *near the earth's surface* can be made zero anywhere only because the gravitational force near the earth's surface is a constant--it isn't *zero* anywhere. If there had been a preferred *zero force position*, that would be where U_{grav} would equal zero.

- A MINOR POINT about coordinate systems that will become important later: In a Cartesian system, $-\boldsymbol{i}$ is a unit vector that is always directed *to the left* in the negative *x direction*. In a **polar spherical coordinate system**, on the other hand, a *radial* unit vector $\boldsymbol{r}$ always proceeds *from the origin out to the point of interest*. That means that a **negative radial vector**, characterized as $-\boldsymbol{r}$, will always be directed *inward* toward the origin. This kind of notation is very useful when dealing with, say, the theoretical expression for the gravitational force produced by the earth. In that case, as gravity is always directed toward the earth's center no matter where you are on the earth, the gravitational force is conveniently characterized theoretically as being oriented in the negative radial (i.e., $-\boldsymbol{r}$) direction.

Chapter 7
Momentum Chapter Review

EQUATIONS:

- $x_{cm} = \frac{\sum_{i=1}^{n} (m_i x_i)}{M}$ [This is the definition of the *x coordinate* of the center of mass of a system of *n* discrete bodies (i.e., point masses). When done for all three axes, the center of mass yields a kind of weighted *average position* for the system, all relative to the coordinate grid being used.]

- *dV* [A differential volume, *dV's* size depends upon the geometry you are considering. For instance, if it is a differentially thin cylindrical shell, *dV* will equal the circumference $2\pi r$ of the shell times the length *L* of the shell times the differential thickness *dr* of the shell, or $dV = (2\pi rL)dr$. Other geometries require different expressions. Note that in all cases, there *has* to be a differential length like *dr* or *dx* in the expression.]

- $\rho = \frac{mass}{unit\ volume} = \frac{dm}{dV}$ [Called a *volume density function*, ρ (rho) is a RATIO that quantifies the amount of *mass per unit volume* associated with a distribution of mass. Differentially, $\rho = \frac{dm}{dV}$ with *dm* being the differential mass associated with the differential volume *dV*. The expression is useful because it allows you to write *dm* in terms of ρ and *dV*, or $dm = \rho dV$. If the structure is homogeneous (i.e., the mass is uniformly distributed throughout), ρ also equals the total mass in the structure divided by the total volume, or *M*/*V*. If, on the other hand, the structure is inhomogeneous, *M*/*V* is nonsense and a function for ρ must be provided (i.e., something like $\rho = kr$, where *k* is a constant and *r* is a variable that defines the distance between *dm* and an axis of interest). In any case, the $dm = \rho dV$ expression is ALWAYS true.]

- *dA* [A differential area, its size depends upon the geometry you are considering. For instance, if it is a flat, differentially thin hoop, *dA* will equal the circumference $2\pi r$ of the hoop times the hoops differential thickness *dr*, or $dA = (2\pi r)dr$. Other geometries require different expressions. Note that in all cases, there *has* to be a differential length like *dr* or *dx* in the expression.]

- $\sigma = \frac{mass}{unit\ area} = \frac{dm}{dA}$ [Called an *area density function*, σ (sigma) is a RATIO that quantifies the amount of *mass per unit area* there is behind a given area on the surface of a body. Differentially, $\sigma = \frac{dm}{dA}$ with *dm* being the differential mass behind the differential surface area *dA*. This function is used whenever there is mass variation in two dimensions, or in uniformly distributed three-dimensional situations that just *look* easier to do in two dimensions--a rectangular solid is a good example. The expression is useful because it allows you to write *dm* in terms of σ and *dA*, or $dm = \sigma dA$. If the structure is homogeneous, σ is equal to the total mass within the structure divided by the total surface

the x direction at *time 2*, and $\sum F_{x,external}\Delta t$ is the net impulse provided by all of the EXTERNAL FORCES acting on the system in the x direction between *times 1* and *2*.]

COMMENTS, HINTS, and THINGS to think about:

- **Momentum**--what is it?
  --If you will remember, **inertial mass** was defined as a *relative measure* of a body's resistance to changing its motion. In other words, if you are told that a certain object has, say, 3 kilograms of mass associated with it, you know that the weight has *three times the resistance to changing its motion than does the standard in Paris*. Because you have had experience with objects measured in kilograms, that has meaning for you.
  --**Momentum** is a lot like inertial mass in the sense that it, too, is **a relative measure of a quality associated with massive objects**. It gives you a relative feel for the force needed to stop the object in a given amount of time. Newton realized that the two parameters that governed the difficulty or ease of changing a body's motion are mass and velocity. By taking the two relevant parameters and multiplying them together, he came up with a quantity that allowed him to quantify this characteristic of matter. That quantity, mv, he called *momentum*.
  --In short, a body with a lot of momentum will be hard to stop in a short period of time. A body with little momentum will be easier to stop in a short period of time.

- If you are dealing with a SINGLE MASS, the **Impulse Equation** relates the mass's *change of momentum* and the *total external impulse* ($F_{net,external}\Delta t$) applied to the body over the allotted time interval Δt.

- For a collision, the **area under** a ***force versus time*** **graph** yields $F\Delta t$--the impulse applied during the collision. This is useful as it is related to the body's *change of momentum*.

- **External forces** are forces that impinge upon pieces of a system but are not due to the interaction of those pieces with other pieces of the system. Gravity, for instance, is normally an external force. The force a wall applies to a ball when the ball is thrown against it is an external force UNLESS the wall is free to move after the collision (i.e., it is on rollers) and the initial and final momentum of the wall can be calculated. In that case, the wall can be considered *part of the system*, the force considered internal, and the total momentum of the ball and wall conserved. If the wall is secured to the ground so that it isn't free to move, it will provide an external force to the ball and the ball's momentum will *not* be conserved through the collision.

- Don't get **internal forces** and **conservative forces** confused. For *energy to be truly conserved*, the work done by each force acting on a system must have a potential energy function associated with it. That can happen only if the work done by each force is path independent. As *path independence* is the defining characteristic of the work done by a conservative force, *conservation of energy* in its strictest form can only happen if the forces in the system are conservative. For *momentum to be truly conserved*, each of the forces within a system must be due to the interaction of the pieces of the system. The forces must be *internal* to the system. What all of this means is that although a force like friction will never be conservative (hence, energy will never truly be conserved with friction present), it could exist as a consequence of the interaction of the pieces within the system (hence, could exist within a system in which *momentum* was conserved).

Chapter 8
Rotational Motion I Chapter Review

EQUATIONS:

- $v = r\omega$ [This is the relationship between the magnitude of the velocity v of a point moving in a circular path of radius r and the magnitude of the motion's angular velocity ω about the center of motion. Note that the center must be a *fixed point* in the system.]

- $a = r\alpha$ [This is the relationship between the magnitude of the acceleration a of a point moving in a circular path of radius r and the magnitude of the motion's angular acceleration α about the center of motion. Note that the center must be a *fixed point* in the system.]

- $\omega = -5\boldsymbol{i}$ [This is an angular velocity vector, where *5* is the magnitude, $\boldsymbol{i}$ is the direction of the *axis about which the rotation occurs*, and the negative sign signifies that the rotation is *clockwise* as viewed from the $+\boldsymbol{i}$ side of the coordinate axis.]

- $\alpha_{instantaneous} = \alpha_{average} = \alpha$ [If the angular acceleration is CONSTANT, the instantaneous angular acceleration and the average angular acceleration are equal. In such cases, both angular acceleration terms are characterized with an α.]

- $\alpha = \frac{\Delta\omega}{\Delta t}$, *or* $\omega_2 = \omega_1 + \alpha t$ [This rotational kinematic equation comes from the fact that the instantaneous angular acceleration α and the average angular acceleration $\frac{\Delta\omega}{\Delta t}$ over any interval will be the same, given the *angular acceleration* is *constant*. REMEMBER whenever you see a *time variable*, whether presented as t or Δt, you are *always* dealing with a TIME INTERVAL.]

- $\Delta\theta = \omega_1 t + \frac{1}{2}\alpha t^2$ *or* $\theta_2 = \theta_1 + \omega_1 t + \frac{1}{2}\alpha t^2$ [This rotational kinematic equation relates angular displacement $\Delta\theta$ (i.e., the change of angular position between *time 1* and *time 2*), the initial angular velocity ω_1 (i.e., the angular velocity at *time 1*), the angular acceleration α, and the time interval t.]

- $\omega_2^{\,2} = \omega_1^{\,2} + 2\alpha(\theta_2 - \theta_1)$ [This rotational kinematic equation relates angular velocities at two different points in time to the angular acceleration α and the angular displacement $\Delta\theta$ during that interval.]

- $\omega_{avg} = \frac{\omega_2 + \omega_1}{2}$ [This rotational kinematic equation is RARELY USED. It relates the average angular velocity ω_{avg} over an interval to the initial and final angular velocities--ω_1 and ω_2--associated with that interval.]

- $\Delta\theta = \omega_{avg}\Delta t$ [This rotational kinematic equation is RARELY USED. It relates the average angular velocity ω_{avg} over time interval Δt to the angular displacement $\Delta\theta$ during that time interval.]

- $I = \sum_{i=1}^{n} m_i r_i^2$ [This is the definition of the moment of inertia of n discrete pieces of mass about a given axis, where m_i is the i^{th} mass in the system and r_i is the perpendicular distance (i.e., the shortest distance) between that mass and the axis in question.]

- $I = mr^2$ [This is the moment of inertia of a point mass m about an axis, where r is the perpendicular distance from the mass to the axis in question.]

- $I = \int r^2 dm$ [This expression is used to derive the moment of inertia of a continuous, solid object about some axis, where r is the perpendicular distance from the mass dm to the axis in question.]

- $\lambda = \frac{mass}{unit\ length} = \frac{dm}{dl}$ [Called a *linear density function*, λ (lambda) is a RATIO that quantifies the amount of *mass per unit length* there is along some linear structure. Differentially, $\lambda = \frac{dm}{dl}$ with dm being the differential mass involved in a differential length dl. The expression is useful because it allows you to express dm in terms of λ and dl, or $dm = \lambda dl$. This function is used whenever there is mass variation in one dimension only, specifically for rod-like structures. If the structure is homogeneous (i.e., the mass is uniformly distributed throughout), λ also equals the total mass in the structure divided by the total length of the structure, or M/L. If the structure is inhomogeneous, M/L is nonsense and a function must be provided for λ (i.e., something like $\lambda = kx$, where k is a constant and x is a variable that defines the distance between dm and an axis of interest). In any case, the $dm = \lambda dl$ relationship is ALWAYS true.]

- $\sigma = \frac{mass}{unit\ area} = \frac{dm}{dA}$ [Called an *area density function*, σ (sigma) is a RATIO that quantifies the amount of *mass per unit area* there is behind a given area on the surface of a body. Differentially, $\sigma = \frac{dm}{dA}$ with dm being the differential mass behind the differential surface area dA. This function is used whenever there is mass variation in two dimensions, or in uniformly distributed three-dimensional situations that just *look* easier to do in two dimensions--a rectangular solid is a good example. The expression is useful because it allows you to write dm in terms σ and dA, or $dm = \sigma dA$. If the structure is homogeneous, σ is equal to the total mass within the structure divided by the total surface area of the structure, or M/A. If the structure is inhomogeneous, M/A is nonsense and a function is required to tell you how the density acts from point to point (something like $\sigma = kr$, where k is a constant and r is a distance variable that makes sense relative to the coordinate system). In any case, the $dm = \sigma dA$ expression is ALWAYS true.]

- $\rho = \frac{mass}{unit\ volume} = \frac{dm}{dV}$ [Called a *volume density function*, ρ (rho) is a RATIO that quantifies the amount of *mass per unit volume* associated with a distribution of mass. Differentially, $\rho = \frac{dm}{dV}$ with dm being the differential mass associated with the differential volume dV. The expression is useful because it allows you to write dm in terms of ρ and dV, or $dm = \rho dV$. If the structure is homogeneous (i.e., the mass is uniformly distributed throughout), ρ also equals the total mass in the structure divided by the total volume, or M/V. If, on the other hand, the structure is inhomogeneous, M/V is nonsense and a function for ρ must be provided (i.e., something like $\rho = kr$, where k is a constant and r is a variable that defines the distance between dm and an axis of interest). In any case, the $dm = \rho dV$ expression is ALWAYS true.]

<u>COMMENTS, HINTS, and THINGS to be aware of</u>:

- The **rotational kinematic equations**--what are they?
  --These are rotational relationships between the angular displacement $\Delta\theta$, angular velocity ω, angular acceleration α, and time t, that are applicable ONLY when the angular acceleration is CONSTANT (i.e., when α is not, say, a function of θ).

- $v = r\omega$ is predicated on the **assumption** that the angular velocity of an object is measured as the object sweeps circularly about a **fixed point**. The same is true of $a = r\alpha$.

- In theory, the **integral form** of the **moment of inertia** expression says the following: Take all of the mass you can find that is located within a tiny differential volume dV a distance r units from the axis in question (or, if you are using an area density function, within a tiny differential area dA a distance r units from the axis . . . or, if you are working with a linear density function, in a tiny differential length dr a distance r units from the axis), call that differential mass dm, multiply dm by r^2, do that for all possible $r's$, then sum (i.e., integrate--the final expression will look like $I = \int r^2 dm$).

Chapter 9
Rotational Motion II Chapter Review

EQUATIONS:

- $\Gamma = \boldsymbol{r} x \boldsymbol{F}$ [This is the formal definition of torque, where $\boldsymbol{r}$ is a position vector that extends *from* the point about which you are taking the torque *to* the point where the force $\boldsymbol{F}$ acts.]

- $|\Gamma| = |r||F|sin\,\theta$ [This defines the magnitude of a torque taken using vectors characterized in polar notation.]

- $|\Gamma| = |r_{\perp}||F|$ [This defines the magnitude of a torque using the force magnitude $\boldsymbol{F}$ and the component of the $\boldsymbol{r}$ vector that is perpendicular to $\boldsymbol{F}$. Often, this is the easiest way to calculate a torque because $r_{\perp}$ turns out to be <u>*the shortest distance*</u> *between the line of the force* $\boldsymbol{F}$ *and the point about which the torque is being taken.*]

- $|\Gamma| = |r||F_{\perp}|$ [This defines the magnitude of a torque using the magnitude of $\boldsymbol{r}$ and the component of $\boldsymbol{F}$ that is perpendicular to $\boldsymbol{r}$.]

- $\Gamma_{net\ about\ point\ P} = I_P \alpha$ [This is the rotational version of Newton's Second Law. It states that the net torque acting about a point P is equal to the product of the moment of inertia I_P about the point and the angular acceleration α of the body about that point.]

- $KE_{rot\ about\ axis\ P} = \frac{1}{2} I_P \omega^2$ [This is the rotational version of kinetic energy. It states that the kinetic energy associated with a body rotating about axis P is equal to one-half the product of the body's moment of inertia I_P about that point and the square of the body's angular velocity ω about that point.]

- $\boldsymbol{L}_{mass\ A\ about\ pt\ P} = \boldsymbol{r}_P\ x\ \boldsymbol{p}_A$ [This is the rotational counterpart to momentum, or *angular momentum*. In translational parameters, the relationship states that the angular momentum L about a point P is equal to the cross product of a position vector $\boldsymbol{r}$, defined as a line *from* the point about which the angular momentum is being taken *to* the body's position, and the body's translational momentum $m\boldsymbol{v}$. As this is a cross product, you can either determine this quantity using the definition approach, an $r_{\perp}$ approach, or an $F_{\perp}$ approach.]

- $L_{mass\ A\ about\ point\ P} = I_P \omega$ [This is the *magnitude* of the angular momentum calculated using rotational parameters. The direction of the angular momentum vector is the same as the direction of the angular velocity vector, or positive if the rotation is counterclockwise and negative if clockwise.]

- $\sum L_1 + \sum \Gamma_{net,external}\Delta t = \sum L_2$ [This is the rotational counterpart to the modified conservation of momentum equation. It states that the total angular momentum associated with the motion of a system of particles at a given point in time will equal the total angular momentum in the system at some later point in time *if* there are no external rotational impulses $\Gamma_{net,ext}\Delta t$ acting on the system between the two points in time. In practice, this latter condition is rarely met and the expression usually reads $\sum L_1 = \sum L_2$.]

- $\pm f\Delta t = (mv_2 - mv_1)$ *and/or* $\mp f\Delta t = \dfrac{I\omega_2 - I\omega_1}{R}$ [These equations will come in handy if you ever encounter a problem in which there is FRICTIONAL SLIPPING going on (in such problems, you will normally be asked to determine some *final velocity*). None of the conservation approaches work, so you have to go back to first causes (i.e., Newton's Second Law in its most elementary form). As the common force will always be friction acting over some time interval Δt, the trick is to generate two N.S.L. expressions in impulse form (whether these are both translational, both rotational, or one translational and one rotational expression will depend upon the system). Having grouped the unknowns *f* and Δt into one unknown ($f\Delta t$) in both expressions, the $f\Delta t$*'s* can be eliminated and the problem solved.]

<u>COMMENTS, HINTS, and THINGS to be aware of</u>:

- One of the nice things about this chapter is that it is essentially a review of all of the approaches and concepts you've run into to date. The problem is that there is a huge amount of material. What you are about to see are some hopefully helpful hints as to how to deal with this.

- If you see any problem that has a **collision**, an **explosion**, or any other situation in which one piece of a **system** changes its motion due to its **interaction** with another piece of the system (assuming all of the pieces within the system can move freely), the first approach you should consider is **conservation of** either **momentum or angular momentum**. Just as small external forces can be ignored *through a collision*, small external torques can be ignored *through a collision* and the total angular momentum of the system before and after the collision will be the same. In cases like this, *angular momentum* is said to be *conserved through the collision.*

- Don't forget, a **point mass** moving in a straight line can have **angular momentum**, relative to a fixed point, even though there appears to be no angular motion happening. That quantity will be $(mv)r_\perp$, where $r_\perp$ is the shortest distance between the line of motion and the point about which you are calculating the angular momentum quantity.

- Whenever you have a **collision** or **explosion** problem, *never* assume that **mechanical energy** is conserved.

- In a system in which there is both rotation and translation, you must include both rotational and translational kinetic energy in the same **conservation of energy expression**.

- In the modified conservation of energy expression, you will essentially never see an **extraneous work** quantity W_{ext} associated with work done by a **torque**, but you will see similar work quantities associated with forces such as friction.

- If you are **looking for** an **acceleration or angular acceleration**, always think first of Newton's Second Law. Kinematics is also possible, but N.S.L. is usually your first, best bet.

- If you are **looking for** a **velocity or angular velocity**, and if there is motion in which potential energy is changing (i.e., a body changes vertical position in a gravitational field, or a spring goes sproing), always think first about energy considerations. Kinematics is also possible, as is N.S.L. if the body is moving in a circular path, but energy considerations are usually your first, best bet.

- If there is a **collision** or **protracted force interaction** between the pieces of the system and you are asked either for a *force of interaction*, a *velocity*, or an *angular velocity*, always think first of **conservation of momentum** or **conservation of angular momentum**.

- The word acceleration (or velocity) *without* the word ANGULAR in front of it ALWAYS denotes *translational* acceleration (or velocity).

- A point of confusion for a lot of people comes from the fact that there is rotational kinetic energy as well as translational kinetic energy wrapped up in the conservation of energy expression, but there are only momentum quantities in the conservation of momentum expression and only angular momentum quantities in the conservation of angular momentum expression. **DON'T MIX momentum** and **angular momentum quantities** in the same, single, governing expression for a situation. They are two different things (whereas energy is energy whether it is rotational or translational in nature).

- Be careful about **unembedding signs** when using the rotational counterpart of N.S.L. If you think a body's angular acceleration is counterclockwise, the sign of its angular acceleration term should be positive. If clockwise, it's negative.

- Don't forget, when a **rope** is strung **over** a **massive pulley**, the tension on either side of the pulley will be different.

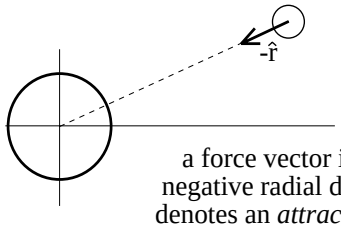
- Remember, you can either look at the motion of a **body** that is **both rotating and translating** as a *rotation about and translation of* the body's center of mass, or as an instantaneous, *pure rotation* about the only point on the mass that is not instantaneously moving (i.e., the point of contact with the support over which the mass is rolling). This is useful both in N.S.L. problems and in conservation of energy problems. Either approach will work in the analysis of a problem (though one is often easier than the other, depending), but don't mix 'em.

- **Don't be buffaloed** by problems that look familiar but are different. A favorite AP trick is to give you a problem you've seen before, but turned around. You'd be surprised how many people get confused when the incline is presented sloping from the right to the left, instead of from the left to the right as is usually shown in texts. Remember, what you have at your disposal is a series of approaches that will provide you with mathematical relationships that are true (i.e., one side of the expression will equal the other side of the expression). You know how to use those approaches. Use them from scratch. Don't get hung up trying to make a test problem fit the form of a problem you've seen before.

Chapter 10
Gravitational Chapter Review

EQUATIONS:

- $\boldsymbol{F} = G\frac{m_1 m_2}{r^2}(-\boldsymbol{r})$ [This is the technical presentation of the magnitude and direction of a gravitational force between masses m_1 and m_2, where r is the distance between each mass's center of mass and G is the Universal Gravitational Constant. Please note that the unit vector $-\boldsymbol{r}$ is a polar-spherical unit vector that denotes the fact that the force is always on line between the two bodies, and is always attractive. It is very rare that you will use this expression as a full blown vector. Most problems are done in a Cartesian coordinate system (not a polar spherical system), so the norm is to use this expression to determine the *magnitude* of the gravitational force, then assign a positive or negative sign depending upon how the force is oriented relative to the Cartesian coordinate axis being used in the problem. It has been presented in all of its glory because the notation is common in textbooks, but the $-\boldsymbol{r}$ notation is rarely explained.]



a force vector in the negative radial direction denotes an *attractive* force

- $|F| = G\left(\frac{m_1 m_e}{r_e^3}\right)r$ [The magnitude of the gravitational force on a mass m_1 when *inside* the earth (assumed homogeneous) at distance r units from the center of the earth. Note that as r_e and m_e are constants--the radius and mass of the earth, respectively--this force is a linear function of the position r. You should be able to derive this expression. That is, you should understand what was done in its derivation so that you could deal with an inhomogeneous earth.]

- $U = -G\frac{m_1 m_2}{r}$ [This defines the amount of gravitational potential energy that exists in a two-body system. You should be able to derive this. It comes from $U(r) - U(\infty) = -\int_{\infty}^{\boldsymbol{r}} \boldsymbol{F} \bullet d\boldsymbol{r}$, where $\boldsymbol{F}$ is the gravitational force $G\frac{m_1 m_2}{r^2}(-\boldsymbol{r})$ and the potential energy is defined where the force is zero (i.e., out at infinity). Note that the direction of the force is in the *negative radial direction*. If you don't like radial coordinates, change to *x's* and *dx's*, do the Calculus, then substitute the *r's* back into the expression.]

- $E_{tot} = -G\frac{Mm_1}{2r}$ [This is the total mechanical energy--*potential* plus *kinetic*--wrapped up in the motion of a single mass m_1 orbiting a considerably more massive object M in a CIRCULAR ORBIT of radius r. Note that this expression makes no sense if the motion is elliptical.]

- $v = \frac{2\pi r}{T}$ [This is the relationship between the magnitude of the velocity of a body moving in *circular orbit* of radius *r* and the period *T* of the motion, the period being the amount of time required to complete one full orbit.]

- $\frac{v^2}{r}$ [In orbital motion, this is the magnitude of the centripetal acceleration an object moving in a circular orbit must experience due to its gravitational interaction with the other body in the system. Note that *r* is the *radius* of the circular orbit.]

- mvr [This is the angular momentum ***r*x*p*** of a mass moving in a CIRCULAR orbit about a much larger mass (a satellite about a planet, for instance). Also, this is the angular momentum of a small mass moving in an elliptical orbit about a large body, where *r* is the distance between the small mass and the system's center of the mass (usually inside the large body). Note also that *v* must be the *component* of velocity vector that is *perpendicular* to ***r***. The angular momentum magnitude can additionally be written as $I\omega$ AS LONG AS ω is defined as $(v_{perpendicular})/r$ and *I* is the moment of inertia of a point mass, or mr^2.]

- $T^2 \alpha r^3$ [This is Kepler's Third Law. It states that the square of the period of a planet orbiting the sun is proportional to the cube of the orbit's semi-major axis. This is the *only one* of Kepler's laws that is an approximation. It isn't true if the masses involved are anywhere near equal in size.]

COMMENTS, HINTS, and THINGS to be aware of:

- This is a nice review chapter because it incorporates most of the concepts you have studied to date into the analysis of one *kind* of system--a gravitational one.

- **Angular momentum** in orbital motion is **conserved** whether the path is circular or elliptical. Stated mathematically, that relationship is $L_1 = L_2$.

- The **total mechanical energy** in orbital motion is **conserved** whether the path is circular or elliptical. Stated mathematically, that relationship is $E_1 = E_2$.

- For all motion in space in which **a** very **small body orbits a** very **large body**:
  --As there is little frictional drag in space, the total mechanical energy is conserved. If the motion is circular, that energy is $E_{tot} = -G\frac{Mm_1}{2r}$. If the motion is elliptical, the total mechanical energy must be written as the *potential* plus *kinetic energy* in the system, relative to the system's center of mass.
  --As there are no external torques acting (the gravitational force is along the line between the two bodies), angular momentum is conserved.
  --Kepler's Third Law holds in this situation ($T^2 \alpha r^3$).

- For motion in space in which **two similar sized bodies orbit** one another:
  --The two bodies orbit about the system's center of mass.

--The total mechanical energy and the angular momentum are conserved, relative to the system's center of mass.

- The fact that the **gravitational potential energy** function is negative shouldn't put you off. The only time you will ever deal with a potential energy function is when you are trying to determine a work quantity. Remember, work is related to potential energy *differences*.

- **Conservation of energy**, **conservation of angular momentum**, and **N.S.L.** coupled with **centripetal acceleration** are the main stays of analysis when dealing with gravitational systems. There will be times when you will be asked to determine quantities that seem to be completely off the wall--the *period* of an orbiting body's motion, for instance. When in doubt, write down every expression you can think of that is true for the system, then go from there.

- An easy way to remember **Kepler's Three Laws** is to remember their names:
  --The Law of Orbits: Planets move in *elliptical orbits* having the *sun at one focal point*.
  --The Law of Areas: A line joining any planet to the sun sweeps out *equal areas in equal times* (i.e., $dA/dt = constant$).
  --The Law of Periods: The *square of the period* of any planet about the sun is proportional to the *cube of the planet's mean distance* from the sun.

- There are two **basic problem-types** to look for when dealing with gravity. The first is **orbital motion**. The second is **free fall**. Be sure you know which you are dealing with. For instance, the total mechanical energy in a free fall problem is NOT $E_{tot} = -G\frac{Mm_1}{2r}$ (that expression relates total mechanical energy in a circular orbit situation where one body is a lot larger than the other). Know which relationships are relevant, and when!

Chapter 11
Vibratory Motion Chapter Review

EQUATIONS:

- $\boldsymbol{F}_{spring} = kx(-\boldsymbol{i})$ [This expression yields the magnitude and directional nature of the force a spring applies to a mass when the spring has been compressed or elongated a distance x units from equilibrium. The negative sign denotes the fact that the force is always toward the equilibrium position (when the mass is on the negative side of equilibrium, x is negative and the force is calculated correctly as being in the positive direction). The variable k is defined as the *spring constant*. Technically, it tells you how many newtons it takes to compress or elongate the spring one meter (its units are nts/m). Qualitatively, it tells you how rigid the spring is.]

- $T = \frac{1}{\nu}$ [This is an expression that is ALWAYS TRUE. It states that the period T of an oscillatory system is the inverse of the frequency ν. Note that the units for the period are *seconds*/*cycle*, whereas the units for the frequency are *cycles*/*second*.]

- $\omega = 2\pi\nu$ [This is an expression that is ALWAYS TRUE. It states that the angular frequency ω of a vibrating body is equal to 2π times the frequency ν of the system. The units of angular frequency are *radians*/*second*, which means the units for the 2π quantity must be *radians*/*cycle*.]

- $\omega_{spring} = \left(\frac{k}{m}\right)^{\frac{1}{2}}$ [True only for an ideal spring system (i.e., one that oscillates with simple harmonic motion), this is the relationship between the angular frequency ω of an oscillating mass m attached to a spring of spring constant k.]

- $E_{tot} = \frac{1}{2}kA^2$ [This is the total energy of a spring system oscillating in simple harmonic motion, where k is the spring constant and A is the amplitude of the motion.]

- $v_{max} = \omega A$ [This is the magnitude of the maximum velocity of any oscillating system, where ω is the angular frequency of the system and A is the system's amplitude. The maximum velocity occurs at equilibrium where the accelerating force is completely exhausted.]

- $a_{max} = \omega^2 A$ [This is the magnitude of the maximum acceleration of any oscillating system, where ω is the angular frequency of the system and A the system's amplitude. The maximum acceleration occurs at the extremes where the accelerating force is the greatest.]

- $(acceleration\ term) + (constant)(position\ term) = 0$ [This is the characteristic equation for simple harmonic motion. If you can manipulate a Newton's Second Law expression into the form $a + (K)x = 0$, or $\alpha + (K)\theta = 0$, where K is some constant (i.e., k/m if the system is an ideal spring, etc.), previous observations and derivations allow us to know that the

motion IS simple harmonic in nature, and that the angular frequency of the motion is $\omega = \sqrt{K}$.]

- $x(t) = A\sin(\omega t + \phi)$ [This is the equation that characterizes the position of an oscillating body as a function of time (example: 3 sin (12t +.7) . . .). Note that A is the amplitude of the motion, ω is the angular frequency, and ϕ is the phase shift (the inclusion of ϕ allows you to start the clock whenever you want within the oscillatory cycle).]

- $\omega_{simple\ pendulum} = \left(\frac{g}{L}\right)^{\frac{1}{2}}$ [True only for a *simple* pendulum system (i.e., a massless string with a point-mass bob at one end) moving in *small amplitude oscillation*, this is the relationship between the system's angular frequency ω, its pendulum length L, and the acceleration of gravity g. Note that this expression suggests a clever way to determine the acceleration of gravity on the moon or some other celestial body.]

<u>COMMENTS, HINTS, and THINGS to be aware of</u>:

- The temptation is to memorize formulas. **Know the relationships**, **BUT** beyond all else, understand the *reasoning* behind the relationships. I remember an AP test problem that assumed a non-linear spring (example: $F = -kx^5$) and asked qualitative questions from there. An example: *If you elongate the spring a distance L and release it, the period is T. What happens to T if the spring is released at 1.5L?* The moral? Understand the concepts. Understand why the relationships are as they are. Don't memorize relationships mindlessly.

- If you are given an exotic oscillating system and asked to **determine** the **system's period**, **the trick is**:
  --Show that the motion is *simple harmonic motion* by writing out N.S.L., then put that equation into the form *(acceleration) + (constant)(position) = 0* . . . (this is the characteristic equation for simple harmonic motion.)
  --The *constant* will equal the square of the motion's angular frequency.
  --You know the relationship between *angular frequency*, *frequency*, and *period*. Use that relationship to derive your *period* expression.

- A **spring constant tells you**:
  --How rigid the spring is.
  --How much force per unit distance is required to elongate or compress the spring.

- To **determine** a **spring constant**:
  --You need to determine the amount of force required to elongate or compress the spring some given distance.
  --You could either be given that information straight away, or you could be told the distance d a given amount of mass m displaces the spring (remember, the force that is being applied in such a case is that of gravity--it is $f_{grav} = mg$) when hung vertically and allowed to slowly lower to its new equilibrium position.

Chapter 12
Wave Motion Chapter Review

EQUATIONS:

- $y = A\sin(kx \pm \omega t)$ [Waves move. That means that their displacement y above or below the medium's equilibrium position at a given time t will vary with position x, AND that displacement at a given position x will vary with time t. In other words, there are two variables that have to be taken into account when trying to determine the displacement of a wave--the position of interest and the time of interest. The function above allows you to determine the displacement (i.e., the y coordinate) of a wave as it exists at a given position x at a given time t. Note: Understanding the constants ω and k can be tricky. Think about them this way. At a given x, as time goes on, the displacement y changes. The angular frequency term ω governs how fast that change occurs (ω's units are *radians*/*second*--a big angular frequency means THE DISPLACEMENT CHANGES QUICKLY IN TIME). Likewise, at a given t, as you move positionally from point to point, the displacement y changes. The wave number k governs how fast that change occurs, relative to position (k's units are *radians*/*meter*---a big wave number means THE DISPLACEMENT CHANGES QUICKLY FROM POINT TO POINT IN SPACE).]

- $T = \frac{1}{\nu}$ [The time T it takes for one full wavelength to pass a point is equal to the inverse of the frequency ν of the wave.]

- λ [The wavelength λ of a wave is defined as the distance between one peak to the next, OR one trough to the next.]

- $v_{wave} = \lambda\nu$ [The wave velocity v equals the product of the wavelength λ in *meters per cycle* and the wave frequency ν in *cycles per second*. Note that with physical waves (i.e., waves on a string, not light waves), the frequency of the wave is the same as the frequency of the vibratory motion that produced the wave.]

- $\nu_{observed/heard} = \frac{(\nu_{original\ from\ source})(v_{sound\ in\ air})}{v_{sound\ in\ air} - v_{of\ source}}$ [Called the Doppler effect, this relationship gives you the frequency $\nu_{observed/heard}$ you actually *hear* when a sound source of frequency $\nu_{original\ source}$ approaches you at velocity $v_{of\ source}$. Note that $v_{sound\ in\ air}$ is the velocity of sound in air. Note also that if we were talking about light being Doppler shifted from a star that is, say, receding from earth, the expression would read $\nu_{observed} = \frac{(\nu_{original\ from\ source})(c)}{c - v_{of\ source}}$, where c is the speed of light in a vacuum, $v_{of\ source}$ is the speed of the star, relative to the earth, $\nu_{observed}$ is the frequency taken in by the telescope making the observation, and $\nu_{original\ from\ source}$ is the frequency actually being given off by the star.]

- $\nu_{beat} = \nu_{higher} - \nu_{lower}$ [This is the relationship between the frequency of the beats (variations in intensity) created when two signals whose very close frequencies superimpose on each other.]

COMMENTS, HINTS, and THINGS to be aware of:

- A **wave** is a **disturbance** that moves through a medium. If you don't have a medium, you can't have a wave (example: sound in space . . . doesn't happen).

- A **transverse wave** is created when a force directed *perpendicular to* the direction of the wave motion is applied to a medium. A pebble breaking the surface of a still pond produces a transverse wave.

- A **longitudinal wave** is created when a force directed *parallel to* the direction of the wave motion is applied to a medium. Sound coming from a speaker is an example of a longitudinal wave.

- **Resonance** occurs when the frequency of a force being applied to an oscillating system matches one of the natural frequencies of the system. When it occurs, the amplitude of the oscillation increases dramatically.

- To **determine** one of the many possibly **natural frequencies** of an oscillating system (i.e., to find a frequency that produces a standing wave when resonance happens):
  --Look to see what the endpoint constraints must be for your system (for ease below, assume that one end point is called *endpoint A* and the other is called *endpoint B*). These will either be nodes (fixed ends), antinodes (free ends), or a combination thereof.
  --Draw a sine wave.
  --On the sine wave, start at a node or an antinode--whichever corresponds to the constraint at *endpoint A*.
  --Proceed down the sine wave until you hit either a node or an antinode, whichever corresponds to the constraint at *endpoint B*.
  --On the wave form you have created, look to see if any internal constraints in the system are being satisfied. That is, if there should be a node halfway between the endpoints, does the wave form you are working with *have* a node halfway between the endpoints?
    --Note that there is no artificial way to ensure that an internal point will act like an antinode, so all internal constraints MUST BE nodes.
    --To check to see if an internal constraint is being satisfied, especially in complicated situations, note that if the product of the *number of quarter wavelengths* in the wave form and the *fractional distance* from a chosen endpoint is A WHOLE NUMBER, you *might* have a satisfied constraint. That is, if the endpoint from which the fractional distance is measured is a node, all that is required is that the product be ANY whole number. If the constraint from which the fractional distance is measured is an antinode, the product has to be an ODD whole number to comply.
  --If all of the internal constraints are met, you have an acceptable wave form. If not, continue down the wave form until *endpoint B's* constraint is again met, and repeat the process outlined above.

--Assume that the length of the system is L.
--Once the wave form has been identified, ask the question, "How many wavelengths are there in L?" (I usually count these in quarter wavelengths.) Writing that relationship down will allow you to determine the wavelength that fits the situation in terms of a physical parameter associated with the system, or L.
--Knowing λ in terms of L, you can now use $v = \lambda\nu$ to determine the required natural frequency (you will usually be given the wave velocity v).
--Note that if a force varying at that frequency impinges on the system, the standing wave produced will look just like the wave sketch, assuming you are dealing with a *transverse wave*.

- The **Doppler Shift** is particularly useful in determining the velocity of an object moving relative to you. Examples: Measuring the speed of a star via spectroscopic analysis, relative to the earth, or measuring the speed of a car via Doppler radar, relative to a cop.

- **Beats** are used by a guitarist to tune a guitar. One string is held down (fretted) and plucked to produce a tone. A second string, one that should produce the same tone (frequency) as the first if the two are really in tune, is then plucked. The superposition of the two frequencies will produce variations in sound intensity (beats) if they are just a little bit out of tune.

Multiple Choice Tests--Prologue

The four multiple choice tests you are about to examine have been provided as an opportunity for you to think about and review the concepts you have come to know and love over the course of this year. I would suggest that you either do as much as you can in an hour (the A.P. test allots 45 minutes to do 35 questions--your semester final will be longer), or do only a few pages at a time. The problems are in random order to make you think broadly, versus focusing on the material of just one chapter at a time. I would strongly suggest you get together with friends and talk about the problems that stumped you. Getting problems right is not important. Allowing the exercise to catapult you into some serious thought about the material so far covered--that *is* important. Remember, it is just as educational to understand why a *wrong answer* is NOT correct as it is to understand why a *right answer* IS correct.

P.S. As this section is designed to be a REVIEW of material you already know, there is no need to treat the Solutions as something to be initially shunned. That is, although you can take each test cold, I would suggest that you simply jump straight into the Solutions if you are strapped for time. Also, please remember that these multiple choice questions are not *all* designed to be like A.P. questions (after all, the A.P. folks do not allow the use of a calculator when doing their multiple choice section). In fact, you will find a number of *Wave Dynamics* questions that have nothing to do with the AP test but might be on your semester final exam.

Good luck.

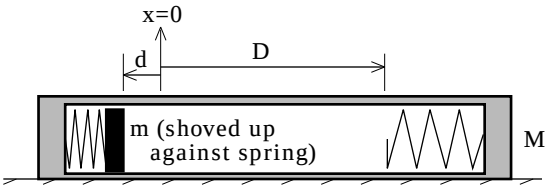
Multiple Choice -- TEST I

1.) The position function for an oscillating body is $x = 20 \sin(.6t - \pi/2)$. At $t = 0$, the magnitude of the body's acceleration is:

a.) 20 m/s^2.
b.) 12 m/s^2.
c.) 7.2 m/s^2.
d.) None of the above.

2.) A satellite following an elliptical path around a planet has an angular velocity ω_{far} when at its maximum distance d units from the planet's center. At its closest point, the distance between the satellite and planet's center is $d/3$. The satellite's angular velocity at that closest point is:

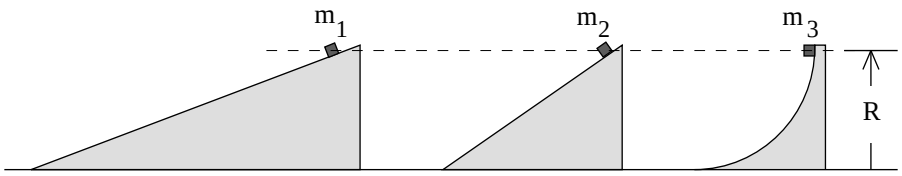
a.) $\omega_{far}/3$.
b.) ω_{far}.
c.) $3\,\omega_{far}$.
d.) $9\,\omega_{far}$.



3.) An ideal spring (spring constant k) is mounted inside a hollowed-out block whose mass is M. With M temporarily held stationary, a second mass m is pushed up against the spring until the spring's displacement is d. A second spring with the same spring constant is positioned at the end of the hollow. The net distance between the ends of the two unsprung springs is D (see sketch). Assume M is *not* fixed to the ground and can move frictionlessly. Assume also that there is no internal friction in the system. The spring is released, accelerating m.

a.) Momentum should be conserved in the system because the spring applies a conservative force.
b.) Just after it leaves the spring, m's velocity will be $(kx^2)/(m+m^2/M)$.
c.) When m reaches the second spring, it will depress it a distance d before coming to rest relative to the spring. Also at that point, M will have come to rest.
d.) All of the above.

The sketch below is for Questions #4 and #5. In the sketch, three inclines are shown. The sketches are drawn approximately to scale with the curved incline being a quarter circle. The masses are different sizes, but the distance traversed by m_2 in getting to the bottom of its incline is the same as the distance traversed by m_3 in getting to the bottom of its incline.



4.) Assuming the inclines are frictionless, the mass with the largest velocity at the bottom will be:

a.) Mass m_1.

b.) Mass m_2.

c.) Mass m_3.

d.) Masses m_2 and m_3.

e.) They will all have the same velocity at the bottom.

5.) Friction is introduced into the system such that the coefficient of friction is the same for all three inclines. The mass with the largest velocity at the bottom will be:

a.) Mass m_1.

b.) Mass m_2.

c.) Mass m_3.

d.) Masses m_2 and m_3.

e.) They will all have the same velocity at the bottom.

--------end of situation--------

6.) *Point A* and *Point B* on a wave are observed to be 7/4 wavelengths apart. At a particular instant, *Point A* is below the axis and moving upward toward equilibrium. At that time, *Point B* is:

a.) Below the axis and moving upward toward equilibrium.

b.) Below the axis and moving downward away from equilibrium.

c.) Above the axis and moving upward away from equilibrium.

d.) Above the axis and moving downward toward equilibrium.

7.) A single, constant force is applied to a body. After the force does 20 joules of work, the body's velocity has changed from zero to 6 m/s. The work required to change the body's velocity from 6 m/s to 12 m/s is:

a.) 20 joules.

b.) 40 joules.

c.) 60 joules.

d.) 400 joules.

8.) A man on a unicycle pedaling northward begins to slow down. The direction of the cycle's angular acceleration vector will be:

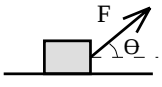
a.) North.

b.) South.

c.) East.

d.) West.

e.) Upward, relative to the ground.

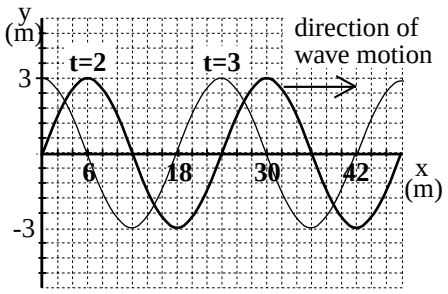


9.) A mass m has a force F applied to it as shown to the right:

a.) The normal force on the block will be mg.

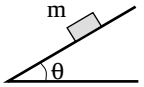
b.) The acceleration of the block will be a function of F only.
c.) If there was kinetic friction acting on the block, it would be directed toward the left.
d.) None of the above.

10.) A graph of a traveling wave as seen at $t = 2$ *seconds* and $t = 3$ *seconds* is shown to the right. The period of the wave is:



a.) (3/4) seconds/cycle.
b.) (4/3) second/cycle.
c.) 4 seconds/cycle.
d.) Can't tell with the information given.
e.) None of the above.

11.) A mass m sits on a frictionless incline plane of angle θ.



a.) The acceleration of mass m will be g.
b.) The acceleration of mass m will be dependent upon the size of the mass.
c.) The acceleration of mass m will be dependent upon θ.
d.) All of the above.

12.) It is known that a single, conservative force that runs along the *x axis* does -25 joules of work on a body moving in the *negative x direction*.
a.) The force is in the negative direction, and the body's change of potential energy is -25 joules.
b.) The force is in the negative direction, and the body's change of potential energy is 25 joules.
c.) The force is in the positive direction, and the body's change of potential energy is -25 joules.
d.) The force is in the positive direction, and the body's change of potential energy is 25 joules.
e.) None of the responses make any sense as gravity is not oriented along the *x axis*.

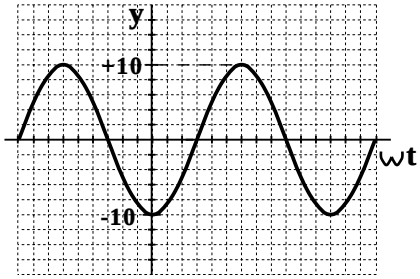
13.) A satellite following an elliptical path around a planet has a velocity v_{far} when at its maximum distance d units from the planet's center. At its closest point, the distance between the satellite and planet's center is $d/3$. The satellite's velocity at that closest point is:
a.) $v_{far}/3$.
b.) v_{far}.
c.) $3v_{far}$.
d.) $9v_{far}$.

14.) A turntable has an angular velocity of .8 rad/sec. It is accelerated at a rate of -4 rad/sec^2.
a.) After 5 seconds, its angular velocity is -19.2 rad/sec, and it will have rotated through a net angular displacement of -46 radians.
b.) It will take twice as long to go from .8 rad/sec to 0 rad/sec as it takes to go from .8 rad/sec to .4 rad/sec.

c.) It will have moved through a greater net angular displacement after the first .2 seconds than after the first .4 seconds.
d.) Both *a* and *b*.
e.) All of the above.

15.) A resting wheel has a constant torque applied to it. After the first 10 seconds, the wheel has turned through an angular displacement of 1.2 radians. After the first 20 seconds, the angular displacement will be:
a.) 2.4 radians.
b.) 4.8 radians.
c.) 7.2 radians.
d.) None of the above.

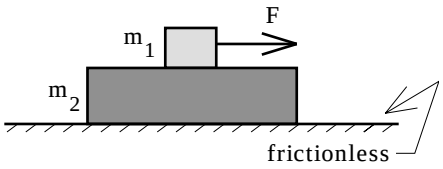
16.) A graph of the position function for a body oscillating with a frequency of $1/(4\pi)$ radians per second in *simple harmonic motion* is shown to the right. The equation that best describes the motion is:



a.) $y = 20 \sin (.5t - \pi/2)$.
b.) $y = 10 \sin ([1/(4\pi)]t - 5)$.
c.) $y = 10 \sin (.5t + 3\pi/2)$.
d.) $y = 20 \sin ([1/(4\pi)]t - \pi/2)$.
e.) None of the above.

17.) A sound wave moving at 330 m/s has a frequency of 220 Hz. Its wavelength is:
a.) 2/3 meter.
b.) 1.5 meters.
c.) 66,000 meters.
d.) None of the above.

18.) A mass m_1 sits on top of a second mass m_2 which sits on a frictionless surface. The coefficient of static friction between the two is .7. A force F is applied to the top mass (m_1).



a.) The maximum force F that m_1 can experience without slipping over m_2 is $\mu_s m_1 g$.
b.) The maximum force m_1 can experience without slipping over m_2 is $2\mu_s m_1 g$.
c.) The maximum force m_2 can experience without slipping relative to m_1 is $\mu_s m_1 g$.
d.) Both *b* and *c*.
e.) None of the above.

19.) A spinning skater's angular speed is 2 rad/sec when her kinetic energy is 20 joules. If the skater's moment of inertia changes by half during the spin, what is her new angular speed?
a.) .5 rad/sec.
b.) 2 rad/sec.
c.) $(8)^{1/2}$ rad/sec.
d.) 4 rad/sec.

20.) A mass m is attached to a string of length L that is, itself, attached to the ceiling. The mass is drawn to the side and released so that it swings back and forth. At the bottom of its arc, the mass's velocity is v_o.

a.) The acceleration of the mass when at the bottom of the arc is *zero*.

b.) The acceleration of the body when at the bottom of the arc is a function of the body's mass.

c.) The magnitude of the force on the mass when at the bottom of the arc is mg.

d.) The net force on the mass when at the bottom of the arc is $(mv_o^2/L)\mathbf{j}$, where $\mathbf{j}$ is a unit vector in the *y-direction*.

21.) A flatbed truck coasts forward frictionlessly along on a flat surface. The surface between the truck bed and a large, massive crate sitting on the bed is *frictional*. At $t = 0$, you begin to push the crate toward the rear of the bed. At $t = 1$ *second*, the crate leaves the truck.

a.) Between $t = 0$ and $t = 1$ *second*, the momentum of the truck is conserved.

b.) Between $t = 0$ and $t = 1$ *second*, the momentum of the truck/crate system is conserved, and the truck will have slowed by the time the crate leaves it.

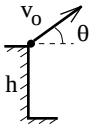
c.) Between $t = 0$ and $t = 1$ *second*, the momentum of the truck/crate system is conserved, and the truck will have sped up by the time the crate leaves it.

d.) Between $t = 0$ and $t = 1$ *second*, the momentum of the truck/crate system is not conserved as *you* are applying an external force to the crate.

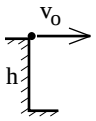
e.) None of the above.

Multiple Choice -- TEST II

--The following information pertains to Problems 1 through 5: Projectiles A, B, C, and D are fired at the same time from a height h meters above the ground. With the exception of Projectile D, which is dropped from rest, all the projectiles (i.e., Projectiles A, B, and C) have the same muzzle velocity v_o, (though each is fired at a different angle--see the sketches below and note that the angle defined as θ is the same in all cases). It takes t_1 seconds for Projectile A to get to the top of its flight. It takes t_2 seconds for Projectile D to reach the ground.

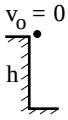


Projectile A



Projectile B

Projectile C



Projectile D

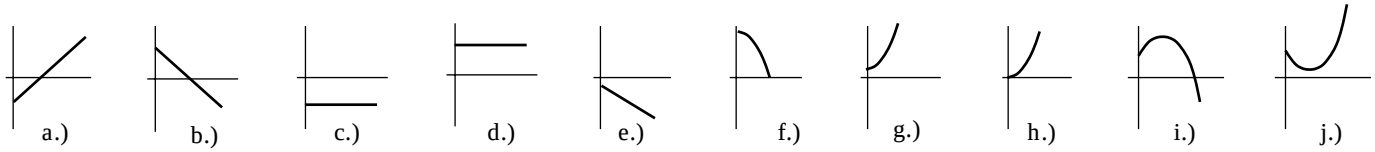
1.) At time t_1, Projectile A's:

a.) Velocity will be perpendicular to its acceleration.
b.) Velocity will be $v_o \cos \theta$.
c.) X-component of acceleration will be twice what it was at t = 0.
d.) All of the above responses are true.

2.) Projectile A's:

a.) Acceleration is greater on the way up than on the way down.
b.) Velocity changes at the same rate going up as going down.
c.) Y-component acceleration sign is the same as its y-component velocity sign while going up.
d.) Velocity, when at h going upward, will be the same as its velocity when at h coming down.
e.) Both b and d.

3.) Consider the graphs:



Projectile A's:

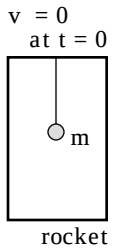
a.) Y-component of Position vs. Time graph looks like graph a.
b.) X-component of Position vs. Time graph looks like graph j.
c.) Y-component of Velocity vs. Time graph looks like graph b.
d.) X-component of Velocity vs. Time graph looks like graph c.
e.) Y-component of Acceleration vs. Time graph looks like graph d.

4.) The time t_2:

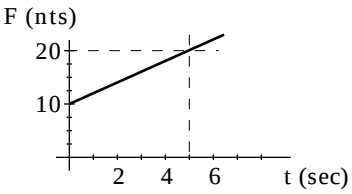
a.) Depends only on h and constant(s).

b.) Is the same time it takes Projectile B to hit the ground.
c.) Is more than the time it takes Projectile C to hit the ground, but less than the time it takes Projectile A to hit.
d.) Both a and b.
e.) All of the above except d.

5.) If h were doubled, Projectile D's:
a.) Time to touch down will double.
b.) Velocity just before touch down will double.
c.) Acceleration just before touch down will double.
d.) None of the above.



6.) A mass m is attached to a string hung from the ceiling of a rocket. The rocket accelerates from rest upward with an acceleration equal to a_1. Just an instant after t = 0:
a.) The tension in the line is mg.
b.) The tension in the line is $m(g + a_1)$.
c.) The tension in the line is $m(g - a_1)$.
d.) None of the above.



7.) A 2.5 kg mass moves one-dimensionally over a frictionless surface. A single, external force F = 2t + 10 is applied (see graph to the right). The body's velocity changes from 0 m/s to 30 m/s during the first 5 seconds:
a.) The net work done by the force is 75 joules (the area under the graph).
b.) The net work done by the force is 1125 joules.
c.) The net work done by the force is -1125 joules.
d.) The net work done by the force is 2813 joules.
e.) The net work done by the force is -2813 joules.

8.) A 20 centimeter diameter ball whose moment of inertia is $(2/5)mR^2$ accelerates down a 30^o incline plane. After traveling .2 meters down the incline, its angular velocity is 14 rad/sec. Assume the magnitude of g is 10 m/s^2.
a.) The velocity of the ball's center of mass at the .2 meter point is 140 m/s.
b.) After traveling an additional .2 meters, the ball's angular velocity will be 28 rad/sec.
c.) The ball's angular acceleration is 35.7 rad/sec^2.
d.) Both a and c are true statements.

9.) A .5 kg ball moving with velocity v = (20 m/s)i strikes a massive wall over a .02 second period and, reversing itself, leaves with a velocity magnitude equal to 16 m/s.
a.) The impulse applied to the ball during the collision is (2 nt·sec)i, and the magnitude of the average applied force is 100 nts.

b.) The impulse applied to the ball during the collision is (18 nt·sec)i, and the magnitude of the average applied force is 900 nts.

c.) The impulse applied to the ball during the collision is (-2 nt·sec)i, and the magnitude of the average applied force is 100 nts.

d.) The impulse applied to the ball during the collision is (-18 nt·sec)i, and the magnitude of the average applied force is 900 nts.

e.) None of the above.

10.) A satellite 3R units from the surface of a planet of radius R has potential energy U_1. Its gravitational potential energy 6R units from the surface (call this potential energy U_2) will be:

a.) Larger and equal to $(1/2)U_1$.

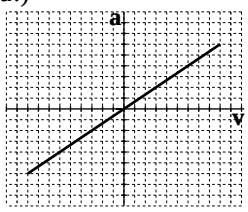
b.) Larger and equal to $2U_1$.

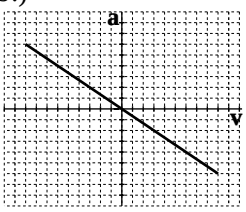
c.) Larger and equal to $(4/7)U_1$.

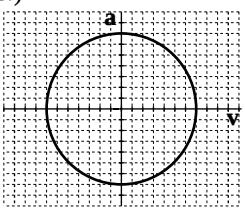
d.) Smaller and equal to $(4/7)U_1$.

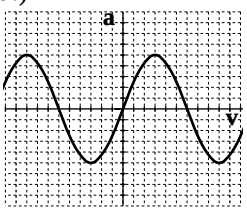
e.) None of the above.

11.) A graph of the acceleration vs. velocity of a body oscillating in simple harmonic motion looks like:

a.) 

b.) 

c.) 

d.) 

e.) None of these.

12.) A 30 newton force acting at 30^o below the horizontal acts over a 12 meter distance. The work done by F:

a.) Is -360 joules.

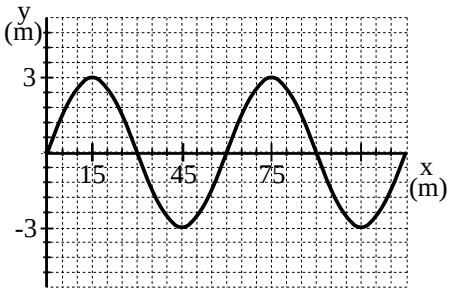
b.) Is +360 joules.

c.) Would be the same if the angle had been above the horizontal.

d.) Both a and c.

e.) Both b and c.

13.) At a particular point in time during the motion of a wave, the displacement vs. position function is as graphed to the right.



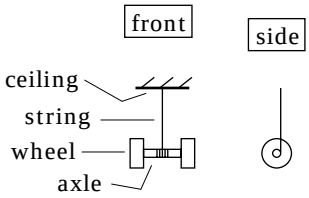
a.) The amplitude of the wave is 3 meters, its wavelength is 60 meters, and its frequency is .5 Hz.

b.) The amplitude of the wave is 6 meters, its wavelength is 30 meters, and its frequency is impossible to tell with the information given.

c.) The amplitude of the wave is 3 meters, its

wavelength is impossible to tell with the information given, and its frequency is impossible to tell with the information provided.

d.) None of the above.



14.) A string attached to the ceiling at one end has its other end wrapped around an axle whose radius is R (see sketch). Attached to the axle are two wheels of mass m each, one at each end. The moment of inertia of the entire assembly along the axle's length is $1.4mR^2$. All numerical values are measured in the MKS system. The system is released to freefall. During the fall:

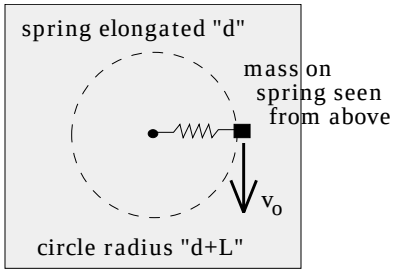
a.) Energy is conserved but angular momentum is not conserved.

b.) The body's angular acceleration α is numerically greater than the acceleration a of the body's center of mass.

c.) The body's angular acceleration α will be numerically less than the acceleration a of the body's center of mass only if the radius of the axle is greater than 1 meter.

d.) Both a and b.

e.) Both a and c.



15.) A spring produces a force whose magnitude is kx, where x is the elongation of the spring and k is a constant. Our spring's unstretched spring length is L. One end of the spring is attached to a mass m and the other end of the spring is hooked frictionlessly over a pin at the center of a table. The spring is elongated a distance d beyond L, and the mass is given a constant velocity magnitude v_o so that the whole system moves in a circular path as shown to the right (the view is looking down from above).

a.) With the velocity magnitude a constant, the acceleration for this motion must be zero.

b.) The acceleration is non-zero and k = mg.

c.) The acceleration is non-zero and $k = (mv_o^2)/[x(d + L)]$.

d.) The acceleration is non-zero and $k = (mv_o^2)/(x)$.

16.) Two 2 kilogram masses collide. The first, Body A, has 25 joules of kinetic energy before the collision. The second mass is initially stationary. The two masses experience a perfectly inelastic collision. After the collision, mass B's kinetic energy is:

a.) 12.5 joules.

b.) 2.5 joules.

c.) 6.25 joules.

d.) None of the above.

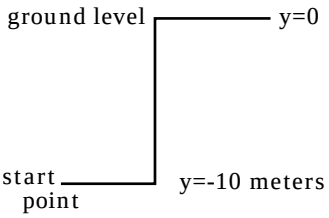
17.) An orbiting 1200 kg satellite has a period of 43 minutes (2580 seconds) in an orbit that is 15,000 meters above a planet of radius 10^6 meters. The satellite's speed is:

a.) 2472 m/s.

b.) 2435 m/s.

c.) 32.5 m/s.

d.) None of the above.



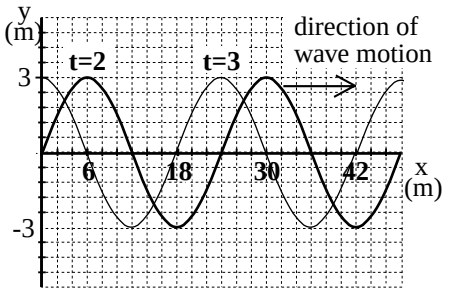
18.) A 10 kg projectile moving vertically has 1500 joules of gravitational potential energy at the same time that it has 1500 joules of kinetic energy. (Assume g's magnitude is 10 m/s^2). The potential energy is defined as zero at a point 10 meters below ground level where the body begins its motion (see sketch):

a.) The work gravity does to the top of the body's flight is -3000 joules, and the body's maximum height is 30 meters above ground level.

b.) The work gravity does to the top of the body's flight is -3000 joules, and the body's maximum height is 20 meters above ground level.

c.) The work gravity does to the top of the body's flight is -4000 joules, and the body's maximum height is 30 meters above ground level.

d.) The work gravity does to the top of the body's flight is -4000 joules, and the body's maximum height is 20 meters above ground level.



19.) A graph of a traveling wave as seen at t = 2 seconds and t = 3 seconds is shown to the right. At the rate it is traveling, how far will it travel between t = 2 seconds and t = 5 seconds?

a.) 12 meters.

b.) 18 meters.

c.) 54 meters.

d.) None of the above.

20.) Newton's Second Law is applied to a system. After a free body diagram is drawn and the forces summed, the equation - $32x^2$= 2a emerges, where x is the position and a is the acceleration of the body at an arbitrary point in time.

a.) This equation does not characterize an oscillatory system.

b.) This equation does characterize an oscillatory system and the motion is simple harmonic in nature.

c.) This equation does characterize an oscillatory system and the motion's frequency is 4 radians per second.

d.) Both Response b and c.

e.) None of the above.

21.) A spring oscillates with frequency 1 cycle per second. What approximate length must a simple pendulum have to oscillate with that same frequency?

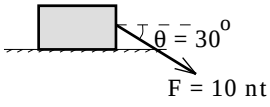
a.) 25 cm.

b.) 50 cm.

c.) 67 cm.

d.) 90 cm.

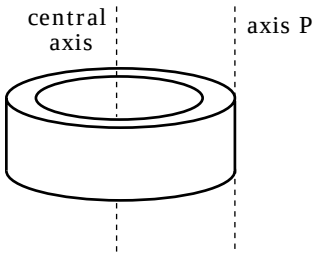
22.) A body moves 25 meters. For the entire motion, a 10 newton force acts on the body as shown to the right.



a.) The work done by the force is 250 joules.

b.) The work done by the force is -250 joules.

c.) The work done by the force is 217 joules.
d.) The work done by the force is -217 joules.
e.) None of the above.



23.) The moment of inertia about the central axis of a ring is $(1/2)m(R_1^2+R_2^2)$, where R_1 is the inside radius and R_2 is the outside radius of the ring (see sketch). The moment of inertia about axis labeled P in the sketch is:

a.) $(1/2)mR_1^2 + (1/2)mR_2^2 + (1/2)m[(R_1+ R_2)/2]^2$.

b.) $(3/2)mR_1^2 + (1/2)mR_2^2$.

c.) $(1/2)mR_1^2 + (3/2)mR_2^2$.

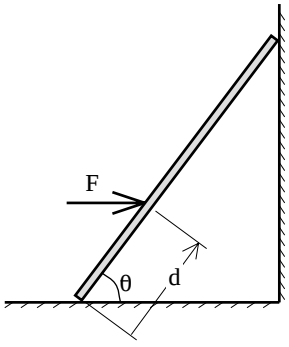
d.) None of the above.

24.) A .3 kg body moves down a 30^o incline. At a point .7 meters from the bottom the body is found to be moving with a velocity magnitude of .2 m/s. At that point a frictional force of .5 newtons begins to act. Assume the magnitude of the gravitational acceleration is 10 m/s^2.

a.) The body will not make it to the bottom of the ramp.
b.) The body will just barely make it to the bottom of the ramp.
c.) The body will make it to the bottom of the ramp, and its velocity at that point will be 4.73 m/s.
d.) None of the above.

25.) The position function for an oscillating body is $x = 20 \sin (.6t - \pi/2)$. The approximate frequency of the motion is:

a.) 20 Hz.
b.) 1.57 Hz.
c.) .6 Hz.
d.) .1 Hz.



26.) A ladder of mass m and length L sits on a frictionless floor perched against a frictionless wall. A force F acting at a distance d units up the ladder (see sketch) keeps the ladder from angularly accelerating.

a.) The torque applied by mg about the point of contact with the floor is negative because mg is directed in the -j direction.
b.) The torque applied by the normal force at the wall about the contact point with the floor is negative because N is directed in the -i direction.
c.) About the floor contact, the torque due to F is negative as it tends to motivate the ladder to angularly accelerate clockwise.
d.) Both a and b.

27.) A mass m blows into two pieces having a mass ratio of 5 to 9 (call these masses m_a to m_b,):

a.) The ratio of the kinetic energies (E_a to E_b) will be 5 to 9 respectively, and the ratio of the momenta (p_a to p_b) will be 5 to 9.

b.) The ratio of the kinetic energies (E_a to E_b) will be 9 to 5 respectively, and the ratio of the momenta (p_a to p_b) will be 9 to 5.

c.) The ratio of the kinetic energies (E_a to E_b) will be 9 to 5 and the ratio of the momenta (p_a to p_b) will be 1 to 1.

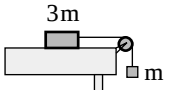
d.) None of the above is true.

28.) Two stars circle in elliptical orbits about their collective center of mass. If one star is three times the size of the second star, which of the following is true?

	momentum of center of mass	system's total angular momentum	system's total energy	for the smaller star, T^2 is proportional to r^3
a.)	conserved	conserved	not conserved	yes
b.)	not conserved	not conserved	not conserved	no
c.)	conserved	conserved	conserved	yes
d.)	conserved	conserved	conserved	no

e.) None of the above.

29.) The frictional force acting on the 3m mass shown to the right is .6mg. If the hanging mass is m, the system's acceleration is:



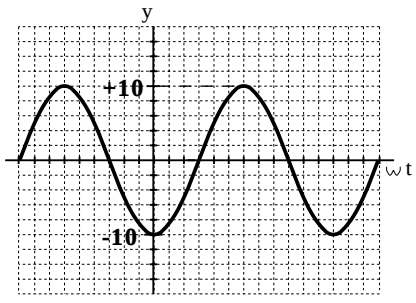
a.) .1g.

b.) .4g.

c.) .133g.

d.) Indeterminable as there isn't enough information.

30.) A graph of the position function for a body oscillating with an angular frequency of π radians per second in simple harmonic motion is shown to the right.



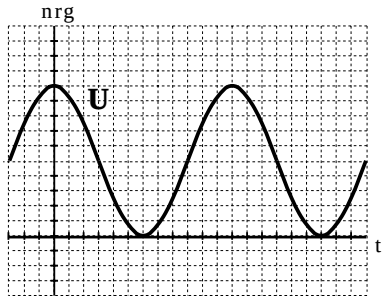
a.) The amplitude is 10, the period is 2 seconds per cycle, and the phase shift is π/2 radians.

b.) The amplitude is 20, the period is 5 seconds per cycle, and the phase shift is -π/2 radians.

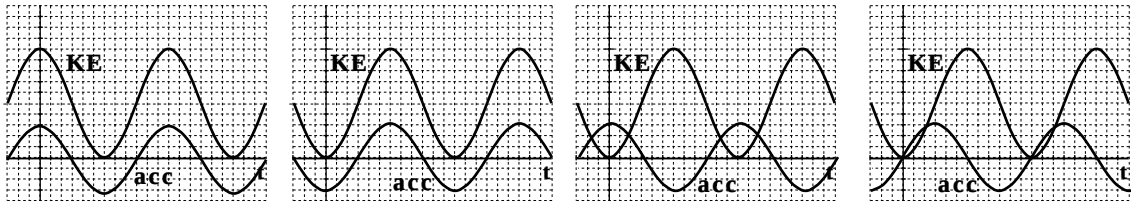
c.) The amplitude is 10, the period is 2 seconds per cycle, and the phase shift is -π/2 radians.

d.) None of the above.

31.) The potential energy vs. time graph for an oscillating system is shown to the right. The kinetic energy vs. time and acceleration vs. time graphs look like:



a.) b.) c.) d.) e.) None of these.



32.) At Point A in its motion, a 10 kg projectile moving vertically has 1500 joules of gravitational potential energy at the same time that it has 1500 joules of kinetic energy. The body starts its motion at ground level, and it takes t_1 seconds to reach the point alluded to above. (Assume the gravitational potential energy function is defined to be zero at ground level, and assume g's magnitude is 10 m/s^2).

a.) The maximum height above the ground the body reaches will be 15 meters.
b.) It will take $2t_1$ seconds for the body to reach the top of its flight.
c.) Point A is exactly halfway to the top of the body's flight.
d.) None of the above.

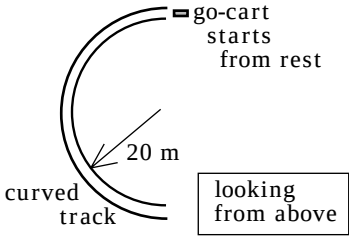
33.) A skater with arms outstretched goes into a spin during which her arms are brought in near her body. During the movement:

a.) Energy is conserved and angular momentum is conserved.
b.) Energy is conserved and angular momentum is not conserved.
c.) Energy is not conserved and angular momentum is conserved.
d.) Energy is not conserved and angular momentum is not conserved.

34.) A sound wave moving at 330 m/s has a frequency of 220 Hz. What is the phase difference between two points .3 meters apart?

a.) 1/5 of a cycle, or .4π radians.
b.) 2/3 of a cycle, or 4π/3 radians.
c.) 3/2 of a cycle, or 3π radians.
d.) None of the above.

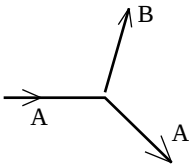
35.) A go-cart sitting on a flat, circular section of track of radius 20 meters accelerates at a rate of 1 m/s^2. If the car's mass is 100 kg and the coefficient of static friction between the road and its tires is .5, approximately how long will it take before friction can no longer hold the cart in circular motion and the cart spins out? Approximate g to be 10m/s^2.

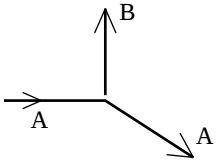


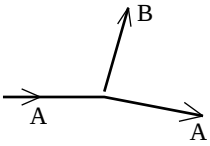
a.) 2.4 seconds.
b.) 8.2 seconds.
c.) 10.0 seconds.
d.) 13.6 seconds.

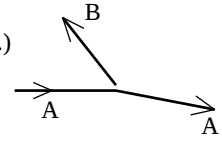
Multiple Choice -- TEST III

1.) An atomic particle A whose mass is 210 atomic mass units collides with a stationary atomic particle B whose mass is 12 atomic mass units. Which of the sketches below most closely approximates a probable collision scenario?

a.) 

b.) 

c.) 

d.) 

2.) A satellite orbits with a constant velocity v in a circular path around a planet. At one point in the flight, a short thruster burst is used to do work on the satellite, motivating it into an elliptical path. After the burst, the average velocity per orbit of the satellite is less than v. In what direction were the thrusters fired?

a.) In the direction of the satellite's motion.
b.) In the direction opposite the satellite's motion.
c.) Perpendicular and inward relative to the satellite's motion.
d.) There is not enough information given to tell.

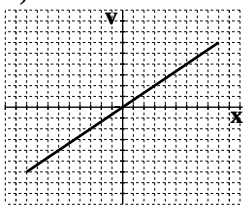
3.) A jackhammer pounds the ground with a frequency of 5 hertz. If each cycle is driven by a hydraulic spring whose effective spring constant is 800 nt/m, and if the throw of the device (i.e., the distance over which the bit is accelerated before contact) is .32 meters:

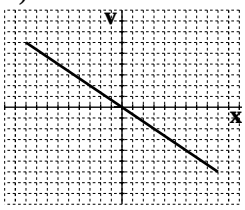
a.) The energy provided by each stroke will be approximately 40 joules.
b.) The work done per stroke will be approximately 40 joules.
c.) The power provided per stroke will be approximately 200 watts.
d.) All of the above.
e.) None of the above.

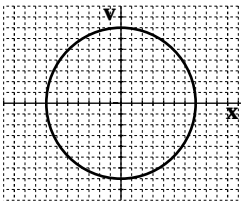
4.) A 12 kg projectile is fired straight up. Halfway to the top of its flight, its kinetic energy is 300 joules. Taking g's magnitude to be 10 m/s^2:

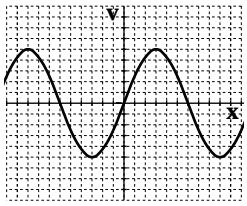
a.) The projectile's velocity at the half-way point is 50 m/s.
b.) The projectile's maximum height is 5 meters.
c.) The projectile's initial velocity (i.e., at ground level) is 8 m/s.
d.) None of the above.

5.) A graph of the velocity vs. position of a body oscillating in simple harmonic motion looks like:

a.) 

b.) 

c.) 

d.) 

e.) None of these.

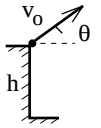
6.) During a 10 second period, a wheel has a constant torque applied to it that diminishes its angular speed from 1 rad/sec to .8 rad/sec. In 20 seconds, the angular speed will be:

a.) .64 rad/sec.
b.) .60 rad/sec.
c.) .40 rad/sec.
d.) .10 rad/sec.

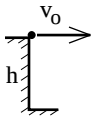
7.) At a given instant, a mass m is observed to be moving with velocity v_o up a frictional incline plane of angle θ.

a.) The direction of the mass's acceleration is opposite the direction of its velocity.
b.) The acceleration would be greater if the block were traveling down the incline.
c.) If θ were made smaller, the coefficient of kinetic friction would change, and the acceleration would be lessened.
d.) Both a and b.
e.) Both a and c.

--The following information pertains to Problems 8 through 10:
Projectiles A and B are fired at the same time from a height h meters above the ground. Both projectiles have the same muzzle velocity v_o. It takes t_1 seconds for Projectile A to get to the top of its flight. A sketch of the two situations is shown below.



projectile A



projectile B

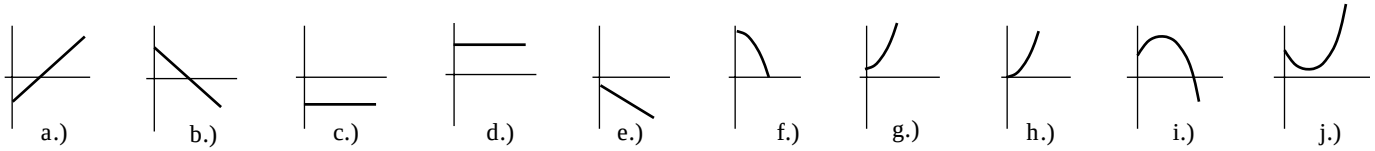
8.) If Projectile A travels a net horizontal distance equal to x_1 meters:

a.) Projectile B will travel a net horizontal distance less than x_1 meters.
b.) Projectile B will travel a net horizontal distance equal to x_1 meters.
c.) Projectile B will travel a net horizontal distance greater than x_1 meters.
d.) The relative distances between Projectiles A and B will depend upon the angle θ.

9.) The initial velocity of Projectile B will:

a.) Be equal to the initial velocity of Projectile A.
b.) Be equal to the velocity of Projectile A at time t_1.
c.) Be equal to the velocity of Projectile A just an instant before touch down.
d.) None of the above.

10.) Consider the graphs shown below:



Projectile B's:

a.) Y-component of Position vs. Time graph looks like graph a.

b.) X-component of Position vs. Time graph looks like graph j.
c.) Magnitude of Velocity vs. Time graph looks like graph g.
d.) X-component of Velocity vs. Time graph looks like graph c.
e.) Y-component of Acceleration vs. Time graph looks like graph b.

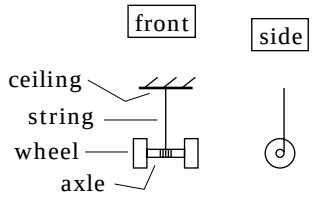
11.) A car moving with velocity v_o rounds a flat curve of radius R. If the car had been moving a bit faster, it would have lost traction and skidded.

a.) If the car had broken traction and skidded, it would have moved away from the curved path it was supposed to take. The force that moved it out of its curved path is called a centrifugal force.
b.) Kinetic friction in this situation will be less than mv_o^2/R.
c.) The coefficient of static friction is equal to mv_o^2/R.
d.) None of the above.

12.) A 10 kg projectile moving vertically has 1500 joules of gravitational potential energy at the same time that it has 1500 joules of kinetic energy. (Assume the gravitational potential energy function is defined as zero at ground level, and assume g's magnitude is 10 m/s^2). At this point:

a.) The body's velocity is exactly half that of its maximum value.
b.) The body's velocity is more than half that of its maximum value.
c.) The body's velocity is less than half of its maximum value.
d.) This cannot be answered as the maximum velocity is not a quantity we can calculate.

13.) A string is attached to the ceiling at one end and has its other end wrapped around an axle of mass m and radius R (see sketch). Attached to the axle are two wheels of mass m and radius 3R each, one at each end. The moment of inertia of the entire assembly along the axle's length is $1.5mR^2$. All numerical values are measured in the MKS system. The system is released to freefall. During the fall, the acceleration of the body's center of mass is:



a.) .40g.
b.) .67g.
c.) 2.0g.
d.) None of the above.

14.) A 1 kg mass is found to be moving 18 m/s up a 30^o incline. How fast is the mass moving 3 seconds later? Take g to be 10 m/s^2.

a.) 2 m/s.
b.) 3 m/s.
c.) 6 m/s.
d.) None of the above.

15.) A satellite of mass m orbits a planet of mass M and radius R in a circular orbit. If the orbit is 3R units above the planet's surface, the velocity of the satellite must be:

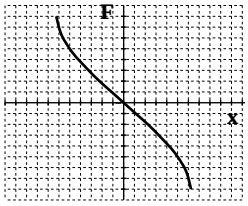
a.) $[GM/(3R)]^{1/2}$.
b.) $[GM/(4R)]^{1/2}$.
c.) $[GM/(16R^2)]^{1/2}$.
d.) $GM/(4R)$.

16.) During time t_1, a single, constant force is applied to body A changing its velocity from zero to 30 m/s. For an unknown period of time t_2, the same force is applied to a second, identical body (call it B). During t_2, B's velocity changes from zero to 15 m/s.

a.) The net work done on A will be twice as much as done on B. Also, the distance over which the force is applied to A is twice that applied to B.
b.) The net work done on A will be twice as much as done on B, but the distance over which the force is applied will not be twice as long.
c.) The net work done on A will be less than twice as much as done on B. Also, the distance over which the force is applied to A is less than twice that for B.
d.) The net work done on A will be more than twice as much as done on B. Also, the distance over which the force is applied to A is more than twice that for B.
e.) None of the above.

17.) A pendulum bob's mass is decreased by a factor of 4 while its length is increased by a factor of 4.

a.) Its frequency will stay the same as will its period.
b.) Its frequency will increase and its period will increase.
c.) Its frequency will decrease and its period will increase.
d.) Its frequency will decrease and its period stays the same.
e.) Its frequency will increase and its period will decrease.

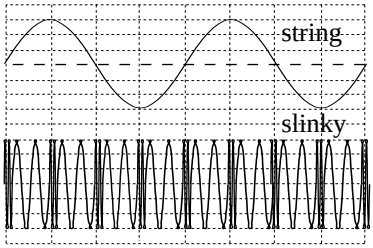


18.) For <u>small amplitude oscillations</u>, the force on an unusually designed spring is very close to being proportional to the displacement of the spring. For <u>large amplitude oscillations</u>, the force becomes non-linear. A graph of the force vs. displacement is shown to the right. When this spring executes large amplitude oscillations:

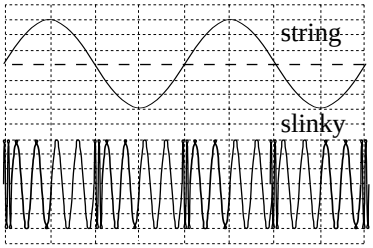
a.) Its period will be the same as in the small oscillations case, and its frequency will also be the same.
b.) Its period will be smaller than in the small oscillations case, and the frequency will also be smaller.
c.) Its period will be smaller than in the small oscillations case, but the frequency will be larger.
d.) Its period will be larger than in the small oscillations case, but the frequency will be smaller.
e.) None of the above.

19.) A string carries a transverse wave down its axis. A slinky carries a longitudinal wave down its axis. At a given instance, the string looks like the top sketches on each grid below. Assuming the slinky's frequency is twice that of the wave on the string, which of the slinky waveforms matches this situation?

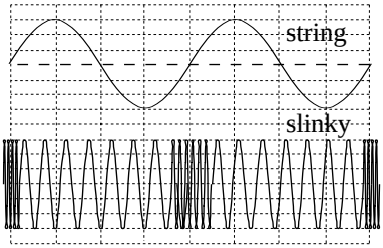
a.)



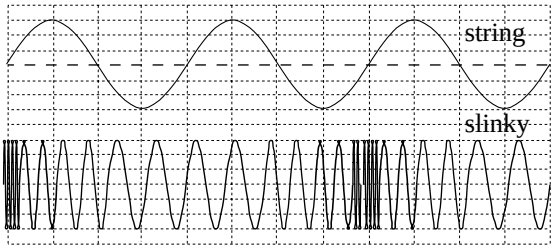
b.)



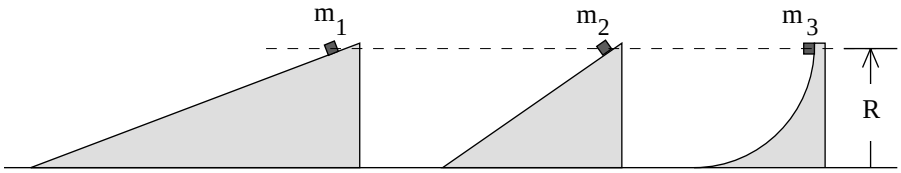
c.)



d.)



e.) There is not enough information to tell.



20.) Three frictionless inclines are shown below. The sketches are drawn approximately to scale with the curved incline being a quarter circle. The masses are different sizes. All three masses are released at once. Mass m_2 travels the same distance as does mass m_3. The first mass to the bottom will be:

a.) Mass m_1.

b.) Mass m_2.

c.) Mass m_3.

d.) Mass m_2 and m_3.

e.) They will all reach the bottom at the same time.

21.) A metal hoop and a hollow rubber ball have the same radius and mass. The two are placed side by side at the top of an incline.

a.) The hoop will arrive at the bottom first, but its rotational kinetic energy will be less than the rotational kinetic energy of the ball when it arrives at the bottom.

b.) The hoop will arrive at the bottom first, and its rotational kinetic energy will be greater than the rotational kinetic energy of the ball when it arrives at the bottom.

c.) The ball will arrive at the bottom first, but its rotational kinetic energy will be less than the rotational kinetic energy of the hoop when it arrives at the bottom.

d.) The ball will arrive at the bottom first, and its angular velocity will be greater than the angular velocity of the hoop when it arrives at the bottom.

22.) A block of mass 2 kg is found sliding with velocity 5 m/s against a frictional, circular wall of radius 3 meters. If the frictional force is a constant 4 newtons:

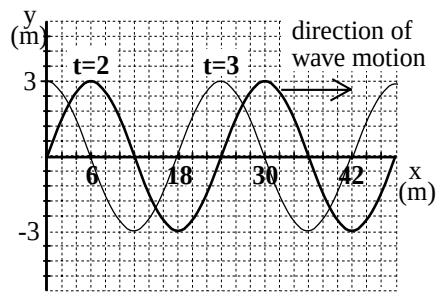
a.) The net force acting on the block at the instant alluded to is 4 newtons, and the distance the block travels before coming to rest is 1.25 meters.

b.) The net force acting on the block at the instant alluded to is 16.7 newtons, and the distance the block travels before coming to rest is 1.25 meters.

c.) The net force acting on the block at the instant alluded to is 17.1 newtons, and the distance the block travels before coming to rest is 6.25 meters.

d.) None of the above.

23.) A graph of a traveling wave as seen at t = 2 seconds and t = 3 seconds is shown to the right. The wave's velocity is:



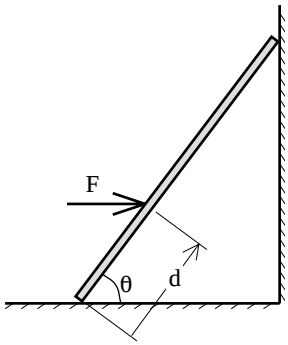
a.) 18 m/s.

b.) 6 m/s.

c.) 4 m/s.

d.) None of the above.

24.) A ladder of mass m and length L sits on a frictionless floor perched against a frictionless wall. A force F acting at a distance d units up the ladder keeps the ladder from angularly accelerating. If F is removed:



a.) There will be only two forces acting on the ladder as it begins to accelerate.

b.) The vertical force V acting at the floor will equal mg.

c.) The normal force at the wall must go to zero.

d.) None of the above.

25.) A spinning skater has rotational kinetic energy equal to 32 joules when spinning with an angular velocity of 4 rad/sec. If the skater had kept the same position but had spun with an angular velocity of only 2 rad/sec, his kinetic energy would have been:

a.) Half the original amount, or 16 joules.

b.) A quarter the original amount, or 8 joules.

c.) Double the original amount, or 64 joules.

d.) Quadruple the original amount, or 128 joules.

26.) A box of mass m sits on the bed of a flatbed pick-up truck. The maximum acceleration the truck can experience without the box breaking loose is a_1. Assume the truck accelerates to the right.

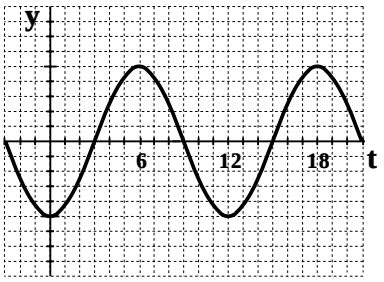
a.) The coefficient of static friction must be equal to a_1/g.

b.) The direction of the static frictional force acting on the box is to the left.

c.) The direction of the static frictional force acting on the box is to the right.

d.) Both a and b.

e.) Both a and c.



27.) A position versus time graph for a body oscillating in simple harmonic motion is shown to the right. The motion's angular frequency is:

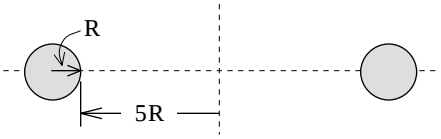
a.) $\pi/12$ radians/second.
b.) $\pi/6$ radians/second.
c.) 1/12 radians/second.
d.) None of the above.

28.) When a 10 kg body is .5 meters up a 30^o incline plane, its velocity up the incline is observed to be 20 m/s. A constant 50 newton frictional force acts on the body.

a.) The net work done on the body will be negative and the body will slow down.
b.) The net work done on the body will be positive and the body will slow down.
c.) The body will have come to rest at the end of 4 meters of motion.
d.) Both b and c.

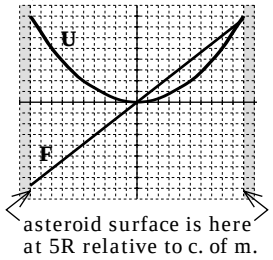
29.) A 2 kg mass with 100 joules of kinetic energy strikes a second body that is half the mass of the first. The collision is perfectly inelastic. What is the approximate velocity of the two bodies after the collision?

a.) 10.0 m/s.
b.) 8.16 m/s.
c.) 6.7 m/s.
d.) 3.3 m/s.

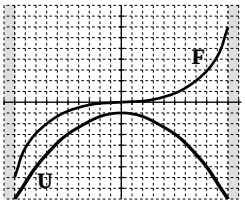


30.) Two asteroids, both of mass M and radius R, orbit one another in circular paths (the origin of the coordinate axis is at the center of mass of the two asteroids). A mass m is moved from the surface of the left asteroid to the surface of the right asteroid. The distance between the two asteroids is 10 R (see sketch). Which of the graphs shown best describes the force on the mass and the mass's potential energy as it makes it trip?

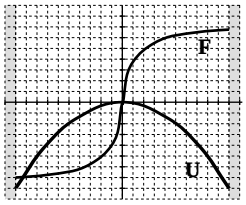
a.)



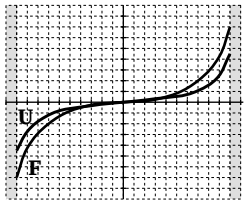
b.)



c.)



d.)



e.) None of these.

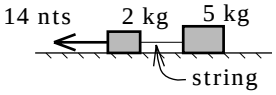
31.) The position function for an oscillating body is $x = 20 \sin (.6t - \pi/2)$. At t = 0, the body is at:

a.) Equilibrium.
b.) Its maximum positive displacement.
c.) Its maximum negative displacement.
d.) None of the above.

32.) The moment of inertia about the central axis of a disk (call it disk A) of radius R and mass m is known to equal I_A. A similar disk (call this disk B) has half the mass of A and half of its radius.

a.) The moment of inertia about disk B's central axis will be the same as that of disk A.
b.) The moment of inertia about disk B's central axis will be $(1/2)I_A$.
c.) The moment of inertia about disk B's central axis will be $(1/4)I_A$.
d.) None of the above.

33.) Two blocks on a frictionless surface are attached by a string (see sketch). If a 14 newton force is applied as shown, the tension in the string will be:



a.) 2.0 nts.
b.) 2.8 nts.
c.) 4.67 nts.
d.) 10.0 nts.

34.) A stationary mass A is struck head on and elastically by a second body B of the same mass and whose velocity before the collision is v_o. After the collision, mass A's velocity is:

a.) Zero.
b.) $v_o/4$.
c.) $v_o/2$.
d.) v_o.
e.) None of the above.

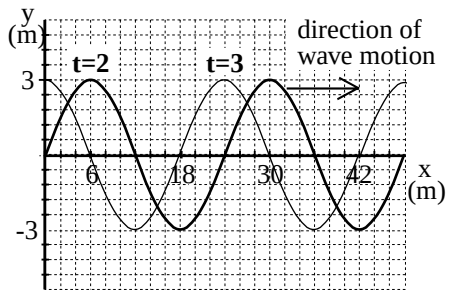
35.) A 1 kg body sits a distance h = 2 meters above a tabletop. The tabletop is d = 1.2 meters high. Three students calculate the body's potential energy. Assuming the magnitude of the gravitational acceleration is 10 m/s^2, which student got it right?

a.) Student A with a calculated potential energy of 20 joules.
b.) Student B with a calculated potential energy of 30 joules.
c.) Student C with a calculated potential energy of 0 joules.
d.) All of the above could be right.

Multiple Choice -- TEST IV

1.) A graph of a traveling wave as seen at t = 2 seconds and t = 3 seconds is shown to the right. The wavelength of the wave is:

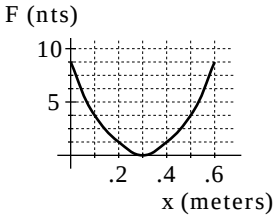
a.) 18 meters.
b.) 24 meters.
c.) 30 meters.
d.) Can't tell with the information given.
e.) None of the above.



2.) The position function for an oscillating body is $x = 20 \sin (.6t - \pi/2)$. At t = 0, the body's velocity is:

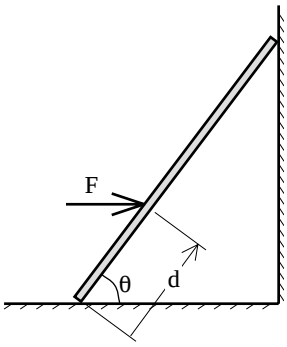
a.) 20 m/s.
b.) 12 m/s.
c.) 2.0 m/s.
d.) None of the above.

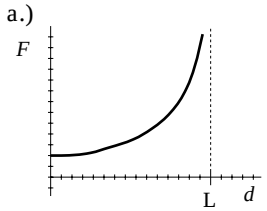
3.) A 4 kg body moves from x = 0 to x = .6 meters one-dimensionally over a frictionless surface. It takes 12 seconds to execute the trip. The single force acting on the body during the motion is graphed to the right. Ignoring the units discrepancy, the force function in this case is $F = (x - 3)^2$.

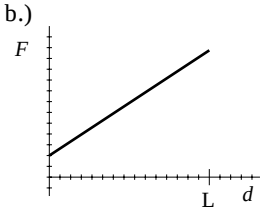


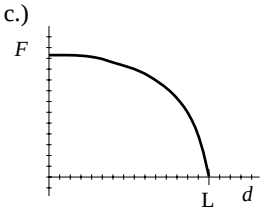
a.) The net work done by the force over the first .3 meters is negative, whereas the work done over the final .3 meters is positive.
b.) The net work over the entire interval is $(x^3/3) - 3x^2 + 9x$.
c.) The work done between x = .1 meters and x = .6 meters is 3.5 joules.
d.) None of the above.

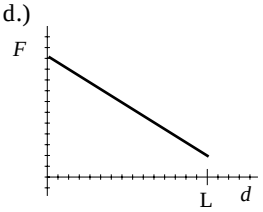
4.) A ladder of mass m and length L sitting on a frictionless floor at an angle θ is perched against a frictionless wall. A horizontal force F acting at a distance d units up the ladder keeps the ladder from angularly accelerating. Which graph characterizes the relationship between F and d?



a.)


b.)


c.)


d.)


e.) None of these.

5.) An ideal spring is hung free in the vertical. A 5 kg mass is attached without elongating the spring, and the mass is released from that point to oscillate up and down. If the mass had been gently lowered (it wasn't, but if it had been), it would have moved 2.5 meters downward before coming to equilibrium. Approximate g to equal 10 m/s^2:

a.) The spring constant is 2 nts/meter, the angular frequency of the motion will be 2 radians/second, and the maximum velocity will be 5 m/s.

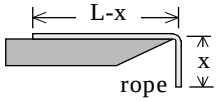
b.) The spring constant is 20 nts/meter, the angular frequency of the motion will be 4 radians/second, and the maximum velocity will be 2 m/s.

c.) The spring constant is 2 nts/meter, and neither the angular frequency nor the maximum velocity can be calculated as there is not enough information to do so as the problem stands.

d.) The spring constant is 20 nts/meter, the angular frequency of the motion will be 2 radians/second, and the maximum velocity will be 5 m/s.

e.) None of the above.

6.) At a given instant, a fat rope of length L and mass density λ (note: mass density is the mass per unit length of rope) is as shown hanging over the edge of a table. Assume the normal force is constant over the table's length, and that N goes to ZERO at the table's edge (versus increasing in magnitude as it would do in real life).



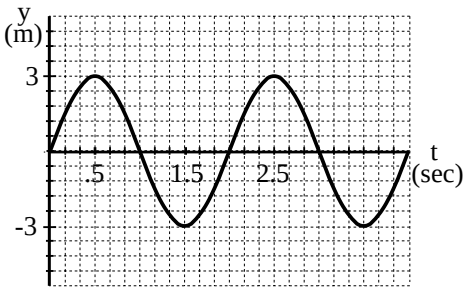
a.) Assuming the table is frictionless, the rope's acceleration will be dependent only upon λ.

b.) Assuming the table is frictional and the rope is stationary and not accelerating, the coefficient of static friction between the rope and the table will be y/(L-y).

c.) Back to the frictionless situation, the acceleration will be a constant.

d.) Both b and c.

7.) At a particular position, the displacement vs. time graph for a wave is shown to the right.



a.) The amplitude of the wave is 3 meters, its wavelength is 2 meters, and its frequency is .5 Hz.

b.) The amplitude of the wave is 6 meters, it is impossible to tell what the wavelength is from the information given, and its frequency is 2 Hz.

c.) The amplitude of the wave is 3 meters, it is impossible to tell what the wavelength is from the information given, and its frequency is .5 Hz.

d.) None of the above.

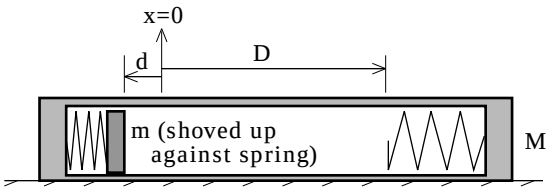
8.) A mass m is released 6R meters from the surface of a moon whose radius is R and whose mass is M. If released from rest, how fast will the mass be traveling just before it hits the surface of the planet?

a.) $[2GM/(6R)]^{1/2}$.

b.) $[-2GM/(7R)]^{1/2}$

c.) $[12GM/(7R)]^{1/2}$.

d.) None of the above.



9.) An ideal spring (spring constant k) is mounted inside a hollowed out block whose mass is M. With M held stationary, a second mass m is pushed up against the spring until the spring's displacement is d. A second spring with the same spring constant is positioned at the end of the hollow. The net distance between the ends of the two unsprung springs is D (see sketch). The shaft applies a constant frictional force f on m only over the distance marked D in the sketch. Assume M is fixed to the ground so that it cannot move. The spring is released accelerating m.

a.) Just after leaving the spring, m's velocity will be $(k/m)^{1/2}d$.

b.) The impulse provided to m by the spring will equal $(km)^{1/2}d$.

c.) If m = .1 kg, D = 1 meter, f = 1 nt, and m's velocity just as it leaves the spring is 7 m/s, the time it takes m to reach the second spring will be approximately .16 seconds.

d.) Both a and b but not c.

e.) Responses a, b, and c..

10.) Moving under the influence of gravity, a body's kinetic energy is KE_o. If, due to its natural motion in the gravitational field, the body's velocity changes to one-third of its original value, the change of the body's potential energy will equal:

a.) $(1/3)KE_o$.

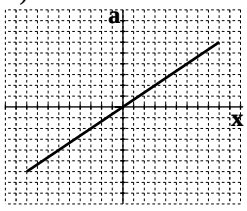
b.) $(1/9)KE_o$.

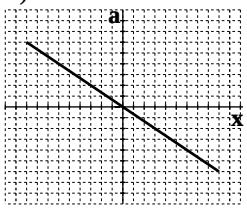
c.) $(8/9)KE_o$.

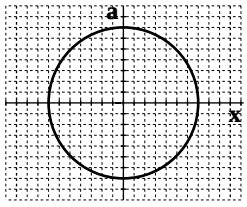
d.) $(3)KE_o$.

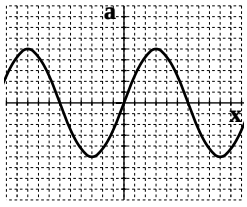
e.) $(9)KE_o$.

11.) The graph of the acceleration vs. position of a body oscillating in simple harmonic motion is:

a.) 

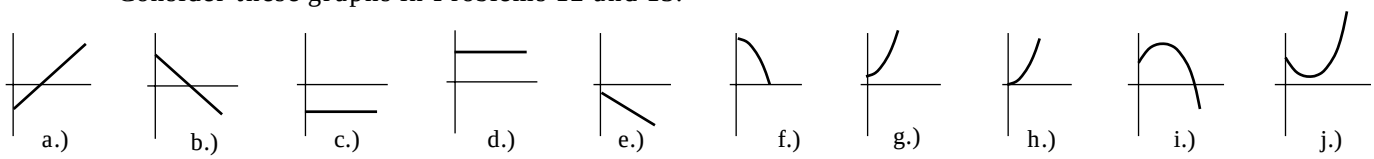
b.) 

c.) 

d.) 

e.) None of these.

--Consider these graphs in Problems 12 and 13:



12.) If all of the graphs were that of Position vs. Time, the only graphs listed below that would fit a kinematic situation would be:

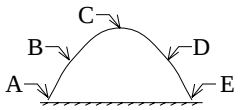
a.) Graphs a, b, and e.
b.) Graphs a, b, c, d, and e.
c.) Graph f, g, h, i, and j.
d.) All of the graphs.
e.) None of the graphs.

13.) If all of the graphs were that of Velocity vs. Time, the only graphs listed below that would fit a kinematic situation would be:

a.) Graphs a, b, and e.
b.) Graphs a, b, c, d, and e.
c.) Graphs f, g, h, i, and j.
d.) All of the graphs.
e.) None of the graphs.

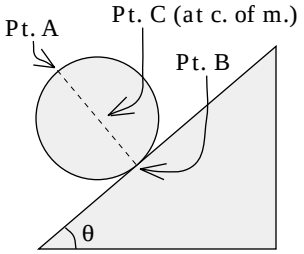
--end of section--

14.) A projectile is fired in a frictionless environment as shown to the right.



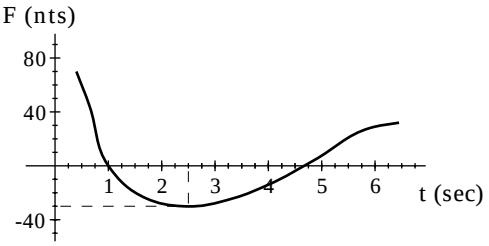
a.) The work gravity does while the body moves from Point B to Point D is zero.
b.) The work gravity does while the body moves in its downward arc between Point C and Point D is negative.
c.) The amount of work gravity does as the body moves from Point A to Point E is twice the amount of work done between Point A and Point C.
d.) Both a and b.
e.) Both a and c.

15.) A ball rolls down an incline. Its angular acceleration is:



a.) $mg(R \cos\theta)/I_{cm}$, where I_{cm} is the moment of inertia about the ball's center of mass.
b.) fR/I_B, where f is the frictional force (equal to $mg \sin\theta - ma_{cm}$) and I_B is the moment of inertia about Point B.
c.) $a_A/(2R)$, where a_A is the acceleration of Point A.
d.) Both a and c.
e.) Both b and c.

16.) A non-linear frictional force in the x direction is graphed to the right. If it is the only force acting on a body, and if the body's mass is 3 kg:



a.) At t = 2.5 seconds, the mass is moving in the -x direction.
b.) At t = 2.5 seconds, the mass's velocity will be approximately zero.
c.) The acceleration of the mass at t = 2.5 seconds is approximately -10 m/s^2.

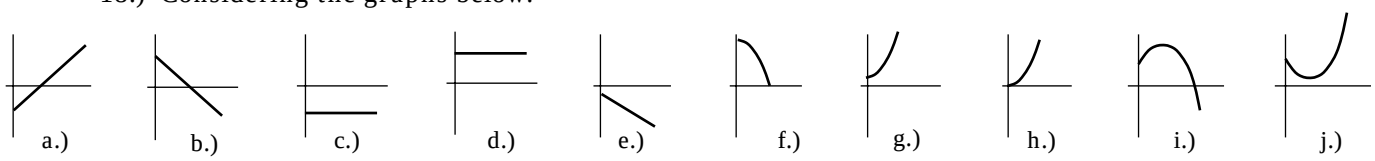
d.) The velocity of the body would either have been positive until t = 1 second, whereupon it would have changed, or vice versa.
e.) None of the above.

--The following information pertains to Problems 17 through 20: Projectiles A, B, C, and D are fired at the same time from a height h meters above the ground. With the exception of Projectile D, which is dropped from rest, all the projectiles (i.e., Projectiles A, B, and C) have the same muzzle velocity v_o, (though each is fired at a different angle--see the sketches below and note that the angle defined as θ is the same in all cases). It takes t_1 seconds for Projectile A to get to the top of its flight. It takes t_2 seconds for Projectile D to reach the ground.

Projectile A Projectile B Projectile C Projectile D

17.) The initial velocity of Projectile C will be:
a.) The same as the initial velocity of Projectile B. The two will also have the same acceleration.
b.) The same as Projectile B after time t_1. The two will also have different accelerations.
c.) The same as Projectile A after time $2t_1$. The two will also have the same acceleration.
d.) None of the above.

18.) Considering the graphs below:



a.) b.) c.) d.) e.) f.) g.) h.) i.) j.)

Projectile C's:
a.) Y-component of Position vs. Time graph looks like graph e.
b.) X-component of Position vs. Time graph looks like graph a.
c.) Y-component of Velocity vs. Time graph looks like graph e.
d.) X-component of Velocity vs. Time graph looks like graph c.
e.) Y-component of Acceleration vs. Time graph looks like graph b.

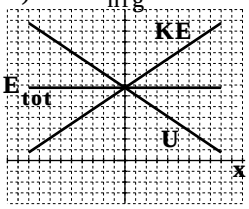
19.) The time t_2:
a.) Depends only on h and constant(s).
b.) Is the same time it takes Projectile B to hit the ground.
c.) Is more than the time it takes Projectile C to hit the ground, but less than the time it takes Projectile A to hit.
d.) Both a and b, but not c.
e.) All of the above except d.

20.) If h were doubled, Projectile D's:

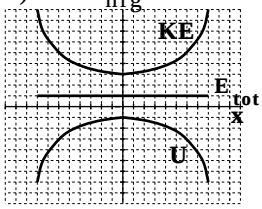
a.) Time to touch down would double.
b.) Velocity just before touch down would double.
c.) Acceleration just before touch down would double.
d.) None of the above.

----------end of section----------

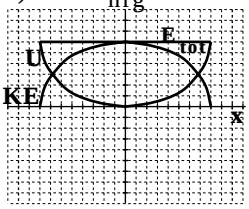
a.)



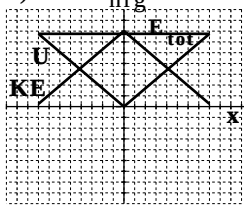
b.)



c.)



d.)



e.) None of these.

21.) Which graph combination depicts the potential energy, the kinetic energy, and the total energy of an ideal vibrating spring?

22.) <u>In theory</u>, a mass must travel how fast to escape the earth's gravitational field?

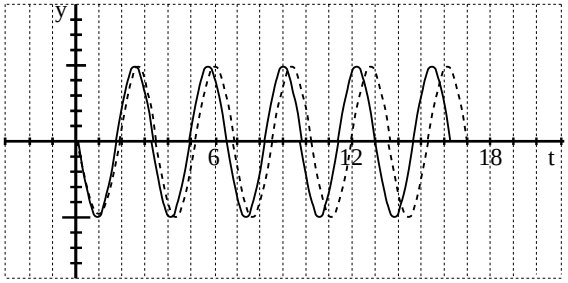
a.) It is not possible to tell for this situation because the mass of the object was not given.
b.) A mass can never escape the earth's gravitational field.
c.) $(2GM/R_e^2)^{1/2}$.
d.) None of the above.

23.) When a 10 kg body is .5 meters up a 30^o incline plane, its velocity is observed to be 20 m/s. A constant 50 newton frictional force acts on the body. Over the next 4 meters (note that a 4 meter run at 30^o will produce a rise of 2 meters):

a.) The work gravity does on the body will be greater than the work friction does. Also, when the body finally gets to the top of its run, it will stop momentarily, then begin to accelerate back down the incline.
b.) The work gravity does on the body will equal the work friction does. Also, when the body finally gets to the top of its run, it will stop and remain there (i.e., it won't be accelerated down the incline).
c.) The work gravity does on the body will be the same as the work friction does. Also, when the body finally gets to the top of its run, it will stop momentarily, then begin to accelerate back down the incline.
d.) The work gravity does on the body will be less than the work friction does. Also, when the body finally gets to the top of its run, it will stop and remain there (i.e., it won't be accelerated down the incline).

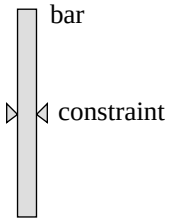
24.) Newton's Second Law is applied to a system. After a free body diagram is drawn and the forces summed, the equation $-32x^3 = 2a$ emerges, where x is the position and a is the acceleration of the body.

a.) This equation does not characterize an oscillatory system.
b.) This equation does characterize an oscillatory system and the motion is simple harmonic in nature.
c.) This equation does characterize an oscillatory system and the motion's frequency is 4 radians per second.
d.) Both Response b and c.
e.) None of the above.



25.) Two waves exist simultaneously in the same medium (they are graphed on the same axis). The beat frequency for the situation is approximately:

a.) (2/13) - (2/17).
b.) (4/13) - (4/17).
c.) (8/13) - (8/17).
d.) None of the above.



26.) A 1.2 meter long bar is clamped at its center is tapped gently once at one end. The bar rings with a frequency equal to 200 hertz. The wave velocity of the disturbance as it moves through the bar will be:

a.) 960 m/s.
b.) 480 m/s.
c.) 240 m/s.
d.) None of the above.

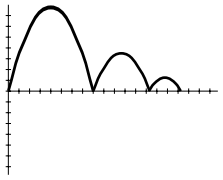
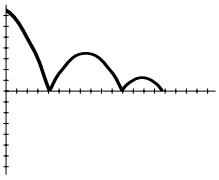
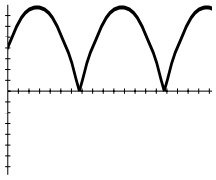
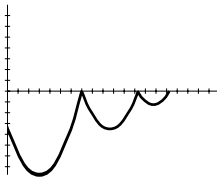
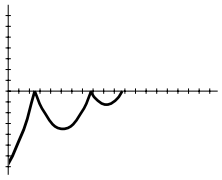
27.) During a 10 second time interval, the power provided to a body by a constant force is 375 watts. If the body was initially at rest:

a.) The amount of work done during the interval is 3750 joules.
b.) Doubling the time over which the power-producing force is applied will quadruple the distance traveled.
c.) Doubling the time over which the power-producing force is applied will double the power provided to the body.
d.) All of the above.

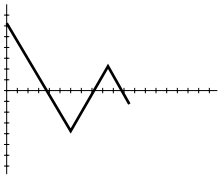
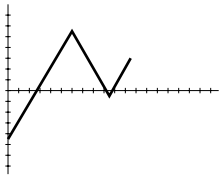
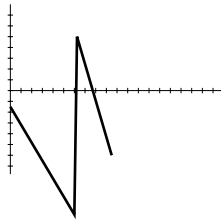
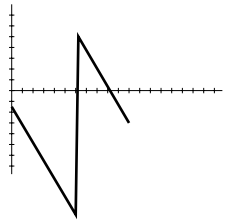
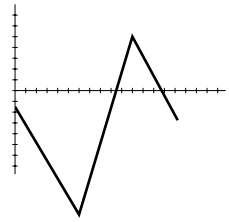
28.) A rotating disk has moment of inertia I and kinetic energy K. The body's angular momentum will be:

a.) K^2I.
b.) $2K^2/I$.
c.) $(K^2I)^{1/2}$.
d.) $(2KI)^{1/2}$.

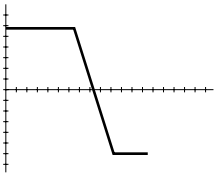
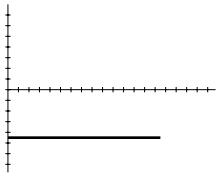
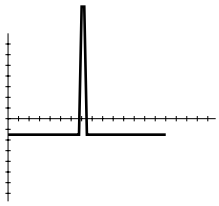
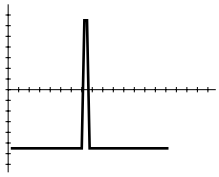
29.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's position as a function of time?

a.)  b.)  c.)  d.)  e.) 

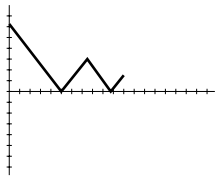
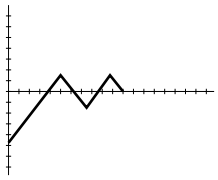
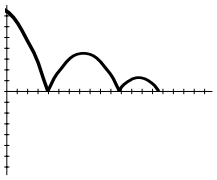
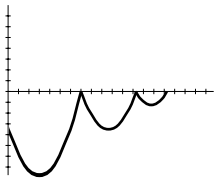
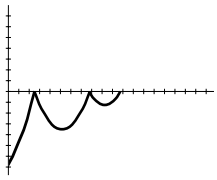
30.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's velocity as a function of time?

a.)  b.)  c.)  d.)  e.) 

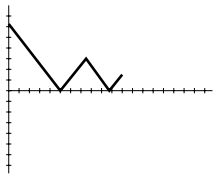
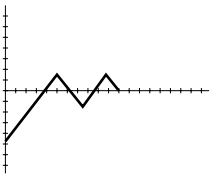
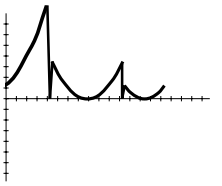
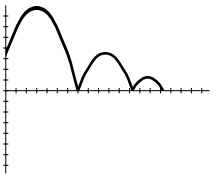
31.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's acceleration as a function of time?

a.)  b.)  c.)  d.)  e.) None of these graphs.

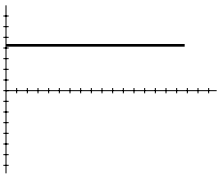
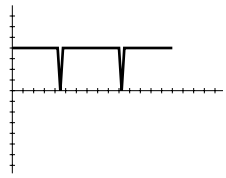
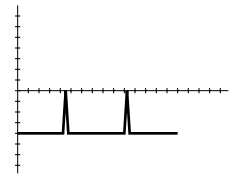
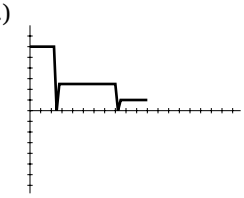
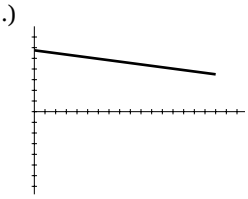
32.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's potential energy as a function of time, assuming $U_{grav} = 0$ at ground level?

a.)  b.)  c.)  d.)  e.) 

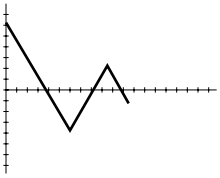
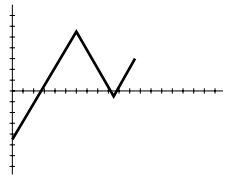
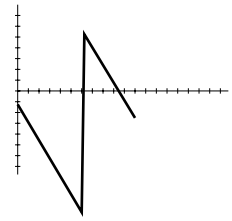
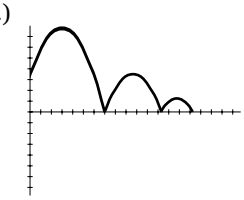
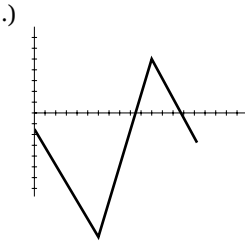
33.) A mass m is fired downward (i.e., vertically) with velocity v_0 off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's kinetic energy as a function of time?

a.)  b.)  c.)  d.)  e.) None of the graphs.

34.) A mass m is fired downward (i.e., vertically) with velocity v_0 off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's total energy as a function of time?

a.)  b.)  c.)  d.)  e.) 

35.) A mass m is fired downward (i.e., vertically) with velocity v_0 off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's momentum as a function of time?

a.)  b.)  c.)  d.)  e.) 

SOLUTIONS

The problems in this book are designed to make you THINK about the ideas being presented. Understand the ideas and you should be able to do the problems.

A couple of suggestions:

1.) Study the book. Try the examples. Once done, attempt the chapter-end problems.

2.) Try to do each chapter-end problem WITHOUT using your book, your notes, or your friends (i.e., treat it like a *test* problem). Above all, do not simply look at a problem, wait fifteen seconds, decide you can't do it, then turn to the *Solutions*.

3.) If you are confused, re-study the book, come see me, or talk to a knowledgeable friend. Use the *Solutions* only after all else has failed. Otherwise, all you will be doing is riding on *my* ability to dispatch the pesky critters. Believe me, that is not going to help you in the long run.

CHAPTER 1 -- MATH REVIEW

1.1) Both *Parts a* and *b* are straight vector addition, graphical style. You need a centimeter stick and protractor to do them.

a.) In this case, the velocity component contributed by the rower is a vector directed straight across the river. It has a magnitude of 5 miles per hour. The velocity component contributed by the river's motion is directed downward. It has a magnitude of 2 miles per hour. Vectorial addition of the two vectors yields the net velocity of the boat during the crossing (see Figure 1). A protractor should be used to measure the angle. It turns out to be -22^{o} (minus because it is *below* the reference line straight across the river). A centimeter stick measures the resultant at 5.4 centimeters. Multiplying by the scaling factor, we get a net velocity vector of 5.4 miles per hour.

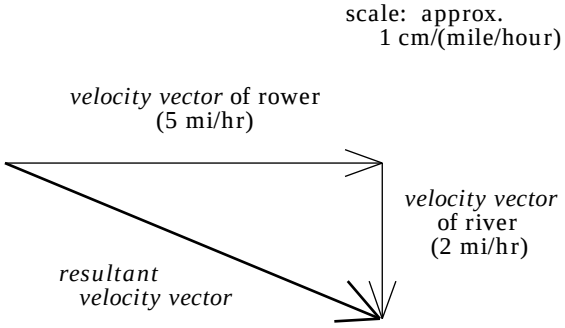


FIGURE 1

Note: The Xerox machine ever-so-slightly miniaturizes whatever it copies. When I used the original to make my measurements, I got the answer quoted above. If you use a centimeter stick and protractor on the Xeroxed copy, you will get an answer that is a bit off. If you do the problem on your own, you should get the same solution as I did to a good approximation.

b.) The direction of any object's *net velocity* is the same as the direction of the object's *net motion*. We want the net motion to be along the direction shown in Figure I on page 22. The question that is really being asked is: "What boat-velocity vector

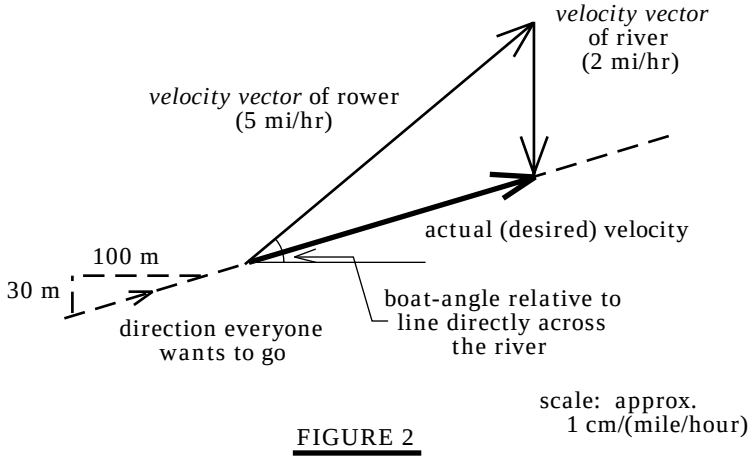


FIGURE 2

must exist if, when added to a water-velocity vector, yields the net-desired-boat-motion?" Clearly it must be upstream. Playing with a compass and protractor, the solution is approximately 40^o (see Figure 2 on the previous page) and the velocity is approximately 4 miles/hour.

1.2) This is also a problem for graphical vector addition.

a.) The vectors involved are shown to the right. A protractor was used to determine an angle of approximately 17^o north of west.

b.) The diagram is to scale. The resultant, measured by a centimeter stick, is approximately 5.6 centimeters. Multiplying by the scaling factor yields (5.6)(20) = 112 miles. This is the net displacement.

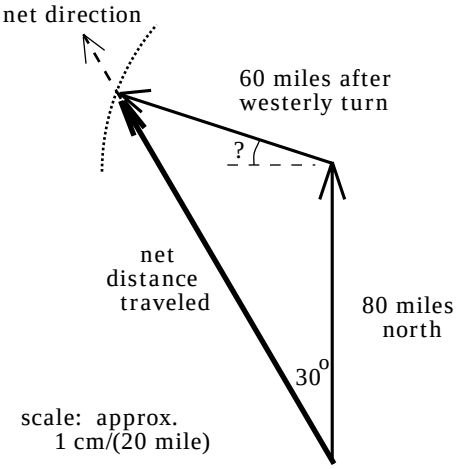


FIGURE 3

Note: Once again, if you use a centimeter stick and the sketch in Figure 3 to follow my steps in doing this problem, you will probably find your result and my result differing just a bit. I did the problem off the original drawing. Xeroxing slightly shrinks the material being copied.

1.3) The graph of vectors ***P*** and ***T*** are shown in Figure 4.

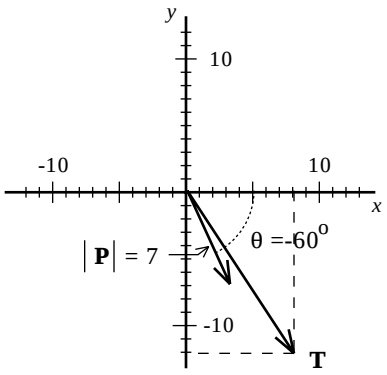


FIGURE 4

1.4) Using the graph given in problem-Figure II:

in unit vector notation: $\mathbf{A} = 2.25\mathbf{i} + .5\mathbf{j}$.
$\mathbf{C} = -.72\mathbf{i} - \mathbf{j}$.

in polar notation: $\mathbf{B} = 1.3 \angle 125^o$.
$\mathbf{D} = 2.6 \angle -20^o$.

Note: I've made the problem tricky as far as the two polar angles go. All angles must be measured either clockwise or counterclockwise FROM THE +*x AXIS*!

1.5) Each axis has its own scale. If the scales had been the same, relative lengths would have had significance. They aren't, hence they don't.

1.6) This is all straight vector manipulation. You should be able to do this kind of thing in your sleep!

a.) $-(1/3)\mathbf{A} = (-1/3)((-8\mathbf{i} + 12\mathbf{j})$
$= (8/3)\mathbf{i} - 4\mathbf{j}$.

b.) $-6\mathbf{E} = (-6)((12 - 225^{o})$
$= (72 \angle 45^{o})$.

Note: in polar notation, a *scalar* multiplies the *magnitude* of the vector while a *negative sign* adds (or subtracts) 180^{o} to the *angle* of the vector.

c.) $\mathbf{A} + \mathbf{B} - \mathbf{C} = (-8\mathbf{i} + 12\mathbf{j}) + (-4\mathbf{i} - 3\mathbf{j}) - (5\mathbf{i} + 6\mathbf{j} - 7\mathbf{k})$
$= -17\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$.

d.) A bit of trig and Figure 5 show ***E's*** unit vector equivalent. Written in *unit vector notation*:

$\mathbf{E} = -8.48\mathbf{i} - 8.48\mathbf{j}$.

Note: The negative signs had to be put in manually.

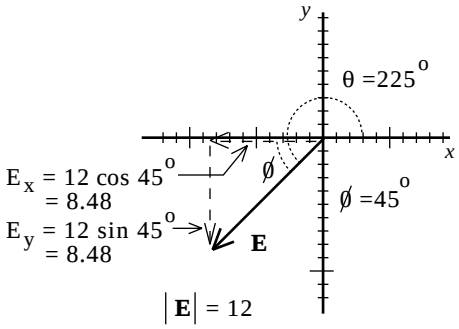


FIGURE 5

e.) A bit of trig and Figure 6 shows **F**'s *unit vector* equivalent to be:

$\mathbf{F} = -.518\mathbf{i} + 1.93\mathbf{j}$.

Note: Again, the negative signs had to be placed manually.

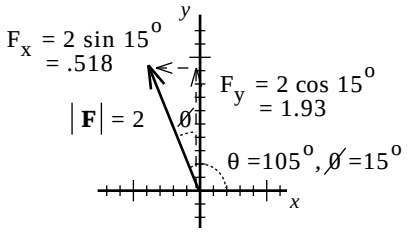


FIGURE 6

f.) The Pythagorean relationship, a bit of trig, and Figure 7 shows **A**'s *polar* equivalent to be:

$$\mathbf{A} = 14.4 \angle 123.7^{o}.$$

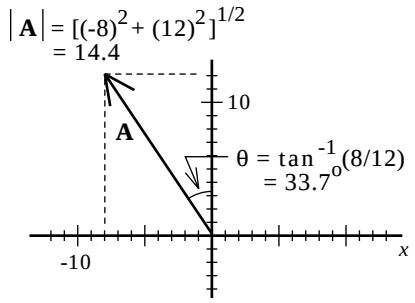


FIGURE 7

Note: *Tan*$^{-1}$ *(8/12) = 33.7*^{o}. Adding that to 90^{o} gives us the calculated 123.7^{o}. An alternate way to do the problem: take the actual components, minus signs and all, and determine *tan*$^{-1}$ (A_y/A_x). That is, *tan*$^{-1}$ *(12/-8) = -56.3*^{o}. This is a *fourth quadrant angle,* whereas our vector is a *second quadrant angle*. To adjust the situation, we must add 180^{o} to get the appropriate 123.7^{o} angle. Either way will do.

g.) The Pythagorean relationship, a bit of trig, and Figure 8 shows **B**'s *polar* equivalent to be:

$$\mathbf{B} = 5 \angle 216.9^{o}.$$

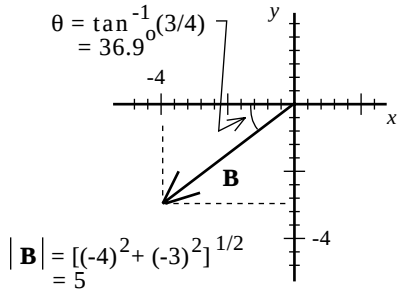


FIGURE 8

Note: *Tan*$^{-1}$ *(3/4) = 36.9*^{o}. Added to 180^{o} gives us the appropriate 216.9^{o}. An alternate way to do the problem: take the actual components, *minus signs and all*, and determine the *tan*$^{-1}$ (A_y/A_x), or *tan*$^{-1}$*(-3/-4) = 36.9*^{o}. This is a *first quadrant angle*, whereas our vector is a *third quadrant angle*. To adjust the situation, we must add 180^{o} to get the appropriate 216.9^{o} angle. Either way will do.

h.) $\mathbf{A}\cdot\mathbf{C} = A_xC_x + A_yC_y + A_zC_z$
$= (-8)(5) + (12)(6) + (0)(-7)$
$= 32.$

i.) $\mathbf{D}\cdot\mathbf{E} = |\mathbf{D}||\mathbf{E}|\cos\phi$
$= (7)\ (12)\cos 75^{o}$
$= 21.74.$

j.)

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 12 & 0 \\ -4 & -3 & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + [(-8)(-3) - (12)(-4)]\mathbf{k}$$
$$= 72\mathbf{k}.$$

k.)

$$\mathbf{C} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 6 & -7 \\ -4 & -3 & 0 \end{vmatrix}$$

$$= [(6)(0) - (-7)(-3)]\mathbf{i} + [(-7)(-4) - (5)(0)]\mathbf{j} + [(5)(-3) - (6)(-4)]\mathbf{k}$$
$$= -21\mathbf{i} + 28\mathbf{j} + 9\mathbf{k}.$$

l.) $\mathbf{D}\times\mathbf{E} = |\mathbf{D}||\mathbf{E}|\sin\phi\ (-\mathbf{k})$
$= (7)\ (12)\ \sin 75^{o}$
$= -81.1\mathbf{k}.$

Note: The *-**k** direction* came from using the *right-hand rule* on **D**x**E**.

1.7.) A *dot product* gives you a scalar equal to the *magnitude of **A*** <u>times</u> the magnitude of the component of **B** *parallel to **A***. OR:

It gives you a scalar equal to the *magnitude of **B*** <u>times</u> the magnitude of the component of **A** *parallel to **B***.

1.8.) A *cross product* gives you a vector whose magnitude is equal to the *magnitude of **A*** <u>times</u> the magnitude of the component of **B** *perpendicular to **A***. OR:

It gives you a vector whose magnitude is equal to the *magnitude of **B*** <u>times</u> the magnitude of the component of **A** *perpendicular to **B***.

The direction of the cross product is perpendicular to the plane defined by **A** and **B**.

CHAPTER 4 -- KINEMATICS

4.1)

a.) The *total* distance traversed (versus the net displacement) divided by the elapsed time. That scalar is:

$$\begin{aligned} s &= \text{dist/time} \\ &= (440\text{ m})/(49\text{ sec}) \\ &= 8.98\text{ m/s}. \end{aligned}$$

b.) The magnitude of the *average velocity* is the *net displacement* divided by the elapsed time. That is:

$$\begin{aligned} \mathbf{v} &= (\text{net disp})/\text{time} \\ &= (0\text{ m})/(49\text{ sec}) \\ &= 0\text{ m/s}. \end{aligned}$$

Making sense of this: The woman finished where she started, so her net displacement is zero. The average velocity tells us the constant velocity she would have to travel to effect that displacement in 49 seconds. That velocity is zero!

c.) You know nothing about her instantaneous velocity at any point in the motion--not even at the beginning (for all you know, she may have had a running start).

4.2) The only options we have are:

a.) The curve could be an *acceleration versus time* graph. For a given *time interval*, the area under such a curve yields the *velocity change* over the interval. As we have *not* been told about *time intervals* or *velocity changes*, we don't have enough information to say *yea* or *nay*.

b.) The curve could be a *velocity versus time* curve. If that be so, the velocity magnitude at $t = 1$ *second* should be $v = -1$ *m/s*. Unfortunately, the graph reads +1 at $t = 1$ *second*. The graph is not a *velocity versus time* graph.

c.) The slope of the tangent to a *distance versus time* graph at any given point equals the *instantaneous velocity* at that point in time. Taking the curve's slope at $t = 1$ *second* we get -1. The curve must, therefore, be a *position versus time* graph.

4.3)

a.) During a given time interval, the *net displacement* of a body is equal to the area under the *velocity versus time* curve. The *area under the curve* between $t = .5$ *seconds* and $t = 3$ *seconds* has two parts: one above the axis and one below the axis.

Noting that one square on the graph is equal to 1/8 meter, the *area above the axis* is eyeballed at approximately (+2.3 squares)(1/8 meter/square), or +.29 meters. The *area below the axis* is eyeballed at approximately (-22 squ)(1/8 meter/squ), or -2.75 meters.

The net distance traveled is approximately (.29 m) + (-2.75 m) = -2.46 meters. That is, the ant travels 2.46 meters to the left of its starting point.

Note 1: This number is *not* the ant's final position-coordinate. It is only the net distance the ant traveled *from its original position* during the time interval.

Note 2: Written out fully, $\Delta \boldsymbol{x} = -2.46\boldsymbol{i}$ *meters*.

b.) *Average velocity* is defined as the *net displacement per unit time* over a time interval. In *Part a*, we determined the ant's *net displacement* between the .5 second and 3 second mark as -2.46 meters. As the *time interval* is 2.5 seconds:

$$\begin{aligned} v_{avg} &= (\text{net disp})/(\text{time interval}) \\ &= (-2.46\mathbf{i} \text{ meters})/(2.5 \text{ seconds}) \\ &= -.98\mathbf{i} \text{ m/s}. \end{aligned}$$

c.) Taking the information directly off the graph, the ant's velocity:

i.) At $t = .5$ *seconds* is approximately $1.3\mathbf{i}$ m/s.

ii.) At $t = 3$ *seconds* is approximately $-2.1\mathbf{i}$ m/s.

d.) At a given point in time, the acceleration (i.e., the change of velocity with time) is the *slope of the tangent to the velocity curve*. Eyeballing it (watch the graph scaling), the slope of the tangent at:

i.) $t = .5$ *seconds* is approximately (-2)/(.5), or $\boldsymbol{a} = -4\boldsymbol{i}\ m/s^2$.

ii.) $t = 3$ *seconds* is approximately (1)/(2.25), or $\boldsymbol{a} = +.44\boldsymbol{i}\ m/s^2$.

e.) When the velocity is positive, the ant is moving in the *+x direction* (the direction of motion is the direction of the instantaneous velocity). This occurs between $t = .3$ *seconds* and $t = 1$ *second*.

f.) When the velocity is zero, the ant is standing still. This occurs at $t = 1$ *second*.

g.) The acceleration is zero when the slope of the tangent to the curve is zero (that is, when the velocity is not changing). This occurs at times $t = 2.9$ *seconds* and $t = 3.6$ *seconds*.

h.) When the *slope* of the velocity graph *changes*, we have what are called *inflection points*. To determine one, we need to observe two things:

i.) When the acceleration is *not changing*, the acceleration is constant.

ii.) A constant acceleration generates a *velocity function* that *changes linearly* (i.e., the velocity changes at a constant rate).

The interval over which the velocity seems to be *changing* linearly is between $t = .3$ *seconds* and $t = .5$ *seconds*, at *3.25 seconds*, and maybe between $t = 4$ *seconds* and $t = 4.3$ *seconds* (though this latter suggestion is debatable).

4.4) The velocity function is $\boldsymbol{v}(t) = (3e^{-1.5kt}\boldsymbol{i} - 4k_1t\boldsymbol{j})$ *m/s*. Noting that the constant $k = 1\ sec^{-1}$ (the only way the exponent in e^{-kt} can be unitless is if *k's* units are *1/seconds*) and $k_1 = 1\ m/s^2$:

a.) Omitting the *k* terms (their magnitudes are "1"--they are present only for the sake of units), we can plug $t = 2$ *seconds* into the *velocity function* yielding:

$$\begin{aligned}\mathbf{v}(t) &= (3e^{-1.5t}\mathbf{i} - 4t\mathbf{j})\ m/s.\\ &= [3e^{-(1.5)(2)}\mathbf{i} - 4(2)\mathbf{j}]\ m/s\\ &= [.15\mathbf{i} - 8\mathbf{j}]\ m/s.\end{aligned}$$

b.) *Acceleration* is the *time derivative* of the *velocity function*, or:

$$\begin{aligned}\mathbf{a}(t) &= d\mathbf{v}(t)/dt\\ &= d(3e^{-1.5t}\mathbf{i} - 4t\mathbf{j}) / dt\\ &= [-4.5e^{-1.5t}\mathbf{i} - 4\mathbf{j}]\ m/s^2.\end{aligned}$$

c.) Plugging $t = 2$ *seconds* into the acceleration function yields:

$$\mathbf{a} = [-4.5e^{-1.5(2)}\mathbf{i} - 4\mathbf{j}]\ m/s^2.$$
$$= [-.22\mathbf{i} - 4\mathbf{j}]\ m/s^2.$$

d.) Using the *kinematic equations* requires a *constant acceleration*. In this case, there are *two* accelerations to consider--one in the *x direction* and one in the *y direction* (they are independent of one another).

i.) The *x component* of acceleration is *time dependent*. That is, it is not a constant. As such, the *kinematic equations* will <u>not</u> be valid for the body's *x-type motion*.

ii.) The *y component* of the acceleration is constant ($a_y = 4\ m/s^2$). Kinematics <u>will</u> work on problems associated with the body's *y-type motion*.

e.) The body's *position function* ***r*** is determined by integrating the *velocity function* ***v(t)*** over an indefinite time interval (i.e., by solving $\int(\boldsymbol{v})dt$ without evaluating between times t_1 and t_2). Executing that operation, we get:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)dt \\ &= \int \left[(3e^{-1.5t}\mathbf{i} - 4t\mathbf{j}\right]dt \\ &= \int \left[(3e^{-1.5t}\mathbf{i}\right]dt + \int [(-4t\mathbf{j}]dt \\ &= \left(\left[-2e^{-1.5t}\mathbf{i}\right] + \left[-2t^2\mathbf{j}\right]\right)\ \text{meters.}\end{aligned}$$

Notice there is no evaluation done here. All the problem asked for was the function that defines the position ***r(t)*** for the body. Note also that the derivative of ***r(t)*** yields ***v(t)***.

f.) To find the *displacement*, we must evaluate the *position function* at *t = 2 seconds* and *t = 3.5 seconds*, then subtract the two to get the net *change* of position (that is what the *net displacement* is). Doing so yields:

$$\begin{aligned}\Delta\mathbf{r} &= \left[\left[-2e^{-1.5t}\mathbf{i}\right] + \left[-2t^2\mathbf{j}\right]\right]_{t=2\ sec}^{3.5} \\ &= \left(\left[-2e^{-1.5(3.5)}\mathbf{i}\right] + \left[-2(3.5)^2\mathbf{j}\right]\right) - \left(\left[-2e^{-1.5(2)}\mathbf{i}\right] + \left[-2(2)^2\mathbf{j}\right]\right) \\ &= (-.01\mathbf{i} - 24.5\mathbf{j}) - (-.10\mathbf{i} - 8\mathbf{j}) \\ &= [.09\mathbf{i} - 16.5\mathbf{j}]\ m.\end{aligned}$$

g.) The change of the body's position between $t = 2$ *seconds,* and $t = 3.5$ *seconds* is $\Delta\boldsymbol{r} = .09\boldsymbol{i} - 16.5\boldsymbol{j}$. This change of position can also be written as $\Delta\boldsymbol{r} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}$. As the *x-type motion* and the *y-type motion* are independent of one another, we can separate the two and write:

for **i** motion:

$$\begin{aligned}(.09\text{ m}) &= (x_2 - x_1)\\ &= x_2 - (-.1\text{ m})\\ \Rightarrow\quad x_2 &= -.01\text{ meters.}\end{aligned}$$

for **j** motion:

$$\begin{aligned}(-16.5\text{ m}) &= (y_2 - y_1)\\ &= y_2 - (-8\text{ m})\\ \Rightarrow\quad y_2 &= -24.5\text{ meters.}\end{aligned}$$

Coordinates at $t = 3.5$ *seconds* will be $x_2 = -.01$ *meters,* $y_2 = -24.5$ *meters*.

h.) The *jerk* is the *time rate of change of the acceleration*. That is:

$$\begin{aligned}\text{jerk} &= d\mathbf{a}/dt\\ &= d[-4.5e^{-1.5t}\mathbf{i} - 4\mathbf{j}] / dt\\ &= [6.75e^{-1.5t}\mathbf{i}]\text{ m/s}^3.\end{aligned}$$

i.) Exponents must be unitless, so k must have the units of $seconds^{-1}$.

4.5) We know the bats' initial vertical velocity $v_1 = 0$, their initial height $h_1 = 100$ *meters*, their pull-out height $h_2 = 3$ *meters*, and their acceleration a is minus the magnitude of the acceleration of gravity, or $a = -g = -9.8\ m/s^2$:

a.) We want v_2. Scanning the kinematic equations (shown below),

$$(x_2 - x_1) = v_1\Delta t + (1/2)a(\Delta t)^2$$
$$(x_2 - x_1) = v_{avg}\Delta t \quad \text{or} \quad v_{avg} = (x_2 - x_1)/\Delta t$$
$$v_{avg} = (v_2 + v_1)/2$$
$$a = (v_2 - v_1)/\Delta t \quad \text{or} \quad v_2 = v_1 + a\Delta t$$

$$(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1),$$

we decide to use:

$$(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1)$$
$$\Rightarrow \quad v_2 = [(0)^2 + 2(-9.8\ m/s^2)[(3\ m) - (100\ m)]]^{1/2}$$
$$= 43.6\ m/s.$$

Note: This equation yields the *magnitude* of the velocity only (the quantity v_2 is *squared* in the equation). If we use this value in subsequent problems, we must make it negative (the body is moving downward in the ***-j*** *direction*).

$$\Rightarrow \quad \mathbf{v}_2 = -43.6\mathbf{j}\ m/s.$$

b.) We know that at 3 meters above the ground, $y_2 = 3\ meters$ and $v_2 = -43.6\ m/s$ (we got the latter from *Part a*). We want the pull-out acceleration a_p executed between $y_2 = 3\ meters$ and $y_3 = 1\ meter$. Scanning the kinematic equations, we find that the same equation used in *Part a* will do the job:

$$(v_3)^2 = (v_2)^2 + 2a_p(x_3 - x_2)$$
$$\Rightarrow \quad (0)^2 = (-43.6\ m/s)^2 + 2a_p[(1\ m) - (3\ m)]$$
$$\Rightarrow \quad a_p = 475\ m/s^2.$$

Note 1: We used the same equation in *Parts a* and *b* even though the two situations are related to entirely different sections of motion. The moral of the story: these equations work between *ANY* two points for which you have information.

Note 2: That the acceleration is positive should not be surprising. The net force and the acceleration are proportional; it will take a positive force (i.e., a force upward in the ***+j*** *direction*) to stop the bats' freefall.

c.) We know the velocity at the beginning of the pull-out ($v_2 = -43.6\ m/s$), the velocity at the end of the pull-out ($v_3 = 0$), and the acceleration through the pull-out ($a_p = 475\ m/s^2$). To determine the time:

$$a = (v_3 - v_2)/\Delta t$$

$$\Rightarrow \quad \Delta t = (v_3 - v_2)/a$$
$$= [(0) - (-43.6 \text{ m/s})]/(475 \text{ m/s}^2)$$
$$= .092 \text{ seconds.}$$

4.6) $x(t) = bt^4 - ct$ (plugging in $b = 6$ and $c = 2$, we get $x(t) = 6t^4 - 2t$).

a.) The body will change its direction when reaches either its maximum or minimum *x position*. At that point it will reverse its direction. The *velocity* at the turn-around point is ZERO. Ignoring *unit vectors* (it is one dimensional motion), the *velocity function* is:

$$v = dx/dt$$
$$= d[6t^4 - 2t] / dt$$
$$= 24t^3 - 2.$$

Putting the body's *velocity at the turn-around point* equal to zero, we get:

$$24t^3 - 2 = 0$$
$$\Rightarrow \quad t = (2/24)^{1/3}$$
$$= .437 \text{ seconds.}$$

b.) If we can determine the time at which $a = +12 \text{ m/s}^2$, we can put that time back into our *velocity function* to determine the velocity of the body at that point in the motion. Knowing the *velocity* (sign and all) will tell us the *direction* of motion. Executing that plan:

$$a(t) = dv(t)/dt$$
$$= d[24t^3 - 2t]/dt$$
$$= 72t^2.$$

Putting this equal to 12 m/s^2 yields our time:

$$72t^2 = 12$$
$$\Rightarrow \quad t = .408 \text{ seconds.}$$

The velocity at $t = .408$ *seconds* is:

$$v = 24t^3 - 2$$
$$= 24(.408)^3 - 2$$
$$= -.37 \text{ m/s.}$$

As the *velocity vector* is negative, the body must be moving in the *negative direction* when the acceleration is *+12 m/s*^{2}.

4.7) The stunt woman's velocity at *Point A* is v_A = *-25 m/s*. The acceleration is still $a = -g$.

a.) After 2 seconds, she has moved a distance:

$$(x_B - x_A) = v_A\Delta t + (1/2)a(\Delta t)^2$$
$$= (-25 \text{ m/s})(2 \text{ sec}) + .5(-9.8 \text{ m/s}^2)(2 \text{ sec})^2$$
$$= -69.6 \text{ meters.}$$

b.) Her velocity at *Point B*:

$$v_B = v_A + a\Delta t$$
$$= (-25 \text{ m/s}) + (-9.8 \text{ m/s}^2)(2 \text{ sec})$$
$$= -44.6 \text{ m/s.}$$

4.8) Because there are two cars in the system, we have two sets of kinematic equations available to us (one for the motion of each car). There are two common quantities that will link the two sets of equations during the time interval between the first pass and the second pass. They are: 1.) both cars will travel for the same amount of time during the interval; and 2.) their last-pass coordinate will be the same (we'll call this the "final" coordinate for simplicity).

If we define the origin (i.e., $x_1 = 0$) at the point where the two cars *first pass* and define the *last pass* position as x_2:

a.) For the *first car* whose initial velocity is 18 m/s and whose acceleration is zero:

$$(x_2 - x_1) = v_{fst,1}\Delta t + (1/2)a_{fst}(\Delta t)^2$$
$$\Rightarrow \quad (x_2 - 0) = (18 \text{ m/s})t + .5(0)t^2$$
$$\Rightarrow \quad x_2 = 18t \qquad \text{(Equation 1).}$$

--For the *second car* whose initial velocity is 4 m/s and whose acceleration is 6 m/s^2:

$$(x_2 - x_1) = v_{sec,1}\Delta t + (1/2)a_{sec}(\Delta t)^2$$
$$\Rightarrow \quad (x_2 - 0) = (4 \text{ m/s})t + .5(6\text{m/s}^2)t^2$$

$$\Rightarrow \qquad x_2 = 4t + 3t^2 \qquad \text{(Equation 2).}$$

Equating the two independent expressions for x_2 (i.e., Equations 1 and 2) yields:

$$18t = 4t + 3t^2.$$

Dividing by t yields:

$$18 = 4 + 3t$$
$$\Rightarrow \quad t = 4.67 \text{ seconds.}$$

b.) Using Equation 1 yields:

$$x_2 = 18t$$
$$= 18(4.67 \text{ seconds})$$
$$= 84.1 \text{ meters.}$$

Using Equation 2 yields:

$$x_2 = 4t + 3t^2$$
$$= 4(4.67 \text{ sec}) + 3(4.67 \text{ sec})^2$$
$$= 84.1 \text{ meters.}$$

EITHER EQUATION WILL DO!

c.) The *second car's* velocity as it passes the first car:

$$v_{sec,2} = v_{sec,1} + a_{sec}\Delta t$$
$$= (4 \text{ m/s}) + (6 \text{ m/s}^2)(4.67 \text{ sec})$$
$$= 32.02 \text{ m/s.}$$

An alternative approach would be:

$$(v_{sec,2})^2 = (v_{sec,1})^2 + 2a_{fst}(x_p - x_o)$$
$$= (4 \text{ m/s})^2 + 2(6 \text{ m/s}^2)[(84.1 \text{ m}) - 0]$$
$$= 32.02 \text{ m/s.}$$

EITHER APPROACH WORKS!

d.) The *second car's* average velocity:

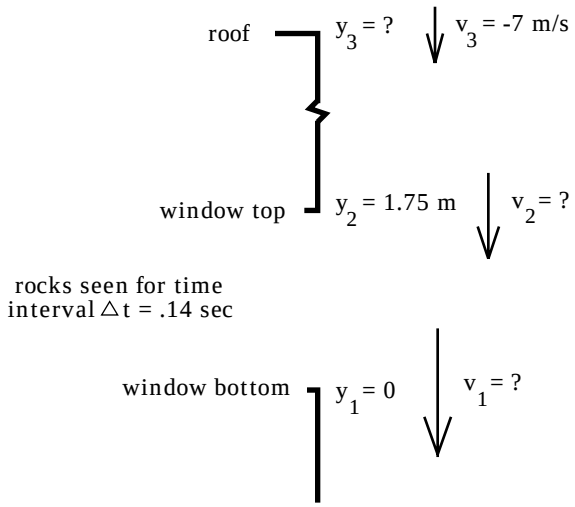
$$\begin{aligned} v_{avg} &= (v_{sec,2} + v_{sec,1})/2 \\ &= (32.02 \text{ m/s} + 4 \text{ m/s})/2 \\ &= 18.01 \text{ m/s}. \end{aligned}$$

e.) Time for the *second car* to travel to a velocity of 100 m/s:

$$\begin{aligned} a_{sec} &= (v_{100} - v_{sec,1})/\Delta t \\ \Rightarrow \quad Dt &= (100 \text{ m/s} - 4 \text{ m/s})/(6 \text{ m/s}^2) \\ &= 16 \text{ seconds}. \end{aligned}$$

4.9)

a.) All the known information has been inserted into the sketch to the right. The coordinate axis has been placed so that $y_1 = 0$ at the bottom of the window. We will proceed by writing out kinematic relationships that will allow us to solve for information we need.



To determine the velocity of the rock when it is at the *top of the window*, we will define the "final" position to be at y_1 with an "initial" position at y_2. With that, we can write:

$$\begin{aligned} (y_1 - y_2) &= v_2 \Delta t + (1/2)a(\Delta t)^2 \\ (0) - (1.75 \text{ m}) &= v_2(.14 \text{ s}) + (1/2)(-9.8 \text{ m/s}^2)(.14 \text{ s})^2 \\ \Rightarrow \quad v_2 &= [(-1.75 \text{ m}) - .5(-9.8 \text{m/s}^2)(.14 \text{ s})^2]/ (.14 \text{ s}) \\ &= -11.8 \text{ m/s}. \end{aligned}$$

Note: The negative sign makes sense considering the fact that the rock is moving *downward*.

b.) We want the distance between the *top of the building* and the *bottom of the window* (this will numerically equal y_3). An equation with known values and y_3 is (note that the "final" position here is y_2):

$$(v_2)^2 = (v_3)^2 + 2\ a\ (y_2 - y_3)$$
$$(-11.8\ \text{m/s})^2 = (-7\ \text{m/s})^2 + 2(-9.8\ \text{m/s}^2)[(1.75\ \text{m}) - y_3)]$$
$$\Rightarrow \quad y_3 = 6.35\ \text{meters.}$$

c.) A change in the initial velocity of the rock does nothing to the acceleration of the rock. Once it becomes free, the rock picks up velocity due to gravity at a rate of *9.8 m/s* <u>*every second*</u>, no matter what. Tricky, eh?

4.10) The car's initial velocity is $v_1 = 40\ m/s$; its pedal-to-the-metal acceleration is $a_{go} = +3\ m/s^2$ and its stopping acceleration is $a_{stop} = -3\ m/s^2$. The light stays yellow for 1.2 seconds before turning. We need to consider two situations: a.) what is the maximum distance the car can be from the restraining line if it is to successfully accelerate all the way through the intersection in 1.2 seconds; and b.) what is the minimum distance the car can be from the restraining line if it is to brake successfully?

a.) Pedal-to-the-metal: Taking Δx_{go} to be the distance from the restraining line to the point at which the car begins its acceleration, the car must go $\Delta x_{go} + 18\ meters$ to make it *through the intersection* without incident (remember, the intersection extends 18 meters beyond the restraining line). With 1.2 seconds to accomplish the feat, we can write:

$$(\Delta x_{go} + 18) = v_1 \Delta t + (1/2)a_{go}(\Delta t)^2$$
$$\Rightarrow \quad \Delta x_{go} = (-18\ \text{m}) + (40\ \text{m/s})(1.2\ \text{sec}) + .5(+3\ \text{m/s}^2)(1.2\ \text{sec})^2$$
$$= 32.16\ \text{meters.}$$

If the car is closer than 32.16 meters, it can accelerate and still make it across the intersection. If the car is farther than 32.16 meters, it will not be able to accelerate completely through the intersection.

b.) Braking: The 1.2 seconds is useless in this section. It doesn't matter whether the car is sliding while the light is red or not. All that matters is that the car stop just behind the restraining line. Putting the origin at the restraining line (i.e., $x_{r.l.} = 0$), and defining x_{brake} to be the

position at which the brakes must be hit to effect the stop right at the restraining line, we can write:

$$v_{r.l.}^{\,2} = v_{brk}^{\,2} + 2a(x_{r.l.} - x_{brk})$$

$$\Rightarrow \quad x_{brk} = -(v_{r.l.}^{\,2} - v_{brk}^{\,2}) / (2 \quad a \quad)$$

$$= -[0^2 - (40 \text{ m/s})^2] / [2(-3 \text{ m/s}^2)]$$

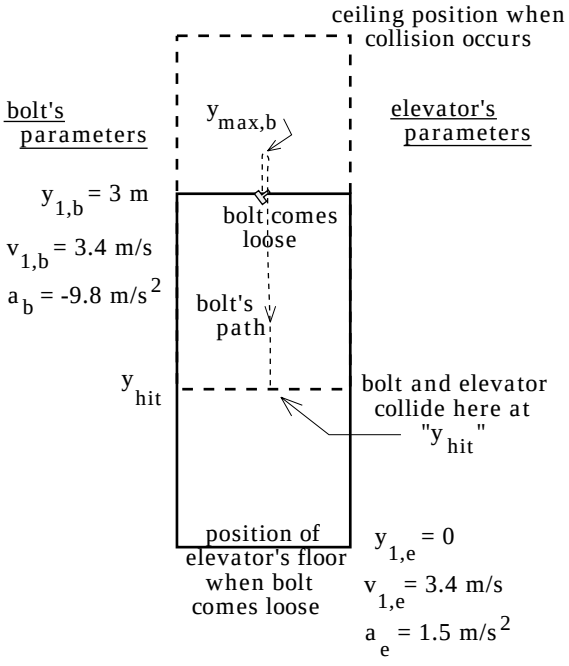
$$= -267 \text{ meters.}$$

Note: The negative sign simply means that a car moving to the right in the positive direction must begin to stop to the *left* of the origin (the restraining line).

c.) Bottom Line: The car eats it no matter what if it is between 267 meters and 32 meters of the restraining line.

4.11)

a.) We need a *fixed* coordinate axis from which to make our measurements. We could take ground level as our $y = 0$ level, but instead we will take the position in space of the *elevator's floor* just as the bolt releases (this will be useful because it is the elevator's floor that will come in contact with the bolt at the *collide* position).



The bolt's initial velocity is the same as that of the elevator at the moment it becomes free. That velocity is upward, so the bolt will move above its *suddenly-free* position to some maximum position y_{max} before beginning to descend. WHENEVER YOU ARE LOOKING FOR y_{max}, ALWAYS USE:

$$v_{top}^{\,2} = v_1^{\,2} + 2a(y_{top} - y_1).$$

As we know the bolt's velocity at the top of its motion is $v_{top} = 0$, we can write:

$$v_{top}^{\,2} = v_{1,b}^{\,2} + 2a_b(y_{top} - y_{1,b}).$$

$$\Rightarrow \quad (0)^2 = (3.4 \text{ m/s})^2 + 2(-9.8 \text{ m/s}^2)[y_{top} - (3 \text{ m})]$$

$$\Rightarrow \quad y_{top} = 3.59 \text{ meters}$$

$$\Rightarrow \quad y_{\text{max above ground}} = y_{top} + 4 = 7.59 \text{ meters .}$$

b. and c.) Finding the *time of flight* Δt and the *final coordinate position* y_{hit} requires two equations solved simultaneously. As such, we will do both *Parts b* and *c* in this section.

The first thing to notice is that this is really two problems happening at the same time. During the bolt's freefall, the elevator is accelerating upward by its motor while the bolt is accelerated downward by gravity. Treating the two entities as individuals:

i.) For the elevator's motion (or, at least, the motion of the elevator's floor), assuming $y = 0$ is the floor's position when the bolt releases:

$$\begin{aligned} y_{hit} &= y_{1,e} + v_{1,e}\,\Delta t + (1/2)\, a_e\, (\Delta t)^2 \\ &= (0) + (3.4\text{ m/s})t + .5\,(1.5\text{ m/s}^2)\,t^2 \end{aligned} \qquad \text{(Equ. A).}$$

ii.) For the bolt's motion, assuming that we are using the same coordinate axis as above so that the bolt's initial position is $y = 3$ *meters*:

$$\begin{aligned} y_{hit} &= y_{1,b} + v_{1,b}\,\Delta t + (1/2)\, a_b\, (\Delta t)^2 \\ &= (3\text{ m}) + (3.4\text{ m/s})t + .5\,(-9.8\text{ m/s}^2)\,t^2 \end{aligned} \qquad \text{(Equ. B).}$$

iii.) Equating *Equations A* and *B* (y_{hit} is the same in both equations):

$$(3.4\text{ m/s})t + .5(1.5\text{ m/s}^2)\,t^2 = 3 + (3.4\text{ m/s})t + .5(-9.8\text{ m/s}^2)t^2$$

$$\Rightarrow \quad 5.65t^2 = 3$$

$$\Rightarrow \quad t = .73 \text{ seconds.}$$

iv.) Using *Equation A* to determine y_{hit}:

$$\begin{aligned} y_{hit} &= y_{1,e} + v_{1,e}\,\Delta t + (1/2)\, a_e\, (\Delta t)^2 \\ &= (0) + (3.4\text{ m/s})(.73\text{ s}) + .5\,(1.5\text{ m/s}^2)\,(.73\text{ s})^2 \\ &= 2.88 \text{ meters.} \end{aligned}$$

v.) The bolt was initially at $y = 3$ *meters*. When it hits the floor, its coordinate is $y_{hit} = 2.88$ *meters*. The distance the bolt actually falls is:

$$\begin{aligned} \Delta y &= y_{final} - y_{initial} \\ &= (2.88\text{ m}) - (3.0\text{ m}) \\ &= -.12 \text{ meters.} \end{aligned}$$

vi.) For the amusement of it, let's check our y_{hit} value by solving *Equation B*:

$$\begin{aligned} y_{hit} &= y_{1,b} + v_{1,b}\ \Delta t + (1/2)\ a_b\ (\Delta t)^2 \\ &= (3\text{ m}) + (3.4\text{ m/s})(.73\text{ s}) + .5\ (-9.8\text{ m/s}^2)(.73\text{ s})^2 \\ &= 2.87\text{ meters.} \end{aligned}$$

. . . close enough for government work.

d.) The magnitude of the velocity of the bolt, relative to a fixed frame of reference (i.e., *not* relative to the elevator's floor) is determined as follows:

$$\begin{aligned} v_{b,bot}^2 &= v_{1,b}^2 + 2\ a_b\ [\ y_{b,bot} - y_{1,b}] \\ &= (3.4\text{ m/s})^2 + 2(-9.8\text{ m/s}^2)[(2.87\text{ m}) - (3\text{ m})] \\ &\Rightarrow\quad v_{b,bot} = 3.76\text{ m/s.} \end{aligned}$$

As a vector, this would be $\mathbf{v}_{b,bot} = -(3.76\text{ m/s})\mathbf{j}$.

4.12)

a.) In projectile motion, questions like, "How far (Δx)" and "How long (Δt)" are usually approached simultaneously. In this case, the *time* variable is most easily determined by looking at the motion in the *y direction* only. Noting that the ball started at $y_1 = 1.3$ *meters*, ended at $y_2 = 0$ (i.e., at ground level), and had a *y component* velocity of $v_y = 41 \sin 50^o$, we can write:

$$\begin{aligned} (y_2 - y_1) &= v_{1,y}t + (1/2)a_y t^2 \\ (0\text{ m} - 1.3\text{ m}) &= (41 \sin 50^o)t + .5(-9.8\text{ m/s}^2)t^2 \\ -1.3 &= 31.4t - 4.9t^2 \\ \Rightarrow\quad 4.9t^2 - 31.4t - 1.3 &= 0. \end{aligned}$$

Using the Quadratic Formula, we get:

$$\begin{aligned} t &= [-b \pm [b^2 - 4ac]^{1/2}]/2a \\ &= [-(-31.4) \pm [(-31.4)^2 - 4\ (4.9)(-1.3)]^{1/2}]/[2(4.9)] \\ &= [31.4 \pm (31.8)]/(9.8) \\ &= 6.45\text{ seconds.} \end{aligned}$$

b.) Assuming $x_1 = 0$ is the *x coordinate* at which the ball is struck, the net horizontal distance traveled will be:

$$(x_2 - x_1) = v_{1,x}t + (1/2)a_x t^2$$
$$(x_2 - 0) = (v_1 \cos q)t + .5(0 \text{ m/s}^2)t^2$$
$$\Rightarrow \quad x_2 = (41 \text{ m/s})(\cos 50^o)(6.45 \text{ sec}) + 0$$
$$= 170 \text{ meters.}$$

c.) Height is a *y* related quantity. Knowing that the *y component* of velocity at the top of the arc (i.e., at y_{max}) will be zero, we can write:

$$(v_{max,y})^2 = (v_{1,y})^2 + 2a_y(y_{max} - y_1)$$
$$\Rightarrow \quad y_{max} = [(v_{max,y})^2 - (v_{1,y})^2 + 2a_y(y_1)]/2a_y$$
$$= [(0)^2 - (41 \sin 50^o)^2 + 2(-9.8 \text{ m/s}^2)(1.3 \text{ m})]/[(2)(-9.8 \text{ m/s}^2)]$$
$$= 51.6 \text{ meters.}$$

d.) At the end of the flight, the ball has an *x component* of velocity that has not changed throughout the motion (we are neglecting air friction and there are no other natural forces out there to make the *x motion* change). That value will be:

$$v_{2,x} = (41 \text{ m/s})(\cos 50^o)$$
$$= 26.4 \text{ m/s.}$$

The *y motion* velocity has changed because gravity has been accelerating the ball throughout the motion. We can get the *y* velocity using:

$$(v_{2,y})^2 = (v_{1,y})^2 + 2a_y(y_2 - y_1)$$
$$\Rightarrow \quad v_{2,y} = [(v_{1,y})^2 + 2a_y(y_2 - y_1)]^{1/2}$$
$$= [[(41 \text{ m/s})(\sin 50^o)]^2 + 2(-9.8 \text{ m/s}^2)(0 - 1.3 \text{ m})]^{1/2}$$
$$= 31.8 \text{ m/s.}$$

Note: This equation yields magnitudes only. The *y component* of the "final" velocity is in the ***-j*** *direction*. That means $v_{2,y} = -31.8$ *m/s*.

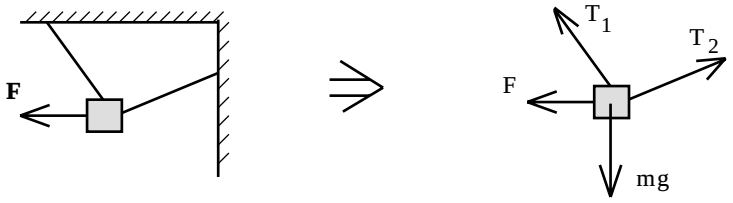
Putting it all together:

$$\mathbf{v}_2 = (26.4\mathbf{i} - 31.8\mathbf{j}) \text{ m/s.}$$

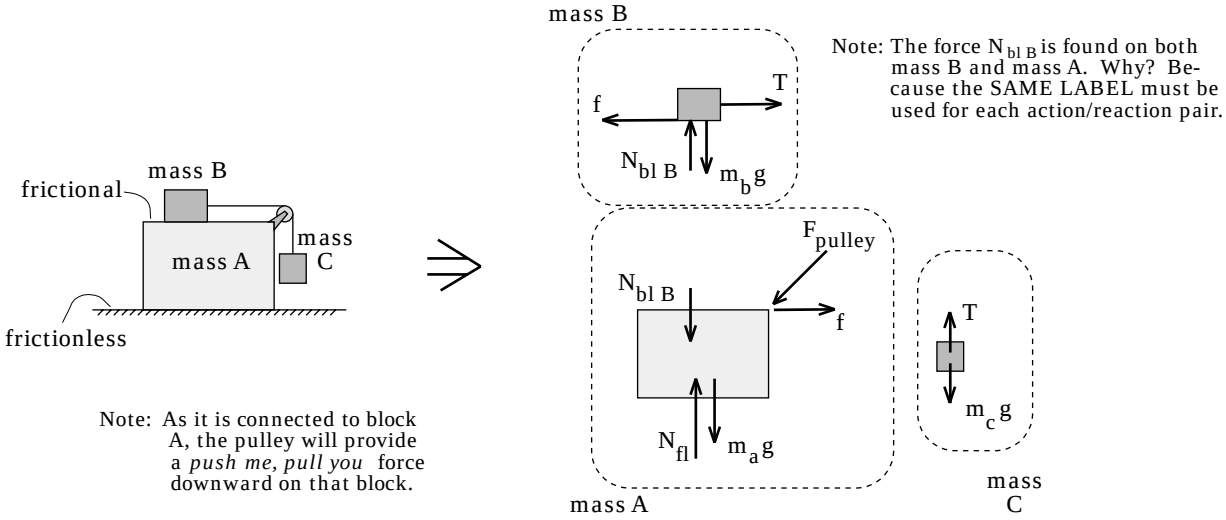
CHAPTER 5 -- NEWTON'S LAWS

5.1) Drawing a *free body diagram* for the force of EACH BODY in each sketch:

a.)



b.)

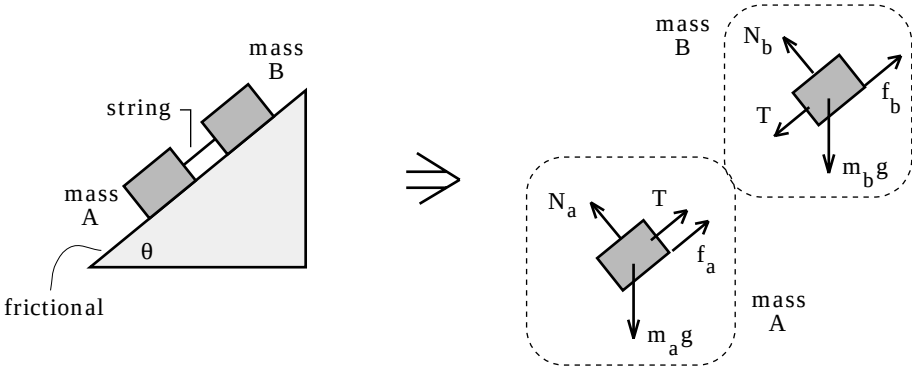

mass B
frictional
mass A
mass C
frictionless
Note: As it is connected to block A, the pulley will provide a push me, pull you force downward on that block.
mass B
f
T
N bl B
m b g
Note: The force N bl B is found on both mass B and mass A. Why? Because the SAME LABEL must be used for each action/reaction pair.
F pulley
N bl B
f
N fl
m a g
mass A
T
m c g
mass C

Note 1: There are two *action/reaction force pairs* between *masses A* and *B*: the normal force $N_{bl\,B}$ that *A* applies to *B* and vice versa, and the frictional force *f* between the two. Be sure you understand what is going on here!

Note 2: Notice that the *magnitude* of the tension force *T* on *mass C* and *mass B* is the same.

Note 3: The pulley mount on *mass A* applies a *downward and to the left* force F_{pulley} on *mass A*. As we are interested in ALL the forces acting on each mass, that force has to be included.

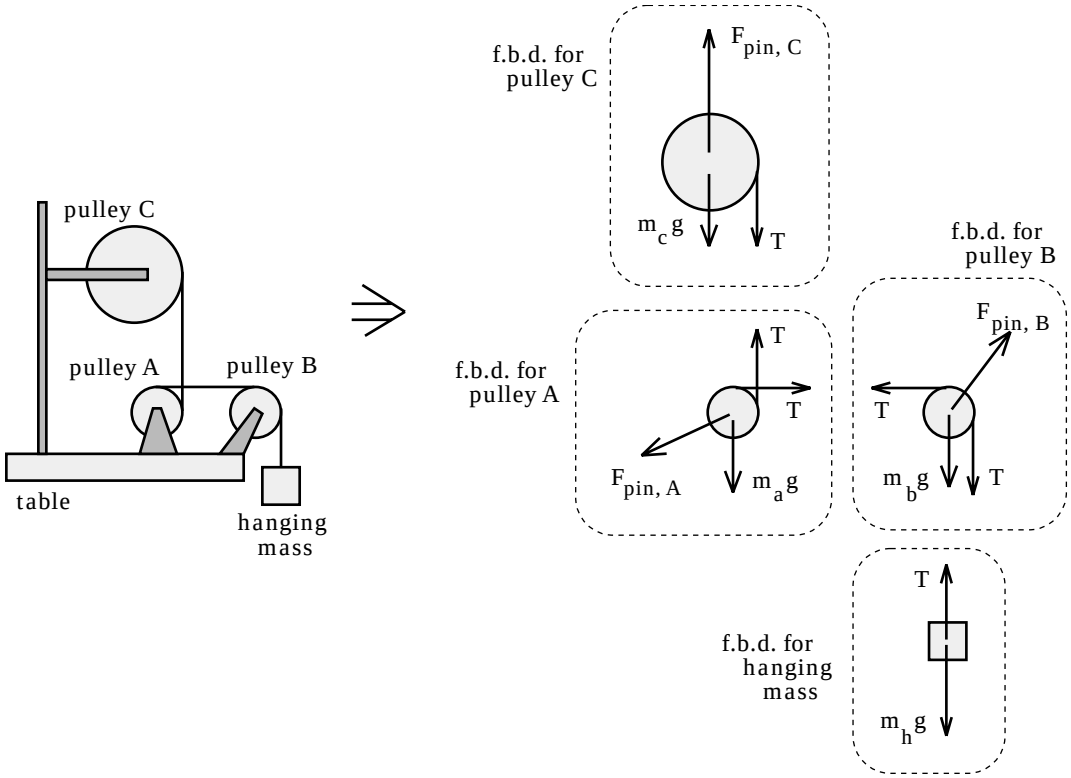
c.)



d.)

Note 1: All the pulleys do here is re-direct the line of the tension *T*.

Note 2: The pin that holds each pulley in place must exert a force that effectively keeps its pulley from flying off into space.



Note 3:
There is a force acting at *the pin of each pulley* to keep the pulleys from falling through the table.

5.2) According to Newton's Third Law:

a.) The reaction to *the force the floor applies to you* is <u>*the force you apply to the floor*</u>.

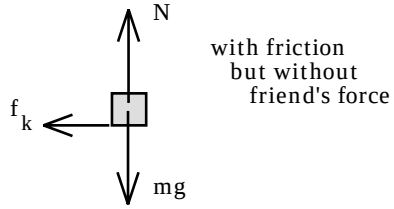
b.) The reaction to *the force a string applies to a weight* is <u>*the force the weight applies to the string*</u>.

c.) The reaction to *the force a car applies to a tree* is <u>*the force the tree applies to the car*</u>.

d.) The reaction to *the force the earth applies to the moon* is <u>*the force the moon applies to the earth*</u>.

5.3)

a.) A *free body diagram* for the situation *before your friend applies his force* (*Part B*) is shown below. Making a_x into a MAGNITUDE by unembedding the negative sign, N.S.L. yields:



$\underline{\sum F_x}$:

$-f_k = -ma_x$

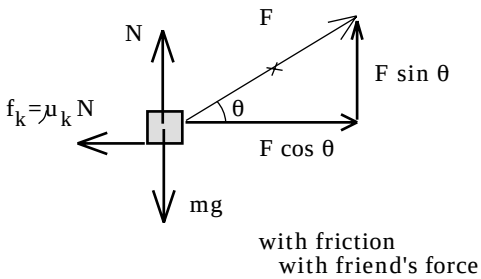
$\Rightarrow \quad -(12 \text{ nt}) = -(30 \text{ kg})\, a$

$\Rightarrow \quad a = .4 \text{ m/s}^2.$

Note 1: Why make a_x into a magnitude by unembedding the negative sign? In certain kinds of problems, doing so will make life easier. Get used to it.

Note 2: In the next question, you are going to need μ_k. From the f.b.d. above, $N = mg = (30 \text{ kg})(9.8 \text{ m/s}^2) = 294 \text{ nts}$. As $f_k = \mu_k N$, we can write $\mu_k = f_k/N = (12 \text{ nt})/(294 \text{ nt}) = .04$.

b.) With the additional force applied by your friend, the *free body diagram* looks like the one shown to the right (note that N has changed). To determine N:



$\underline{\sum F_y}$:

$N + F \sin 40^o - mg = -ma_y \qquad (= 0 \text{ as } a_y = 0)$

$\Rightarrow \quad N = -F \sin 40^o + mg$

$= -(60 \text{ nt}) \sin 40^o + (30 \text{ kg})(9.8 \text{ m/s}^2)$

$= 255 \text{ nts}.$

$\underline{\Sigma F_x}$:

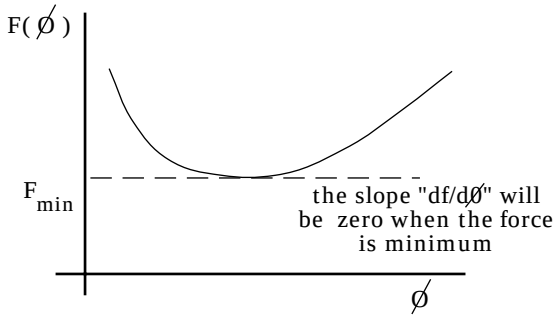
$$- \mu_k N + F \cos 40^o = - ma_x.$$

Note 1: In this case, I have assumed that your friend's force will not overcome that of friction and the *direction* of the sled's *acceleration* will still be negative (i.e., to the left). As such, I have unembedded the negative sign in front of the *ma* term. If I am wrong, the SIGN of the calculated acceleration will be negative. Continuing:

$$- \mu_k N + F \cos 40^o = - ma_x$$
$$\Rightarrow - (.04)(255 \text{ nt}) + (60 \text{ nt})(.766) = - (30 \text{ kg}) a$$
$$\Rightarrow \qquad a = -1.19 \text{ m/s}^2.$$

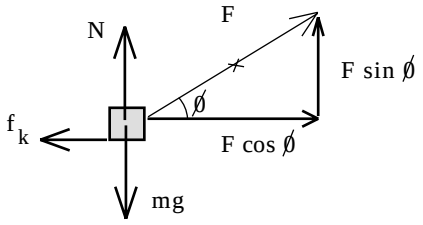
Note 2: The negative sign means that I've assumed the wrong direction for *a*. Evidently, your friend's force was greater than the frictional force and the acceleration was really in the *+x direction* (if this ever happens to you, what I've just said is all you will have to state to make the problem OK).

c.) The graph of $F(\phi)$ *vs.* ϕ looks something like the graph shown in the figure to the right. Notice that at the minimum, the slope of $F(\phi)$ is ZERO (that is, $dF(\phi)/d\phi = 0$ at that point). All we have to do is generate an expression for the force as a function of ϕ, then put its *derivative* equal to zero and solve for ϕ under that condition. That will produce the angle at which the force is a minimum.



Executing that operation:

i.) The f.b.d for the situation is shown to the right.



ii.) Noting that we will need to use $\mu_k N$ for the *frictional force* f_k, we will start with N.S.L. in the *y direction* to determine *N*:

$\underline{\sum F_y}$:

$N + F \sin \phi - mg = ma_y$

$= 0$ (as $a_y = 0$)

$\Rightarrow \quad N = -F \sin \phi + mg$ (Equation A).

iii.) Using N.S.L. in the *x direction*, we get:

$\underline{\sum F_x}$:

$-\mu_k N + F \cos \phi = ma_x$

$= 0$ (as the velocity is constant).

iv.) Substituting *Equation A* into this expression yields:

$-\mu_k(-F \sin \phi + mg) + F \cos \phi = 0$

$\Rightarrow \quad F = [\mu_k mg] / [\mu_k \sin \phi + \cos \phi]$.

v.) Given that the body is moving with a constant velocity (i.e., it isn't accelerating), we now have a function that defines the *force applied* in terms of the *angle of the force*. Taking the derivative of that function and setting it equal to zero yields an expression from which the angle of minimum force can be determined. Using the Chain Rule to determine the expression, we get:

$$\frac{dF(\phi)}{d\phi} = \frac{d\left[\dfrac{\mu_k mg}{\mu_k(\sin\phi)+\cos\phi}\right]}{d\phi}$$

$$= \frac{d\left[\mu_k mg\left[\mu_k(\sin\phi)+\cos\phi\right]^{-1}\right]}{d\phi}$$

$$= \mu_k mg(-1)\left[\mu_k(\sin\phi)+\cos\phi\right]^{-2}\left[\mu_k(\cos\phi)-\sin\phi\right]$$

$$= \frac{-\mu_k mg\left[\mu_k(\cos\phi)-\sin\phi\right]}{\left[\mu_k(\sin\phi)+\cos\phi\right]^2}$$

vi.) As ungodly as this may look, the criterion for this expression equaling zero is relatively simple. All that must be true is that the numerator equal zero. That will be satisfied if $\mu_k(\cos \phi) - (\sin \phi) = 0$. With that observation:

$$\mu_k(\cos\ \phi) - (\sin\ \phi) = 0,$$
$$\Rightarrow \quad \mu_k = [\sin\ \phi]/[\cos\ \phi]\ .$$

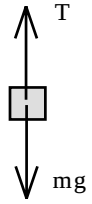
vii.) As *sin (ϕ)/cos (ϕ) = tan ϕ,* we can write:

$$\phi = \tan^{-1}(\mu_k).$$

This is the optimal angle at which the force *F* will be a minimum.

5.4)

a.) A stationary elevator will feel no friction; the f.b.d. for the situation is shown in the sketch to the right. Using N.S.L.:

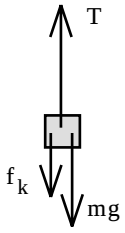


$\underline{\Sigma F_y}$:

$$T - mg = ma$$
$$= 0 \qquad \text{(as elevator's acc. } a_e = 0\text{)}$$

$$\Rightarrow \quad T = mg$$
$$= (400\ \text{kg})(9.8\text{m/s}^2)$$
$$= 3920\ \text{nts.}$$

b.) With the upward acceleration of the elevator, the frictional force will be applied downward as shown in the f.b.d. to the right. The acceleration term *a* is a magnitude whose sign (manually placed) is positive. N.S.L. yields:



$\underline{\Sigma F_y}$:

$$T - mg - f_k = +ma$$
$$\Rightarrow \quad T = mg + f_k + ma$$
$$= (400\ \text{kg})(9.8\text{m/s}^2)+(80\ \text{nt})+(400\ \text{kg})(2.8\ \text{m/s}^2)$$
$$= 5120\ \text{nts.}$$

c.) The only difference between this problem and *Part b* is that the acceleration is zero (constant velocity means zero acceleration). It makes no difference what the velocity actually is; the forces acting on the elevator are the same as in *Part b* so the f.b.d. from *Part b* is still valid. Using it, we get:

$\underline{\sum F_y}$:

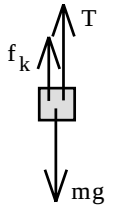
$T - mg - f_k = ma$

$\Rightarrow \quad T = mg + f_k + m(0)$

$\Rightarrow \quad = (400\text{ kg})(9.8\text{m/s}^2) + (80\text{ nt})$

$= 4000$ nts.

d.) With the downward velocity, friction is upward as shown in the f.b.d. to the right. N.S.L. yields:



$\underline{\sum F_y}$:

$T - mg + f_k = -ma$

$\Rightarrow \quad T = mg - f_k - ma$

$= (400\text{ kg})(9.8\text{m/s}^2) - (80\text{ nt}) - (400\text{ kg})(2.8\text{ m/s}^2)$

$= 2720$ nts.

Note: Whenever you can, make the acceleration term *a* a magnitude. That is what I've done above (the acceleration's negative sign has been unembedded). Be careful when you do this, though. Don't put a negative sign in front of the *a*, then proceed to use ***-2.8 m/s²*** when it comes time to put in the numbers.

e.) Moving with a constant velocity means that the acceleration *a* is zero. Friction is still acting (upward in this case), so the f.b.d. used in *Part d* is still valid (the forces haven't changed, there is just no acceleration).

$\underline{\sum F_y}$:

$T - mg + f_k = -ma$

$\Rightarrow \quad T = mg - f_k - m(0)$

$= (400\text{ kg})(9.8\text{m/s}^2) - (80\text{ nt})$

$= 3840$ nts.

5.5) The scale in this case is measuring the net force you apply to the scale (or the ground). If the acceleration is upward, this force F_{scale} will be *greater than mg*; if downward, it will be *less than mg*. To determine the acceleration direction, we need to determine *mg*:

$mg = (60\text{ kg})(9.8\text{ m/s}^2)$

$= 588$ newtons.

As this is less than the scale reading of 860 newtons, the elevator must be accelerating upward and the acceleration's sign must be *positive*.

By Newton's Third Law, the force you apply to the scale must be equal and opposite the force the scale applies to you. As such, using an f.b.d. and N.S.L. on yourself (see to right) yields:

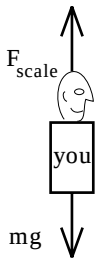
$\underline{\Sigma F_y}$:

$F_{scale} - mg = ma$

$\Rightarrow \quad a = (F_{scale}/m) - g$

$= (860 \text{ nt})/(60 \text{ kg}) - (9.8 \text{m/s}^2)$

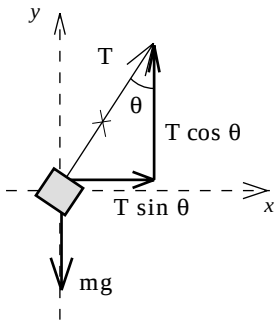
$= 4.53 \text{ m/s}^2$.



Note: If we had assumed a downward acceleration (i.e., an acceleration that was *negative*), we would have gotten a negative sign in front of the calculated *a* term above. The *negative sign* in an answer like that does not identify direction. By unembedding the sign, we have made the acceleration term a *magnitude*. As such, it should be positive. The negative sign in front of an answer in such instances means we have assumed the *wrong* direction for the acceleration, nothing else!

5.6)

a.) *An f.b.d.* for the forces on the mass is shown to the right. Noting that the acceleration is to the right, I have put one coordinate axis along the horizontal. N.S.L. in the *x direction* yields:



$\underline{\Sigma F_x}$:

$T \sin \theta = ma$

$\Rightarrow \quad a = (T \sin \theta)/m$ (Equation A).

We need to determine *T* to solve this. Using N.S.L. in the *y direction* yields:

$\underline{\Sigma F_y}$:

$T \cos \theta - mg = ma_y$

$= 0$ (as $a_y = 0$)

$\Rightarrow \quad T = mg/(\cos \theta)$.

Re-writing, then substituting back into *Equation A* yields:

$$a = [T] (\sin \theta)/m$$
$$= [mg/(\cos \theta)] (\sin \theta)/m.$$

The *m's* cancel and *(sin θ)/(cos θ)* is *tan θ*, so we end up with

$$a = g \tan \theta.$$

For our problem, the numbers yield:

$$a = (9.8 \text{ m/s}^2)(\tan 26^o)$$
$$= 4.78 \text{ m/s}^2.$$

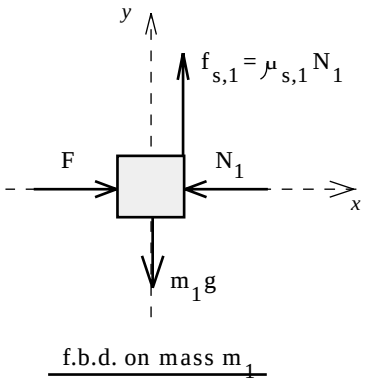
b.) At constant velocity, there is no acceleration and, hence, no swing observed. The string and mass should hang completely vertical. Note: That is exactly what the equation in the *x direction* suggests. The only time the acceleration will equal zero in $T \sin \theta = ma$ is when $\theta = 0$.

Note: One intrepid student whose father was a pilot pointed out that airplane floors (and ceilings) are not horizontal (she observed that when she walks to the bathroom at the rear of a plane, she always walks down hill). In any case, that idiosyncracy isn't important here as the angle is measured relative to the vertical.

5.7)

a.) We are interested in finding the *coefficient of static friction* between both m_1 and m_2 (call this $\mu_{s,1}$) and between m_2 and the wall (call this $\mu_{s,2}$), when $F = 25$ *newtons.*

--To the right is the f.b.d. for m_1. N.S.L. yields:



f.b.d. on mass m_1

$\underline{\sum F_x}$:

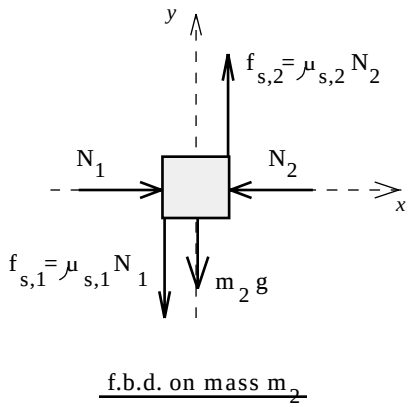
$$F - N_1 = m_1 a_x$$
$$= 0 \quad (\text{as } a_x = 0)$$
$$\Rightarrow \quad N_1 = F \quad (\text{equal to 25 nts}).$$

$\underline{\sum F_y}$:

$$\mu_{s,1} N_1 - m_1 g = m_1 a_1$$
$$= 0 \qquad (\text{as } a_1 = 0).$$
$$\Rightarrow \quad \mu_{s,1} = (m_1 g)/ N_1$$

$= [(2\text{ kg})(9.8\text{ m/s}^2)] / (25\text{ nt})$
$= .784$ (note that the coefficient is unitless).

--The f.b.d. for m_2 is shown to the right. A number of observations need to be made before dealing with N.S.L.:



f.b.d. on mass m_2

i.) Look at m_1*'s* f.b.d. on the previous page. Notice that it experiences a *normal force* N_1 due to its being jammed up against m_2. As such, m_2 must feel a *reaction force* (Newton's Third Law) of the same magnitude (i.e., N_1) in the *opposite direction*. That force has been placed on m_2*'s* f.b.d.

ii.) Look again at m_1*'s* f.b.d. on the previous page. Notice that it experiences a frictional force $f_{s,1}$ due to its rubbing up against m_2. As such, m_2 must feel a *reaction force* of magnitude $f_{s,1}$ in the direction opposite that of the frictional force on m_1. That force has been placed on m_2*'s* f.b.d.

iii.) Having made those observations, N.S.L. yields:

$\underline{\Sigma F_x}$:

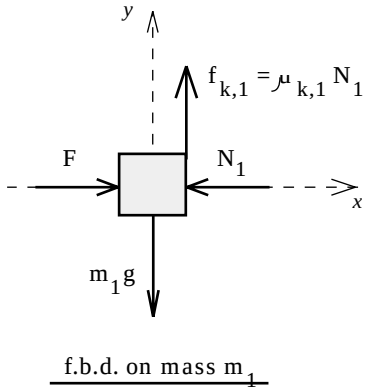
$N_1 - N_2 = m_2 a_x$
$= 0$ (as $a_x = 0$)
$\Rightarrow \quad N_1 = N_2$ (equal to $F = 25\ nts$ as $N_1 = F$) .

$\underline{\Sigma F_y}$:

$\mu_{s,2}N_2 - \mu_{s,1}N_1 - m_2 g = m_2 a_2$
$= 0$ (as $a_2 = 0$)
$\Rightarrow \quad \mu_{s,2} = [\mu_{s,1}N_1 + m_2 g] / N_2$
$= [(.784)(25\text{ nt}) + (7\text{ kg})(9.8\text{ m/s}^2)] / (25\text{ nt})$
$= 3.528.$

b.) The force F is now 20 newtons. That means there is not enough force associated with F to keep the bodies pinned to the wall. That being the case, they begin to accelerate downward. Assume the *coefficients of kinetic friction* are $\mu_{k,1} = .15$ and $\mu_{k,2} = .9$ respectively.

As innocuous as this scenario may seem, the problem has the potential to be a real stinker. Why? Because the direction of a frictional force on a body depends upon the direction of its slide *relative to the other body*. We don't know the *acceleration* of each of the bodies. We do know that if m_2 accelerates downward faster than m_1, then m_1*'s* motion *relative to* m_2 will be upward and the frictional force on m_1 will be *downward*. If m_2 accelerates downward more slowly than m_1, then m_1*'s* motion *relative to* m_2 will be downward and the frictional force on m_1 will be *upward*. Not knowing the acceleration of either body means we don't know which body will be moving faster and, hence, what direction the frictional force will be on either object. In short, we have to do the problem both ways to see which ends up making sense.



f.b.d. on mass m_1

We will start by assuming m_1 accelerates faster than m_2. In that case, the frictional force on m_1 will be upward and the f.b.d. for the situation will be as shown to the right. Using N.S.L. on m_1, we get:

$\underline{\Sigma F_x}$:

$$F - N_1 = m_1 a_x$$
$$= 0 \quad (\text{as } a_x = 0)$$
$$\Rightarrow \qquad F = N_1 \quad (\text{equal to 20 nt}).$$

$\underline{\Sigma F_y}$:

$$\mu_{k,1} N_1 - m_1 g = -m_1 a_1.$$
$$\Rightarrow \quad a_1 = [-\mu_{k,1} N_1 + m_1 g]/m_1$$
$$= [-(.15)(20 \text{ nt}) + (2 \text{ kg})(9.8 \text{ m/s}^2)] / (2 \text{ kg})$$
$$= 8.3 \text{ m/s}^2.$$

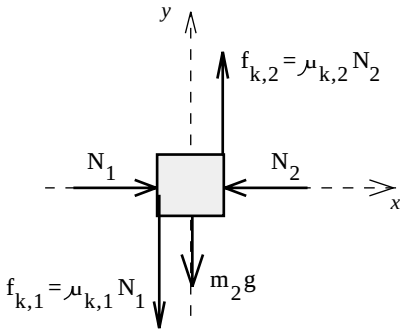
--The f.b.d. for the forces acting on m_2 are shown on the next page. N.S.L. yields:

$\underline{\Sigma F_x}$:

$N_1 - N_2 = m_2 a_x$

$= 0$ (as $a_x = 0$)

$\Rightarrow \quad N_1 = N_2$ (equal to $F = 20$ *nts*).



f.b.d. on mass m_2

$\underline{\Sigma F_y}$:

$\mu_{k,2} N_2 - \mu_{k,1} N_1 - m_2 g = -m_2 a_2$

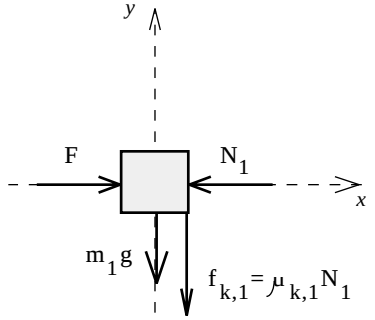
$\Rightarrow \quad a_2 = [-\mu_{k,2} N_2 + \mu_{k,1} N_1 + m_2 g] / m_2$

$= [-(.9)(20 \text{ nt}) + (.15 \text{ kg})(20 \text{ nt}) + (7 \text{ kg})(9.8 \text{ m/s}^2)] / (7 \text{ kg})$

$= 7.66 \text{ m/s}^2$.

Note 1: Yes! We've lucked out. We assumed m_1 accelerates faster than m_2, and that is just what our calculations have verified. If we had been wrong, we would have gotten senseless results. As we got it right on the first try, we needn't go further.

Note 2: For the amusement of it, let's go further. That is, assume that m_1 accelerates *more slowly* than m_2. That means m_1 will slide *upward* relative to m_2 and the frictional force will be downward (this is exactly opposite the situation we outlined above). With the direction of the frictional force reversed, the f.b.d. on m_1 look as shown to the right. N.S.L. yields:



f.b.d. on mass m_1

$\underline{\Sigma F_x}$:

$F - N_1 = m_1 a_x$

$= 0$ (as $a_x = 0$)

$\Rightarrow \quad F = N_1$ (= 20 nts).

$\underline{\Sigma F_y}$:

$-\mu_{k,1} N_1 - m_1 g = -m_1 a_1$.

$\Rightarrow \quad a_1 = [\mu_{k,1} N_1 + m_1 g]/m_1$

$= [(.15)(20 \text{ nt}) + (2 \text{ kg})(9.8 \text{ m/s}^2)] / (2 \text{ kg})$

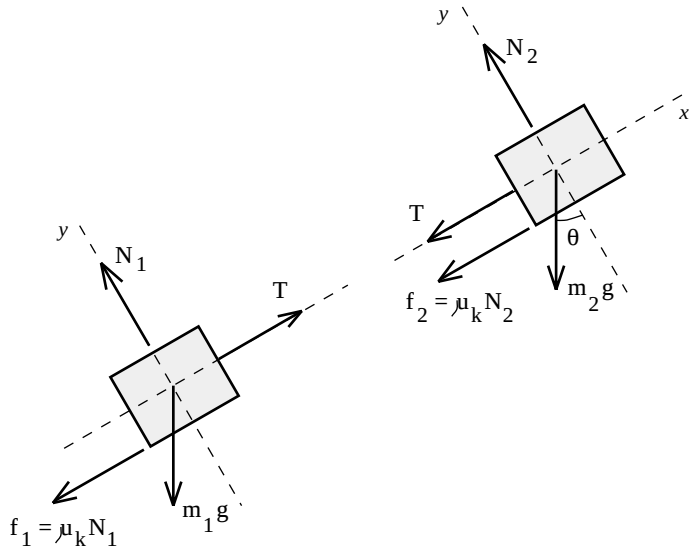
$= 11.3 \text{ m/s}^2$.

Yikes! According to our calculations, block m_1 is accelerating faster than the *acceleration of gravity* ($g = 9.8\ m/s^2$). That isn't possible in this situation. Conclusion? We made bad assumptions about the acceleration of m_1 and m_2.

c.) The reason the accelerations are different? They have different forces acting on them!

5.8)

a.) The *free body diagrams* for this situation are shown to the right.



b.) We need the frictional forces in both cases, which means we need both N_1 and N_2. Using N.S.L. in the *y direction*:

$\underline{\sum F_y}$:

$$N_1 - m_1 g \cos\theta = m_1 a_y$$

$$\Rightarrow \quad N_1 = m_1 g \cos\theta \qquad (\text{as } a_y = 0).$$

Likewise, $N_2 = m_2 g \cos\theta$.

--Using N.S.L. for the *x-motion* of m_1, noting that the acceleration is in the *negative* direction, relative to our coordinate axis (the body is slowing, hence the acceleration is *opposite* the direction of the velocity):

$\underline{\sum F_x}$:

$$T - \mu_k N_1 - m_1 g \sin\theta = -m_1 a.$$

Substituting in for N_1 and solving for $m_1 a$, we get:

$$m_1 a = [-T + \mu_k (m_1 g \cos\theta) + m_1 g \sin\theta] \qquad \text{(Equation A)}.$$

--To get rid of the tension term, consider the *x motion* of m_2:

$\underline{\Sigma F_x}$:

$$-T - \mu_k N_2 - m_2 g \sin\theta = -m_2 a.$$

Substituting in for N_2 and solving for the tension T, we get:

$$T = -\mu_k(m_2 g \cos\theta) - m_2 g \sin\theta + m_2 a.$$

Substituting the tension term into *Equation A* yields:

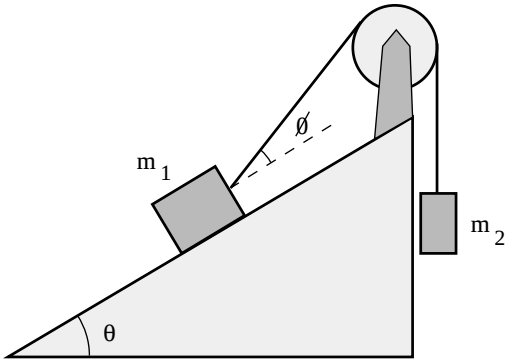
$$m_1 a = [-(-\mu_k(m_2 g \cos\theta) - m_2 g \sin\theta + m_2 a) + \mu_k(m_1 g \cos\theta) + m_1 g \sin\theta].$$

Solving for the acceleration yields:

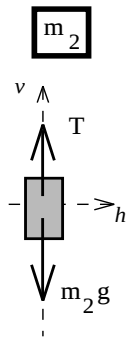
$$a = [\mu_k(m_2 g \cos\theta) + m_2 g \sin\theta + \mu_k(m_1 g \cos\theta) + m_1 g \sin\theta]/(m_1 + m_2)$$
$$= \mu_k g \cos\theta + g \sin\theta.$$

c.) Plugging the expression for a back into *Equation A* allows us to determine T. I'll save space by leaving the exercise to you.

5.9) This is an important situation because it requires you to face all the pitfalls that can occur when doing incline-plane problems.



We know m_1 is moving *down* the incline. That means m_2 is moving upward. Unfortunately, we have not been told the direction of *acceleration* for either m_1 or m_2. For the sake of amusement, let's assume m_1*'s* acceleration is *up* the incline (i.e., it's slowing). That will make m_2*'s* acceleration (remember, it's physically moving upward) *downward* (i.e., it's also slowing). Consider m_2*'s* f.b.d. first. N.S.L. allows us to write:

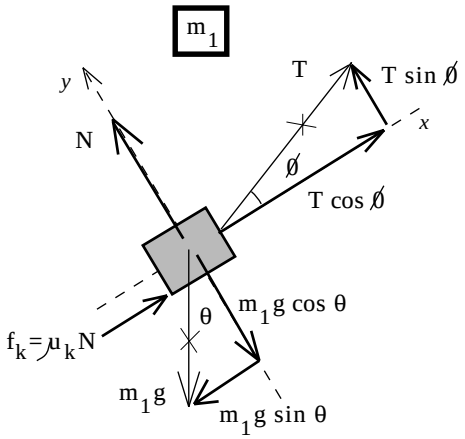


$\underline{\Sigma F_y}$:

$$T - m_2 g = -m_2 a_2$$

$$\Rightarrow \quad T = m_2 g - m_2 a_2 \qquad \text{(Equation 1).}$$

Remembering that the magnitude of m_1's acceleration is numerically equal to $a_2 \cos\theta$ (this was pointed out in the original set-up), now consider m_1's f.b.d. N.S.L. yields:



$\underline{\Sigma F_x}$:

$$T\cos\phi + \mu_k N - m_1 g\sin\theta = m_1 a_1$$

$$\Rightarrow \quad T\cos\phi + \mu_k N - m_1 g\sin\theta = m_1(a_2\cos\theta)$$

(Equation 2).

At this point, we have three unknowns *N*, *a*, and *T*. To determine an expression for *N*, consider N.S.L. in the *y direction* for m_1. Doing so yields:

$\underline{\Sigma F_y}$:

$$T\sin\phi + N - m_1 g\cos\theta = m_1 a_y = 0 \qquad (\text{as } a_y = 0)$$

$$\Rightarrow \quad N = -T\sin\phi + m_1 g\cos\theta \qquad \text{(Equation 3)}$$

--Note that although the problem did not ask you to do so, solving for a_2 is done in the following manner.

Plugging *Equation 1* into *Equation 3* yields:

$$N = -(m_2 g - m_2 a_2)\sin\phi + m_1 g\cos\theta \qquad \text{(Equation 4)}$$

Plugging *Equation 1* and *Equation 4* into *Equation 2* yields:

$$T\cos\phi + \mu_k N - m_1 g\sin\theta = m_1(a_2\cos\phi)$$

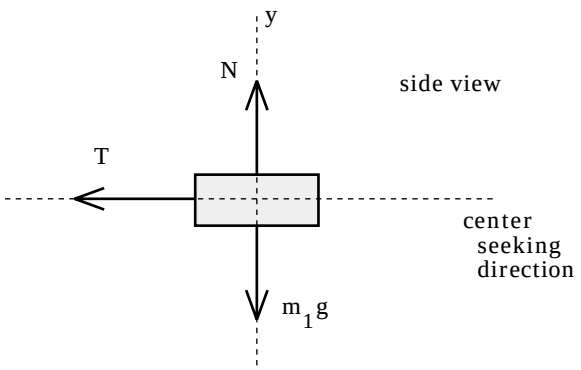
$$(m_2 g - m_2 a_2)\cos\phi + \mu_k[-(m_2 g - m_2 a_2)\sin\phi + m_1 g\cos\theta - m_1 g\sin\theta = m_1(a_2\cos\phi)$$

Rearranging and solving for a_2 yields:

$$a_2 = \frac{m_2 g\cos\phi - \mu_k m_2 g\sin\phi + \mu_k m_1 g\cos\theta - m_1 g\sin\theta}{m_1\cos\phi + m_2\cos\phi - \mu_k m_2\sin\phi}.$$

Interesting Note: There are positive and negative parts of the denominator, but it's OK because the two amounts will never add to zero.

5.10) This is a circular motion problem. There must be a natural force somewhere in the system that acts to change the direction of m_1's motion. That is, there must be a *gravitational* or *normal* or *tension* or *friction* or *push-me-pull-you* force that is *center-seeking*. In this case, that force provided by the system is the tension in the string. The



problem proceeds:

Using N.S.L. on mass m_1 (see f.b.d. to right):

$$\underline{\Sigma F_c}:$$

$$-T = -m_1 a_c$$

$$\Rightarrow \quad T = m_1 (v^2/R)$$

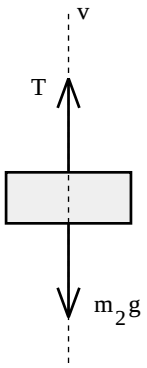
$$\Rightarrow \quad v = (TR/m_1)^{1/2}.$$

This equation has two unknowns, v and T. To get rid of the tension term, consider N.S.L. applied to mass m_2 (see f.b.d. to right):

$$\underline{\Sigma F_v}:$$

$$T - m_2 g = 0 \qquad (\text{as } a_y = 0)$$

$$\Rightarrow \quad T = m_2 g.$$



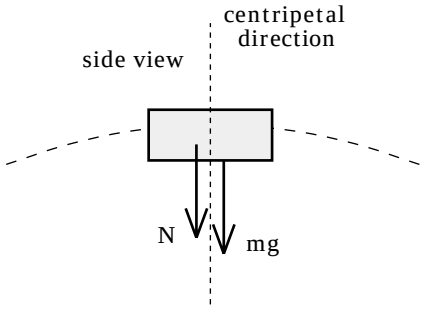
Substituting back into our expression for v, we get:

$$v = [TR/m_1]^{1/2}$$

$$= [(m_2 g)R/m_1]^{1/2}.$$

This is a nice problem as it requires you to deal with more than one body. The approach is the same as it has always been. Do *an f.b.d.* for one body in the system. In this case, notice that the body is moving in a circular path. As such, orient one axis so that it is *center-seeking* (i.e., along the radius of the arc upon which the bob is moving). Use N.S.L. to generate as many equations as needed. If you haven't enough equations to solve for the desired unknown, pick a second mass and repeat the approach.

5.11) *An f.b.d.* for the forces acting on the cart when at the top of the loop is shown to the right. N.S.L. yields:



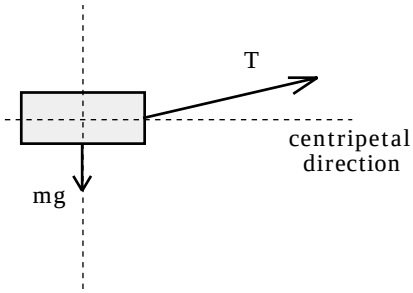
$$\underline{\Sigma F_c}:$$
$$-N - mg = -m\,a_c$$
$$= -m\,(v^2/R)$$
$$\Rightarrow \quad v = [(N + mg)R/m]^{1/2}.$$

When the cart just freefalls through the top of the arc, the normal force goes to zero. In that case:

$$v = [gR]^{1/2}.$$

5.12)

a.) To begin with, the tension vector must have a *vertical component* (see f.b.d. to the right). If it doesn't, there will be nothing to counteract gravity and the rock must accelerate downward--something our object is not doing. As such, that vertical force will ALWAYS equal *mg*. BUT, if the rock is moving fast, the angle will be small and the vertical component will be very small *in comparison to T*. In that case, we can assume the tension force *T* is *wholly centripetal* and $r = L$. Using those assumptions:

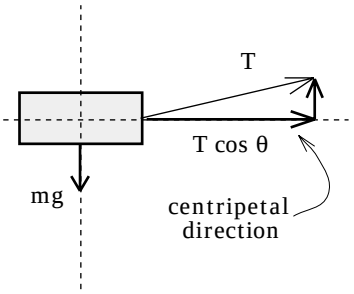


$$\underline{\Sigma F_c}:$$
$$T = m\,a_c$$
$$= m\,(v^2/L)$$
$$\Rightarrow \quad v = [TL/m]^{1/2}.$$

Putting in the numbers and using T_{max}, this yields:

$$v = [TL/m]^{1/2}$$
$$= [(50\text{ nt})(1.2\text{ m})/(.2\text{ kg})]^{1/2}$$
$$= 17.32\text{ m/s}.$$

b) Because there is centripetal motion going on here, the temptation is to draw *an f.b.d.* like the one shown to the right and then sum the forces in the *center-seeking direction*. Noting that the radius r of the body's motion is $L \cos \theta$, we write:



$$\underline{\Sigma F_c}:$$
$$T \cos \theta = m a_c$$
$$= m [v^2/r]$$
$$= m [v^2/(L \cos \theta)]$$
$$\Rightarrow \quad (\cos \theta)^2 = [mv^2/LT].$$

This equation would be great if we knew the velocity and wanted the angle (or vice versa). Unfortunately, we know neither. In other words, for this particular question, summing in the *center-seeking direction* is going to be no help at all (at least not initially). With that in mind, let's use N.S.L. in the *vertical direction* and pray it gives us an equation we can use.

$$\underline{\Sigma F_v}:$$
$$T \sin \theta - mg = 0 \qquad (\text{as } a_y = 0)$$
$$\Rightarrow \quad \sin \theta = mg / T_{max}$$
$$= (.2 \text{ kg})(9.8 \text{ m/s}^2) / (50 \text{ nt})$$
$$= .039$$
$$\Rightarrow \quad \theta = 2.247^o.$$

c.) We now know the angle that corresponds to the velocity at which the string will give up and break. With that information we can use N.S.L. in the *center-seeking* direction to bring the velocity term into play (that equation was derived above--it is re-derived below for your convenience). Doing so yields:

$$\underline{\Sigma F_c}:$$
$$T \cos \theta = m a_c$$
$$= m [v^2/(L \cos \theta)]$$
$$\Rightarrow \quad v = [LT(\cos \theta)^2/m]^{1/2}$$
$$= [(1.2 \text{ m}) (50 \text{ nt}) (\cos 2.247^o)^2 / (.2 \text{ kg})]^{1/2}$$
$$= 17.3 \text{ m/s}.$$

Notice how close this is to the solution determined in *Part a*. The reason for this should be obvious. The string-breaking velocity is high which means the string-breaking angle is small. Being so, the vertical tension component (this must equal *mg*) will be small in comparison to the overall tension *T* and the *horizontal tension component* will very nearly equal *T*. The assumption we made in *Part a* was that the tension was all in the center-seeking (horizontal) direction--in this case, that wasn't a bad assumption to make.

d.) For this part, we must incorporate the velocity into our analysis (we didn't do that when we were looking for the angle in *Part b*; you should understand the difference between these two situations). Using the f.b.d. shown in *Part b-i*, we can use N.S.L. to write:

$$\underline{\textstyle\sum F_c}:$$

$$T\cos\theta = m\,a_c$$

$$= m\,[v^2/(L\cos\theta)]$$

$$\Rightarrow \quad v = [TL(\cos\theta)^2/m]^{1/2} \qquad \text{(Equation A).}$$

In this case, we don't know *T*. Looking at the vertical forces yields:

$$\underline{\textstyle\sum F_v}:$$

$$T\sin\theta - mg = 0 \qquad (\text{as } a_y = 0)$$

$$\Rightarrow \quad T = mg/\sin\theta.$$

Substituting *T* into Equation A:

$$v = [T(\cos\theta)^2 L/m]^{1/2}$$

$$= [(mg/\sin\theta)(\cos\theta)^2 L/m]^{1/2}$$

$$= [(g/\sin\theta)(\cos\theta)^2 L]^{1/2}$$

$$= [g(\cot\theta)(\cos\theta)L]^{1/2}.$$

NOTE: If you don't like the use of the cotangent function (cos/sin), forget it and simply use the sine and cosine terms as presented.

Putting in the numbers, we get:

$$v = [(9.8\ m/s^2)(\cot 30^o)(\cos 30^o)(1.2\ m)]^{1/2}$$

$$= 4.2\ m/s.$$

5.13)

a.) The gravitational force between you and the earth, using Newton's general gravitational expression, is:

$$F_g = G\, m_{you} m_e / r^2$$
$$= (6.67\text{x}10^{-11}\ m^3/kg{\cdot}s^2)\ (70\ kg)\ (5.98\text{x}10^{24}\ kg)\ /\ (6.37\text{x}10^6\ m)^2$$
$$= 688\ nts.$$

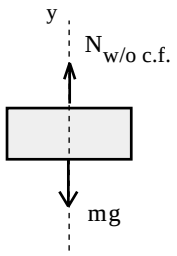
Using $m_{you}g$:

$$F_g = m_{you} g$$
$$= (70\ kg)\ (9.8\ m/s^2)$$
$$= 686\ nts.$$

The discrepancy is due to round-off error.

Note: The reason we can get away with using *mg* when near the earth's surface is due to the fact that the earth's radius is so large. That is, it really doesn't matter whether you are on the earth's surface or 200 meters above the earth's surface. For all intents and purposes, the distance between you and the center of the earth is going to be, to a very good approximation, the same.

b.) Let's begin by determining the amount of normal force ($N_{w/o\ c.f.}$) the earth must apply to you *when you stand at the poles*. The f.b.d. for the situation is shown to the right. Noting that there is no centripetal acceleration at the poles (at the poles the rotational speed of the earth is zero), a_y is zero and N.S.L. yields:



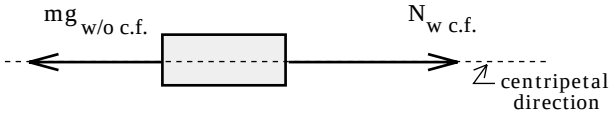
$\underline{\Sigma F_y}$:

$$N_{w/o\ c.f.} - mg_{w/o\ c.f.} = 0 \qquad (\text{as } a_y = 0)$$
$$\Rightarrow \quad N_{w/o\ c.f.} = mg_{w/o\ c.f.}$$
$$= (70\ kg)(9.83\ m/s^2)$$
$$= 688.1\ \text{newtons}.$$

Note that at the equator, the earth's rotational speed is equal to *the distance a point on the equator travels in one day* (i.e., the circumference = $2\pi R = (2)(3.14)(6.37\text{x}10^6\ m) = 4\text{x}10^7\ m$) divided by *the time it takes to do the traveling* (i.e., *24 hours = 86,400 seconds*), or:

$v_{eq} = d / t$
$= (4x10^7 \text{ m}) / (86{,}400 \text{ sec})$
$= 463.2 \text{ m/s}$ (this is around 1000 mph).

Let's now determine the amount of normal force ($N_{w\ c.f.}$) the earth must apply to you *when you stand at the equator*. The f.b.d. for the situation is shown to the right. Noting that as there is centripetal acceleration at the equator (at the equator there is rotational speed in the amount calculated above), a_y is non-zero and N.S.L. yields:



$\underline{\sum F_c}$:

$N_{w\ c.f.} - mg_{w/o\ c.f.} = -ma_c$
$= -m(v^2/R)$
$\Rightarrow \quad N_{w\ c.f.} = mg_{w/o\ c.f.} - m(v^2/R).$

Put in a different context, the normal force required at the equator will be equal to the normal force required without centripetal force (remember, $N_{w/o\ c.f.} = mg_{w/o\ c.f.}$ from above) minus the centripetal force (this will numerically equal mv^2/R) required to move you into circular motion. Putting in the numbers yields:

$N_{w\ c.f.} = mg_{w/o\ c.f.} - m(v^2/R)$
$= (688.1 \text{ nts}) - (70 \text{ kg})[(463.2 \text{ m/s})^2/(6.37x10^6 \text{ m})$
$= 685.7 \text{ nts}.$

If we wanted to define a gravitational constant g_{equ} that, when multiplied by your mass gives the amount of force the earth must exert on you when you stand at the equator (that is exactly how the g value you have come to know and love was originally determined), g_{equ} will be:

$N_{w\ c.f.} = mg_{w\ c.f.}$
$\Rightarrow \quad g_{w\ c.f} = N_{w\ c.f.}/m$
$= (685.7 \text{ nts}) / (70 \text{ kg})$
$= 9.796 \text{ m/s}^2.$

c.) Defining the distance between the earth and moon to be *r* and using N.S.L., we get:

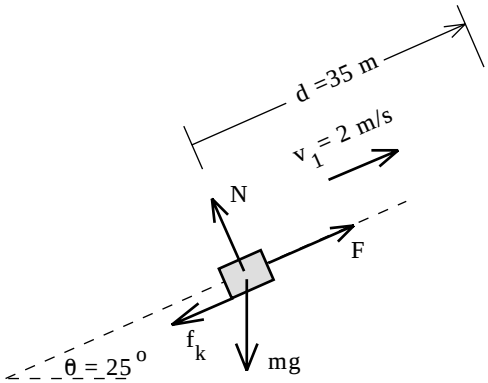
$\underline{\sum F_c}$:

$$-G\, m_e m_m / r^2 = -m_m a_c$$
$$= -m_m v^2 / r$$
$$\Rightarrow \quad v = (G m_e / r)^{1/2} \qquad \text{(Equ. A)}$$
$$= [(6.67\text{x}10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2)\ (5.98\text{x}10^{24}\ \text{kg}) / (3.84\text{x}10^{8}\ \text{m})]^{1/2}$$
$$= 1019\ \text{m/s}.$$

Interesting Note: Just as two objects will accelerate at the same rate (assuming neither gets close to its terminal velocity) under the influence of gravity, the velocity required to pull a mass in a given-radius circular path does NOT depend upon the mass of the object being so motivated. This might not be immediately obvious (just as the first statement in this NOTE wasn't obvious back when you first ran into it), but it is supported by the math. The moon is the mass being centripetally accelerated, and the m_m terms do cancel out in our velocity equation as derived above.

CHAPTER 6 -- ENERGY

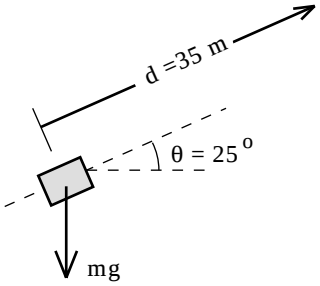
6.1) The f.b.d. shown to the right has been provided to identify all the forces acting on the body as it moves up the incline.



a.) To determine the work done by gravity as the body moves up the incline, there are two approaches. For your convenience, the force, velocity, and displacement are pictured below and also to the right.

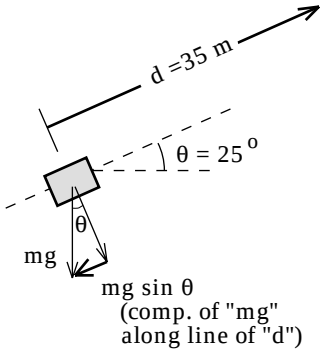
Approach #1: Using the definition of work and the angle between *mg* and *d* as ϕ:

$$W_{grav} = \mathbf{F}_{grav} \cdot \mathbf{d}$$
$$= (mg)\,(d)\cos\phi$$
$$= (mg)\,(d)\cos(25^o + 90^o)$$
$$= (3\text{ kg})(9.8\text{ m/s}^2)(35\text{ m})\,(-.423)$$
$$= -434.87\text{ joules.}$$



Approach #2: Using the component of *mg* along the line of *d*:

$$W_{grav} = \mathbf{F}_{grav} \cdot \mathbf{d}$$
$$= \pm(F_{\text{mg parallel to "d"}})\,(d)$$
$$= -(mg \sin 25^o)\,(d)$$
$$= -[(3\text{ kg})(9.8\text{ m/s}^2)(.423)]\,(35\text{ m})$$
$$= -434.87\text{ joules.}$$

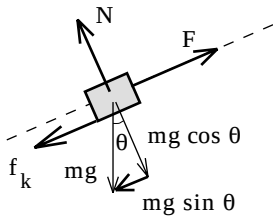


Note: In this case, the force is OPPOSITE the direction of the displacement which means the work must be negative. The *negative sign* in this case must be inserted *manually*. An alternative would be to notice that the angle ϕ between *d* and *F's* component along *d's* line is 180^o and determine the work quantity using:

$$W_{grav} = \mathbf{F}_{grav} \cdot \mathbf{d}$$
$$= (F_{mg \text{ parallel to "d"}})\ (d)\ \cos\ \phi$$
$$= (mg \sin 25^o)\ (d)\ \cos 180^o.$$

In doing so, the cosine function will give you the *-1* automatically.

b.) The frictional force is equal to $\mu_k N$. To determine *N*, we need to use an f.b.d. and N.S.L. in the *normal direction*. The f.b.d. is shown to the right. N.S.L. yields:



$\underline{\sum F_N}$:

$N - mg \cos\ \theta = 0 \quad (\text{as } a_N = 0)$

$\Rightarrow \quad N = mg \cos\ \theta$

$\Rightarrow \quad f_k = \mu_k N$

$= \mu_k\ (mg \cos\ \theta)$

$= (.3)\ (3 \text{ kg})(9.8 \text{ m/s}^2) \cos 25^o$

$= 7.99$ nts.

Friction is always opposite the direction of motion. The *work* friction does will be:

$$W_f = \mathbf{f}_k \cdot \mathbf{d}$$
$$= (f_k)\ (d)\ \cos 180^o$$
$$= -\ f_k d$$
$$= -(7.99 \text{ nts})(35 \text{ m})$$
$$= -279.65 \text{ joules}.$$

c.) The angle between *d* and *N* is 90^o. The cosine of 90^o is zero. That means that the *net work* done by the *normal force* will be zero . . . ALWAYS!

d.) Kinetic energy is defined as $(1/2)mv^2$. Using that expression we get:

$$KE_1 = (1/2)mv^2$$
$$= .5\ (3 \text{ kg})\ (2 \text{ m/s})^2$$
$$= 6 \text{ joules}.$$

e.) The work/energy theorem states:

$$W_{net} = \Delta KE.$$

For this case, that means:

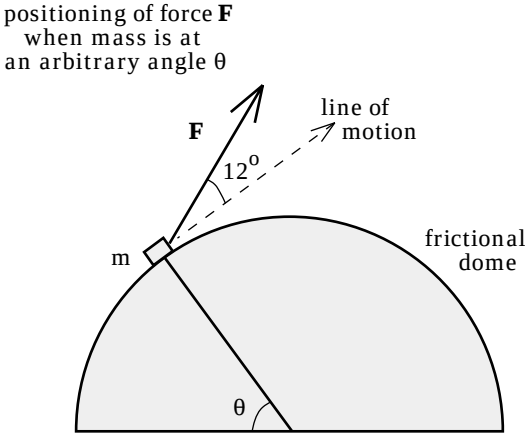
$$\begin{array}{ccccccc} W_{net} & & & = & & \Delta KE. \\ W_f & + & W_F & + & W_{mg} & = & KE_2 - KE_1. \end{array}$$

$$(f_k)\,(d)\cos 180^o + F\,(d)\cos 0^o + (mg)\,(d)\cos\phi = (1/2)mv_2^{\,2} - (1/2)mv_1^{\,2}.$$

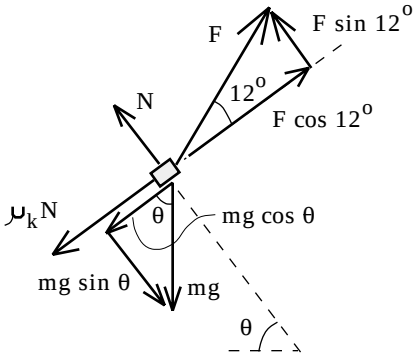
Plugging in the numbers, we get:

$$(-279.65\text{ J}) + F(35\text{ m}) + (-434.87\text{ J}) = (1/2)(3\text{ kg})(7\text{ m/s})^2 - (1/2)(3\text{ kg})(2\text{ m/s})^2$$
$$\Rightarrow \quad F = 22.34\text{ nts.}$$

6.2) The situation is shown in the sketch to the right. We need to derive a *general algebraic expression* for the force ***F*** acting on the block, given the fact that that force is always oriented at an angle of 12^o relative to the direction of motion.



Consider the f.b.d. (shown below) for the forces acting on the body when located at an arbitrary angle. As v is small and constant, both the centripetal acceleration (i.e., v^2/R) and translational acceleration (i.e., dv/dt) are zero or approximately zero. Therefore, N.S.L. and the f.b.d. yields:



$\underline{\sum F_x}$:

$$N - mg\sin\theta + F\sin 12^o = 0 \quad (\text{as } a_N = 0)$$
$$\Rightarrow \quad N = (mg\sin\theta) - F\sin 12^o.$$

Knowing the normal force, the frictional force follows as:

$$f_k = \mu_k N$$
$$= \mu_k\,(mg\sin\theta - F\sin 12^o).$$

$\underline{\sum F_{tang}}$:

$-\mu_k N - mg\cos\theta + F\cos 12^o = 0$

$\Rightarrow \quad -\mu_k(mg\sin\theta - F\sin 12^o) - mg\cos\theta + F\cos 12^o = 0$

$\Rightarrow \quad F = [\mu_k mg\sin\theta + mg\cos\theta] / [\mu_k \sin 12^o + \cos 12^o].$

Rewriting this with the constants in front of the variable expression, we get:

$$F = \left[\frac{mg}{\cos 12^o + \mu_k \sin 12^o}\right][\mu_k \sin\theta + \cos\theta].$$

Knowing the force in general terms, we can use $\int \mathbf{F}\cdot d\mathbf{r}$ to determine the work the force does as the body rises from $\theta = 20^o$ to $\theta = 60^o$. Noting that:

--The angle ϕ between the *line of* $\boldsymbol{F}$ and the *line of* $d\boldsymbol{r}$ is always 12^o;

--The magnitude of a differential displacement $d\boldsymbol{r}$ along an arc equals $R\,d\theta$, where R is the radius of the arc and $d\theta$ is the differential angle through which the motion occurs;

--And $\mu_k = .2$, $m = .5$ *kg*, and $R = .3$ *meters*, we can write:

$$W = \int \mathbf{F}\bullet d\mathbf{r}$$

$$= \int |\mathbf{F}||d\mathbf{r}|\cos\phi$$

$$= \left[\frac{mg}{\cos 12^o + \mu_k \sin 12^o}\right]\int_{\theta=20^o}^{60^o}[\mu_k \sin\theta + \cos\theta][R\,d\theta]\cos 12^o$$

$$= \left[\frac{mgR\cos 12^o}{\cos 12^o + \mu_k \sin 12^o}\right]\left[\int_{\theta=20^o}^{60^o}[\mu_k \sin\theta]d\theta + \int_{\theta=20^o}^{60^o}[\cos\theta]d\theta\right]$$

$$= \left[\frac{mgR\cos 12^o}{\cos 12^o + \mu_k \sin 12^o}\right]\left[[\mu_k(-\cos\theta)]_{\theta=20^o}^{60^o} + [(\sin\theta)]_{\theta=20^o}^{60^o}\right]$$

$$= \left[\frac{(.5\text{ kg})(9.8\text{ m/s}^2)(.3)\cos 12^o}{\cos 12^o + (.2)\sin 12^o}\right]\left[\left[(.2)\left(-\left[\cos 60^o - \cos 20^o\right]\right)\right] + \left[(\sin 60^o - \sin 20^o)\right]\right]$$

$= .86$ joules.

6.3) All the energy is stored in spring potential energy. Using the potential energy function for a spring we get:

$$\begin{aligned} U_{sp} &= (1/2)kx^2 \\ &= .5(120 \text{ nt/m})(.2 \text{ m})^2 \\ &= 2.4 \text{ joules.} \end{aligned}$$

6.4) The relationship between the field's *potential energy* function (assumed known) and its associated *force* function is:

$$\begin{aligned} \mathbf{F}(x,y,z) &= -\vec{\nabla}\, U(x,y,z) \\ &= -\left(\frac{\partial(U)}{\partial x}\mathbf{i} + \frac{\partial(U)}{\partial y}\mathbf{j} + \frac{\partial(U)}{\partial z}\mathbf{k} \right) \\ &= -\left(\frac{\partial\left(-\frac{k_1}{x}e^{-ky}\right)}{\partial x}\mathbf{i} + \frac{\partial\left(-\frac{k_1}{x}e^{-ky}\right)}{\partial y}\mathbf{j} + \frac{\partial\left(-\frac{k_1}{x}e^{-ky}\right)}{\partial z}\mathbf{k} \right) \\ &= -\left[\left(\frac{k_1}{x^2}\right)e^{-ky}\mathbf{i} + \frac{k_1}{x}\left(ke^{-ky}\right)\mathbf{j} \right] \end{aligned}$$

6.5) Looking at the function, the force will equal zero when $y = 0$ and when $\ln x = 3$ (i.e., when $x = 20.086$). Using this and our force/potential energy relationship, we get:

$$U(x, y) - U(x=20.086, y=0) = -\int \mathbf{F}\cdot d\mathbf{r}.$$

Observing that $U(x = 20.086, y = 0) = 0$ and that $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, and realizing that if this was a test question, 90% of the points would be wrapped up in the *first two lines* of what follows (i.e., the layout), we can write this as:

$$\begin{aligned} U(x,y) &= -\int_{x=20.086,y=0}^{x,y} \left[\left(k_1 \ln(x) - 3\right)\mathbf{i} - \left(k_2 y^2\right)\mathbf{j} \right] \bullet \left[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \right] \\ &= -\int_{x=20.086,y=0}^{x,y} \left[\left(k_1 \ln(x) - 3\right)dx - \left(k_2 y^2\right)dy \right] \\ &= -\int_{x=20.086}^{x} \left[k_1 \ln(x) - 3 \right]dx \; + \; \int_{y=0}^{y} \left[k_2 y^2 \right]dy \\ &= -\int_{x=20.086}^{x} \left[k_1 \ln(x) \right]dx \; + \; \int_{x=20.086}^{x} [3]dx \; + \; \int_{y=0}^{y} \left[k_2 y^2 \right]dy. \end{aligned}$$

Setting the *k* terms equal to *one* and noting that $\int(lnx)dx = (x)lnx - x$, we can evaluate these integrals as:

$$U(x,y) = -\left[x\ln x - x\right]_{x=20.086}^{x} - \left[-3x\right]_{x=20.086}^{x} + \left[\frac{y^3}{3}\right]_{y=0}^{y}$$

$$= -\left[(x\ln x - x) - (20.086\ln 20.086 - 20.086)\right] - \left[\left([-3x]\right) - \left([-3(20.086)]\right)\right] + \left[\left(\frac{y^3}{3}\right) - \left(\frac{(0)^3}{3}\right)\right]$$

$$= -x\ln x + 4x + \frac{y^3}{3} - 20.086.$$

Note: A perfectly legitimate follow-up question to this problem might be: How much *work* does the force field do as the body goes from *x = 1, y = 2* to *x = 4, y = -1*. The answer is:

$W_{cons.forc} = -\Delta U$

$= -[\ U_{pt2}\ -\ U_{pt1}\ ]$

$= -[[-4\ln4 + 4(4) + (-1)^3/3 - 20.09)] - [-\ln(1) + 4(1) + (2)^3/3 - 20.09)]]$

$= -3.45$ joules.

6.6)

a.) We could use the *work/energy theorem* on this problem, but the *modified conservation of energy equation* is so much easier to use that we will use it here. Noting that the tension in the line is always *perpendicular to the motion* (i.e., the work done due to tension is zero), we can write:

$$KE_1 + \sum U_1 + \sum W_{extra} = KE_2 + \sum U_2$$

$$(1/2)mv_1^2 + mgy_1 + T\cos 90^o = (1/2)mv_2^2 + mgy_2$$

$$(0) + m(9.8\ m/s^2)(12\ m) + (0) = (1/2)mv_2^2 + m(9.8\ m/s^2)(5\ m).$$

Being careful not to confuse mass terms denoted by *m* and the units of length (meters, abbreviated *m*), we can cancel the mass terms and get:

$$v_2 = 11.71\ m/s.$$

b.) At the bottom of the arc, Tarzan's velocity can again be found using the *modified conservation of energy* expression (we need that velocity

because he is moving under the influence of a center-seeking force--a velocity driven function--at that point). Using the approach:

$$\begin{array}{cccccc} KE_1 & + & \sum U_1 & + & \sum W_{extra} = & KE_{bot} + \sum U_{bot} \\ (1/2)mv_1^2 & + & mgy_1 & + & (0) = & (1/2)mv_{bot}^2 + mgy_2 \end{array}$$

$$(0) + m(9.8\ m/s^2)(12\ m) + (0) = (1/2)mv_{bot}^2 + m(9.8\ m/s^2)(2\ m).$$

Canceling the mass terms yields:

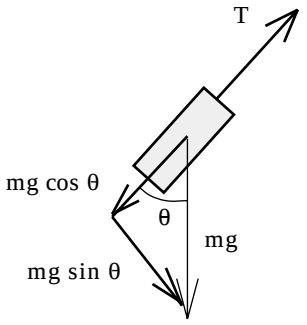
$$v_{bot} = 14\ m/s.$$

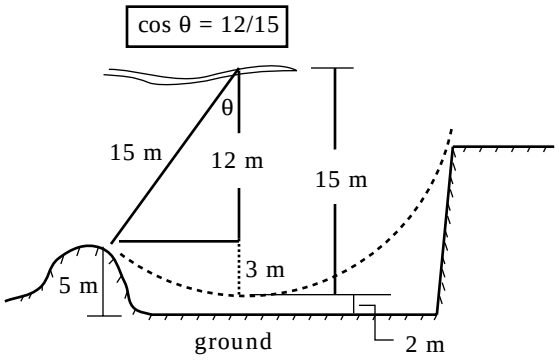
There is a *tension force* upward and *gravity* downward when Tarzan is at the bottom of the arc (you might want to draw an f.b.d. to be complete). N.S.L. implies:

$\underline{\sum F_c}$:

$$\begin{aligned} T - mg &= ma_c \\ &= m(v^2/R) \qquad \text{(as } a_c \text{ is a centripetal acc.)} \\ \Rightarrow \quad T &= mg + mv^2/R \\ &= (80\ kg)(9.8\ m/s^2) + (80\ kg)(14\ m/s)^2/(15\ m) \\ &= 1829.3\ nts. \end{aligned}$$

c.) At the molehill, Tarzan's velocity is 11.71 m/s. An f.b.d. for that situation is shown below-left. The only thing that is really tricky about the problem is determining the angle θ. The diagram below-right will do that.





With θ, N.S.L. yields:

$\underline{\sum F_c}$:

$$T - mg\cos\theta = ma_c$$
$$= m(v^2/R).$$

As a_c is a centripetal acceleration:

$$\Rightarrow \quad T = mg\cos\theta + mv^2/R$$
$$= (80\text{ kg})(9.8\text{ m/s}^2)(12/15) + (80\text{ kg})(11.71\text{ m/s})^2/(15\text{ m})$$
$$= 1358.5\text{ nts.}$$

6.7) Both gravity and friction do *work* as the body slides down the incline. Using the *modified conservation of energy equation*, we get:

$$KE_1 + \sum U_1 + \sum W_{extra} = KE_2 + \sum U_2$$
$$(1/2)mv_1^2 + mgy_1 + (-f_k d) = (1/2)mv_2^2 + mgy_2$$
$$(0) + mgR - (f_k d) = (0) + (0)$$
$$\Rightarrow \quad f_k = mgR/d.$$

In this case, *d* is the total distance over which the frictional force acts. That is, the 18 meters along the horizontal surface AND the quarter circumference down the curved incline (that will equal *(1/4)(2πR)*). That total distance is:

$$d = 18 + .5\pi R$$
$$= 18 + .5\pi(2\text{ m})$$
$$= 21.14\text{ m.}$$

Plugging this into our expression, we get:

$$f_k = mgR/d$$
$$= (12\text{ kg})(9.8\text{ m/s}^2)(2\text{ m})/(21.14)$$
$$= 11.13\text{ nts.}$$

6.8) The fact that the angle is 85^o makes no difference, assuming the velocity is great enough to allow the dart to make it to the monkey. What is important is that the dart has enough energy in the beginning to effect a pierce at the end. Using *conservation of energy*:

$$KE_1 + SU_1 + SW_{ext} = KE_2 + SU_2$$
$$(1/2)mv_1^2 + (0) + (0) = (1/2)mv_2^2 + mgh_2.$$

Dividing out the masses and multiplying by *2* yields:

$$v_1 = [v_2^2 + 2gh_2]^{1/2}$$
$$= [(4 \text{ m/s})^2 + 2(9.8 \text{ m/s}^2)(35 \text{ m})]^{1/2}$$
$$= 26.5 \text{ m/s}.$$

6.9) Assuming an average frictional force of 27 newtons:

a.) Let the *potential energy equals zero* level be the ground. That means that below-ground level, the *h* in *mgh* will be *negative*. Using *conservation of energy*:

$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$
$$(1/2)mv_1^2 + (0) + (-f_k d) = (1/2)mv_C^2 + mgh_C.$$

or

$$.5(1800 \text{ kg})(38 \text{ m/s})^2 - (27 \text{ nts})(130 \text{ m}) = .5(1800 \text{ kg})v_C^2 + (1800 \text{ kg})(9.8 \text{ m/s}^2)(-15 \text{ m})$$
$$\Rightarrow \quad v_C = 41.64 \text{ m/s}.$$

b.) The only thing that is tricky about this is finding the total distance *d* traveled (we need that to determine the amount of work friction does). Noting that the distance traveled while moving through the loop is $2\pi R$, *conservation of energy* yields:

$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$
$$(1/2)mv_1^2 + (0) + (-f_k d_{tot}) = (0) + mgh_{ramp}$$
$$(1/2)mv_1^2 + 0 - f_k(70+60+40+2\pi(20)+d) = 0 + mgd \sin\theta$$
$$.5(1800 \text{ kg})(38 \text{ m/s})^2 - [(27 \text{ nts})(295.7 \text{ m}) + 27d] = (1800)(9.8 \text{ m/s}^2)d \sin 30^o$$
$$\Rightarrow \quad d = 146 \text{ m}.$$

c.) Along with the *conservation of energy*, this problem requires N.S.L. We know that at the top of the arc, the *vertical forces* are centripetal at the

velocity required AND the *normal force* goes to zero. From the *conservation of energy* we get:

$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$
$$(1/2)mv_1^2 + (0) + (-f_k d_{top}) = (1/2)mv_{top}^2 + mgh_{top}$$
$$(1/2)mv_1^2 + 0 - f_k[70+60+40+(2\pi R)/2] = (1/2)mv_{top}^2 + mg(2R)$$
$$.5(1800\text{ kg})v_1^2 + 0 - (27\text{ nts})(232.8\text{ m}) = .5(1800\text{ kg})v_{top}^2 + (1800)(9.8\text{ m/s}^2)[2(20\text{ m})]$$
$$\Rightarrow \quad v_1^2 = v_{top}^2 + 791.$$

Using N.S.L., we get:

$$-N - mg = -ma_c$$
$$= -m(v_{top}^2/R).$$

When the velocity is correct, *N* goes to zero and:

$$v_{top}^2 = gR.$$

Substituting that into our expression, we get:

$$v_1^2 = v_{top}^2 + 791$$
$$= gR + 791$$
$$= (9.8\text{ m/s}^2)(20\text{ m}) + 791$$
$$\Rightarrow \quad v_1 = 31.42\text{ m/s}.$$

6.10)

a.) Assuming the spring is depressed a distance *x*, let's define the *gravitational potential energy equals zero* level to be at that point (i.e., where the spring is depressed a distance *x*). As such, the crate travels a distance $d + x$, where *d* is the crate's initial distance from the bumper. The *modified conservation of energy theorem* implies:

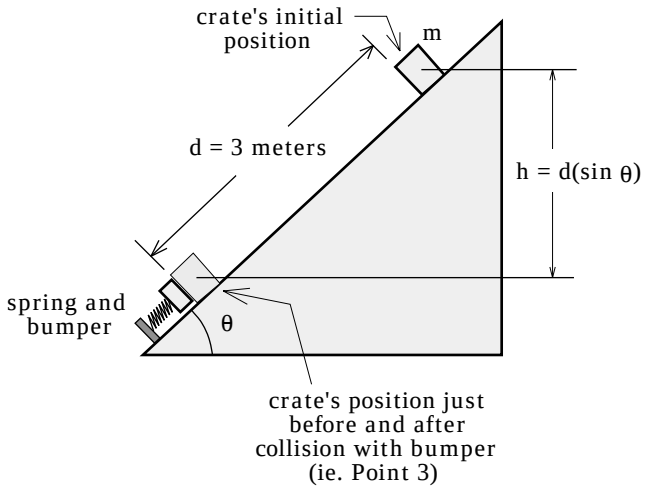
$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$
$$(0) + [U_{grav,1} + U_{sp,1}] + \Sigma W_{ext} = (0) + [U_{grav,2} + U_{sp,2}]$$
$$(0) + [mg(d+x)\sin\theta + (0)] - f_k(d+x) = (0) + [(0) + .5kx^2]$$
$$\Rightarrow \quad mg(d+x)\sin\theta - f_k(d+x) = .5kx^2$$
$$\Rightarrow \quad (60\text{ kg})(9.8\text{ m/s}^2)(3 + x)\sin 55^o - (100\text{ nt})(3 + x) = .5(20{,}000)x^2.$$

Rearranging:

$$10{,}000x^2 - 381.7x - 1145 = 0.$$

Using the Quadratic Formula, we get $x = .36$ *meters*.

b.) In this section, we really are not interested in x--we want to know how much energy the block has just before hitting the spring, and how much energy the block loses by the time it leaves the spring. In other words, this is really a brand new problem. As such, let's redefine the *zero gravitational potential energy level* to be at the point when the block is just about to come into contact with the spring. If that be the case, the *conservation of energy* allows us to determine the energy of the block just before striking the spring at *Point 3*:


crate's initial position
m
d = 3 meters
h = d(sin θ)
spring and bumper
θ
crate's position just before and after collision with bumper (ie. Point 3)

$$\begin{aligned} KE_1 + \quad \Sigma U_1 \quad + \Sigma W_{ext} &= KE_3 + \Sigma U_3 \\ (0) + mg\,d\sin\theta \; - \; f_k d \quad &= KE_3 + \quad (0) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad KE_3 &= mgd\sin\theta - f_k d \\ &= (60\text{ kg})(9.8\text{ m/s}^2)(3\sin 55^o) - (100\text{ nt})(3\text{ m}) \\ &= 1144.98\text{ joules.} \end{aligned}$$

If 3/4 of the kinetic energy is lost, then *1/4* is left. That means the block has kinetic energy *(1/4)(1144.98 j) = 286.2 joules* as it starts back up the incline. If we let L be the distance the block travels up the incline to rest, and if we remember that the *U = 0 level* is at the spring's end, we have:

$$\begin{aligned} KE_4 \; + \; \Sigma U_4 + \quad \Sigma W_{ext} &= KE_5 \; + \quad\quad \Sigma U_5 \\ (286.2\text{ j}) + \; (0) \quad - (100\text{ nt})L \; &= \; (0) \quad + (60\text{ kg})(9.8\text{ m/s}^2)(L\sin 55^o) \end{aligned}$$

$$\Rightarrow \quad L = .49\text{ meters.}$$

6.11) Using the potential energy function provided, *conservation of energy* implies:

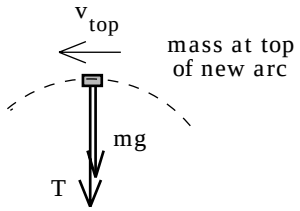
$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$
$$(1/2)m_s v_1^2 + [-Gm_e m_s/(r_e + d_1)] + (0) = (1/2)m_s v_2^2 + [-Gm_e m_s/(r_e + d_2)].$$

Noticing that the m_s*'s* cancel out, we can write:

$$.5(1500 \text{ m/s})^2 + [-(6.67\text{x}10^{-11} \text{ m}^3/\text{kg}\cdot\text{s}^2)(5.98\text{x}10^{24} \text{ kg})/(6.37\text{x}10^6 \text{ m} + 1.2\text{x}10^5 \text{ m})] =$$
$$.5v_2^2 + [-(6.67\text{x}10^{-11} \text{ m}^3/\text{kg}\cdot\text{s}^2)(5.98\text{x}10^{24} \text{ kg})/(6.37\text{x}10^6 \text{ m} + .9\text{x}10^5 \text{ m})].$$

SOLVING yields $v_2 = 1680$ *m/s*.

6.12) The trick to this problem is in recognizing the fact that at the top of its arc, the mass is executing *centripetal acceleration* where *tension* is acting as one of the *centripetal forces* in the system. Using N.S.L. and remembering that the radius of the mass's motion is *L/3*, we get:

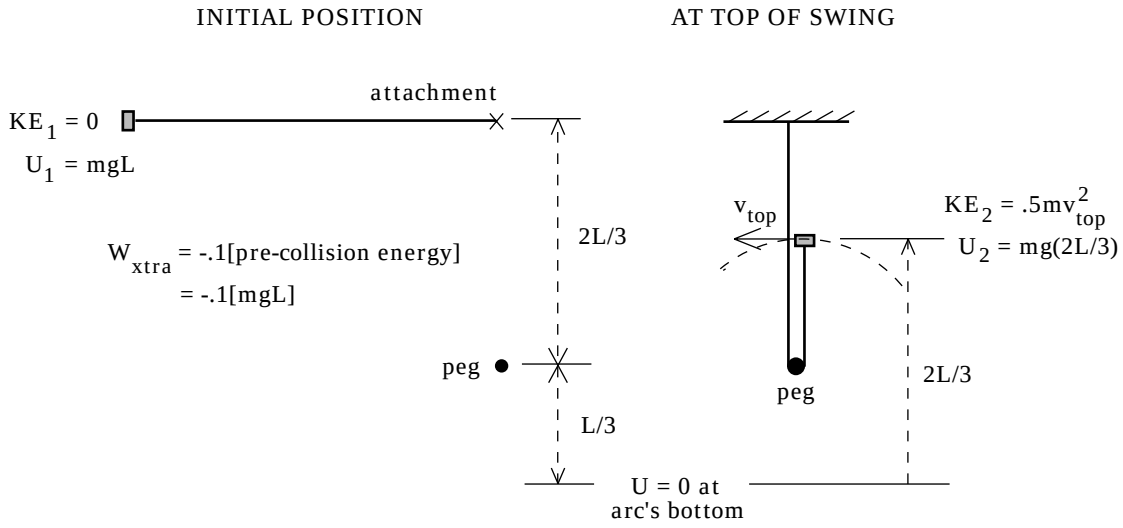


$\underline{\Sigma F_c}$:

$$-T - mg = -ma_c$$
$$= -m(v_{top}^2/R)$$
$$= -m(v_{top}^2/(L/3))$$
$$\Rightarrow \quad T = (3m/L)(v_{top})^2 - mg.$$

To solve this expression for *T*, we need the velocity of the mass at the top of its flight.

Enter the *modified conservation of energy* approach--an approach designed specifically to determine velocities when conservative force fields are doing work on bodies. If we take the *potential energy equal to ZERO* point to be at the bottom of the arc, the information given in the sketch on the next page tells it all.



Using our information, we get:

$$KE_1 + \Sigma U_1 + \Sigma W_{ext} = KE_2 + \Sigma U_2$$
$$(0) + mgL + [-(.1)(mgL)] = (1/2)mv_{top}^2 + mg(2L/3)$$
$$\Rightarrow \quad v_{top} = [.46gL]^{1/2}.$$

Plugging this into our *tension* expression yields:

$$\begin{aligned} T &= (3m/L)(v_{top})^2 - mg \\ &= (3m/L)([.46gL]^{1/2})^2 - mg \\ &= 1.38mg - mg \\ &= .38mg. \end{aligned}$$

CHAPTER 7 -- M O M E N T U M

7.1)

a-i.) The *y component* of momentum before the collision is directed upward while the *y component* of momentum after the collision is directed downward. Clearly, momentum in the *y direction* is not conserved through the collision. This is due to the fact that the force at the ceiling is large enough to change the ball's motion over a minuscule amount of time.

a-ii.) Even if friction was acting in the *x direction* during the collision, the frictional force would be small enough and would be applied over a small enough time to allow the momentum-change during the collision to be negligible. As such, momentum is conserved in the *x direction* through the collision.

b.) As the velocity-magnitude is the same just before and just after the collision, energy was not lost and the collision must have been elastic.

c.) The impulse absorbed by the ceiling as a consequence of the ball's collision with it will be *equal and opposite* to the impulse received by the ball from the ceiling. The ball receives no impulse in the *x direction* (its momentum in that direction is the same before as after the collision) but does receive a change of momentum Δp in the *y direction*. Noting that the ball's initial momentum in the *y direction* (i.e., $p_{1,y}$) is upward (i.e., positive) and its final momentum is downward (i.e., negative), we can write:

$$\begin{aligned}\Delta p_y &= p_{2,y} - p_{1,y} \\ &= (-mv_2 \cos 30^o) - (+mv_2 \cos 30^o) \\ &= -2(\ m\ \ \ v_2\ \ \cos 30^o) \\ &= -2(.4\text{ kg})(11\text{ m/s})\ (.86) \\ &= -7.62\text{ nt·sec.}\end{aligned}$$

If the ball's impulse is -7.62 nt·sec, the ceiling's impulse will be +7.62 nt·sec in the *y direction*. This makes sense as the ball's force on the ceiling will be *upward*, hence the *positive sign* on the impulse applied to the ceiling.

d.) The force the ceiling applies to the ball is (-3200 nt)**j**:

$$\begin{aligned}\mathbf{F}\ \Delta t &= \Delta \mathbf{p} = -7.62\ \mathbf{j}\text{ nt·sec} \\ \Rightarrow\quad \Delta t &= \Delta \mathbf{p}/\mathbf{F} \\ &= (-7.62\ \mathbf{j}\text{ nt·sec}) / (-3200\ \mathbf{j}\text{ nt}) \\ &= 2.4\text{x}10^{-3}\text{ sec.}\end{aligned}$$

7.2) Assuming that player #1 is the 60 kg kid and assuming the runners are moving in the *x direction*:

a.) $p_{1,x} = (60 \text{ kg})(10 \text{ m/s}) = (600 \text{ kg·m/s})$

$p_{2,x} = (120 \text{ kg})(5 \text{ m/s}) = (600 \text{ kg·m/s})$.

Both players will have the same amount of *momentum*.

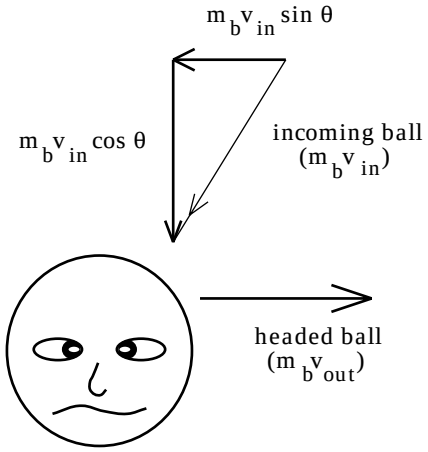
b.) $KE_1 = (1/2)m_1v_1^2 = .5(60 \text{ kg})(10 \text{ m/s})^2 = 3000 \text{ joules}$

$KE_2 = (1/2)m_2v_2^2 = .5(120 \text{ kg})(5 \text{ m/s})^2 = 1500 \text{ joules}$.

The players have different amounts of energy.

c.) Energy is what can hurt you. Energy is directly proportional to the mass of the moving object, but it is also directly proportional to the SQUARE of the object's velocity. The lesser amount of energy will be imparted by the larger player moving at the slower speed. It should be noted that although it may be more blessed to give than receive, both parties are going to hurt from the collision (Newton's third law--for every action there is an EQUAL and opposite reaction). Both players will feel the same force. The trick, assuming you want to play a sport predicated on the desire to kill someone, is to make the other guy absorb his blow in a more tender place than where you receive yours. That is, your head impacting his knee is not the way to go.

7.3) The sketch shows the incoming and outgoing ball, complete with momentum magnitudes and momentum components.



a.) It is easiest to do problems like this by first writing out what's happening in the *x direction*, then writing out the momentum equation for what's happening in the *y direction*. Subtracting the balls *incoming momentum* from its *outgoing momentum* in both directions yields the *change of momentum* in both directions. Sooo . . .

<u>x direction</u>:

$$\Delta p_x = p_{out,x} - p_{in,x}$$
$$= (mv_2) - (-mv_1\sin\theta)$$

$= [(.5 \text{ kg})(18 \text{ m/s}) + (.5 \text{ kg})(25 \text{ m/s})(\sin 30^o)]$
$= 15.25 \text{ kg·m/s}.$

y direction:

$\Delta p_y = p_{out,y} - p_{in,y}$
$= (0) - (-mv_1 \cos \theta)$
$= (.5 \text{ kg})(25 \text{ m/s})(\cos 30^o)$
$= 10.83 \text{ kg·m/s}.$

As a vector, $\Delta \mathbf{p} = (15.25\mathbf{i} + 10.83\mathbf{j})$ kg·m/s.

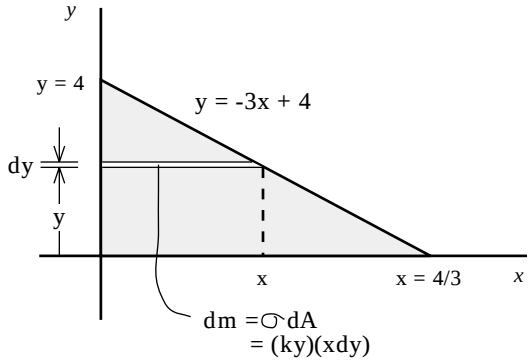
b.) The relationship between *force, change of momentum, and time* is wrapped up in the *impulse equation*. Specifically for the ball:

$\mathbf{F} = \Delta \mathbf{p}/\Delta t$
$= (15.25\mathbf{i} + 10.83\mathbf{j}) / (.08 \text{ sec})$
$= (190.6\mathbf{i} + 135.4\mathbf{j})$ nts.

This will be equal and opposite the force on your head (N.T.L.).

Note that this is a considerable amount of force. Its magnitude is 270 newtons, or approximately 50 pounds. Also, note that the longer the ball is in contact with the head, the smaller the force is. The moral: from the point of view of your head, it is better to play with an under-inflated ball than an over-inflated ball.

7.4) The sketch shows the structure along with a differential strip of width *dy* a distance *y* units above the *x-axis*. The differential area *dA* of that strip will equal:



$dA = (\text{length})(\text{height})$
$= (x)(dy).$

Writing *x* in terms of *y* can be done using the equation of the line. That is, if $y = -3x+4$, then $x = (-y+4)/3$. As such:

$dA = (x)dy$
$=[(-y+4)/3]dy$
$=[-.33y+1.33]dy.$

We need the differential mass associated with the differential area *dA*. We know the area density function is $\sigma = ky$, where *k* is a constant of magnitude one

and units appropriate to the situation (we will assume that $\sigma = y$ from here on as the k term will do nothing to the final numerical solution). That means the differential mass dm can be written as:

$$\begin{aligned} dm &= \sigma dA \\ &= (y)[(-.33y+1.33)dy] \\ &= (-.33y^2+1.33y)dy. \end{aligned}$$

Using the definition of *center of mass*, we can write:

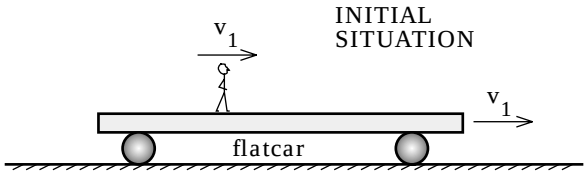
$$y_{cm} = \frac{\int y dm}{\int dm}$$

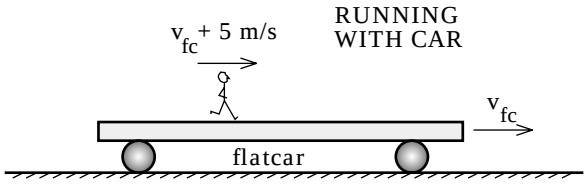
$$\begin{aligned} y_{cm} &= \frac{\int_{y=0}^{4} y\left[(-.33y^2 + 1.33y)dy\right]}{\int_{y=0}^{4}\left[(-.33y^2 + 1.33y)dy\right]} \\ &= \frac{\int_{y=0}^{4}\left[-.33y^3 + 1.33y^2\right]dy}{\int_{y=0}^{4}\left[-.33y^2 + 1.33y\right]dy} \\ &= \frac{\left[-.33\left(\frac{y^4}{4}\right) + 1.33\left(\frac{y^3}{3}\right)\right]_{y=0}^{4}}{\left[-.33\left(\frac{y^3}{3}\right) + 1.33\left(\frac{y^2}{2}\right)\right]_{y=0}^{4}} \\ &= \frac{\left[-.33\left(\frac{(4)^4}{4}\right) + 1.33\left(\frac{(4)^3}{3}\right)\right]}{\left[-.33\left(\frac{(4)^3}{3}\right) + 1.33\left(\frac{(4)^2}{2}\right)\right]} \\ &= \frac{7.125}{3.56} \\ &= 2.0 \text{ m}. \end{aligned}$$

Note: Students will occasionally try to use the *average value theorem* from Calculus to do a problem like this. That approach works in some cases but not here. Why? Because the *average value theorem* deals only with the outline of the object--it does not take into account the fact that the mass density of the object may not be the same everywhere (i.e., inhomogeneous).

7.5) The bum applies a force to the car; the car applies a force to the bum. As long as the forces are in the direction of the car's motion, all the forces in the direction of motion will be *internal* and momentum will be conserved (note that the velocities are all relative to the stationary track).

a.) Assuming the car is moving in the *x direction*, the bum's mass is m_b, the car's mass is m_c, the initial velocity of both the bum and the car is v_1 in the *+x direction*. After the bum starts running, the final velocity of the car (relative to the ground) is v_{fc} and the final velocity of the bum RELATIVE TO THE GROUND is v_{fc} *+ 5 m/s:*





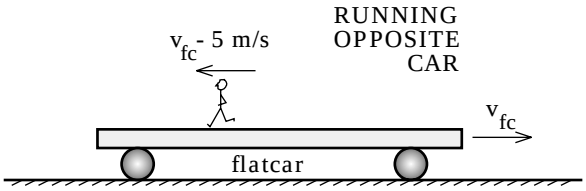
$$\Sigma p_{init} = \Sigma p_{final}$$
$$m_b v_1 + m_c v_1 = m_b(v_{fc} + 5\ m/s) + m_c v_{fc}$$

$$\Rightarrow \quad v_{fc} = [m_b v_1 + m_c v_1 - m_b(5)]/[m_b + m_c]$$
$$= [(60\ kg)(15\ m/s) + (800\ kg)(15\ m/s) - (60\ kg)(5\ m/s)]/[60\ kg + 800\ kg]$$
$$= 14.65\ m/s.$$

(It wasn't requested, but this means that $v_b = v_{fc} + 5\ m/s = 19.65\ m/s$).

Does this make sense? Sure it does. The bum pushes off the car making himself go faster. In doing so, he slows the car just a bit.

b.) With the bum running opposite the direction of the car, the bum's final velocity relative to the car is v_{fc} *- 5 m/s*. Following the same steps used in *Part a*:



$$\Sigma p_{init} = \Sigma p_{final}$$
$$m_b v_1 + m_c v_1 = m_b(v_{fc} - 5\ m/s) + m_c v_{fc}$$

$$\Rightarrow \quad v_{fc} = [m_b v_1 + m_c v_1 + m_b(5)]/[m_b + m_c]$$
$$= [(60\ kg)(15\ m/s) + (800\ kg)(15\ m/s) + (60\ kg)(5\ m/s)]/[60\ kg + 800\ kg]$$
$$= 15.35\ m/s.$$

(Again, this means that $v_b = v_{fc} - 5\ m/s = 10.35\ m/s$).

Does this make sense? Again, it does. The bum pushes off the car which makes himself go slower relative to the ground. In doing so, he forces the car ahead.

c.) As the bum runs in a direction perpendicular to the car's motion (say, in the *y direction*), nothing changes in the *x direction*--the car's momentum stays the same. Additionally, because there is an *external force* being provided *by the tracks on the train*, momentum is NOT conserved in the *y direction* as the bum picks up speed.

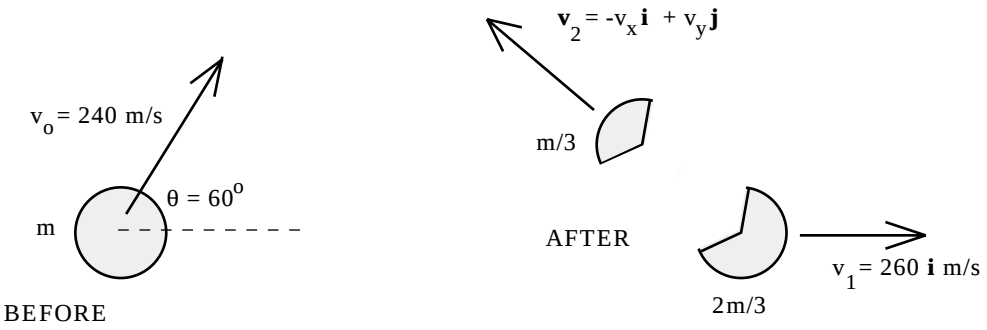
d.) Energy before:

$$KE_{bef} = (1/2)\, m_b v_1^2 + (1/2) m_c v_1^2$$
$$= .5(60\text{ kg})(15\text{ m/s})^2 + .5(800\text{ kg})(15\text{ m/s})^2$$
$$= 96{,}750 \text{ joules.}$$

$$KE_{aft} = (1/2)\, m_b (v_{fc} + 5)^2 + (1/2) m_c v_{fc}^2$$
$$= .5(60\text{ kg})(19.65\text{ m/s})^2 + .5(800\text{ kg})(14.65\text{ m/s})^2$$
$$= 97{,}433 \text{ joules.}$$

Where did the extra energy come from? The bum did work, burning chemical energy in his muscles as he exerted himself. Some of that energy showed itself as *kinetic energy*.

7.6) A sketch of the situation is shown below:



a.) Although there is gravity acting in the *y direction*, the explosion happens so quickly (i.e., Δt is so small) that momentum will be "conserved through the explosion" in all directions. Writing out momentum considerations in both the *x* and *y directions*, and noting that the signs in v_x and v_y are unembedded, we can write:

--In the *x direction*:

$$\Sigma p_{before,x} = \Sigma p_{after,x}$$
$$m(240 \cos 60^o) = [(2/3)m]\,(260) + [(1/3)m](-v_x)$$
$$\Rightarrow \quad v_x = 160 \text{ m/s.}$$

Note: This is the *magnitude* of the *x component* of the velocity.

--In the *y direction*:

$$\Sigma p_{before,y} = \Sigma p_{after,y}$$
$$m(240 \sin 60^o) = [(1/3)m](v_y)$$
$$\Rightarrow \quad v_y = 623.5 \text{ m/s.}$$

As a vector, final velocity of the second piece is, then,

$$\mathbf{v}_2 = (-160\mathbf{i} + 623.5\mathbf{j}) \text{ m/s.}$$

The magnitude of this vector is 643.7 m/s at an angle of 104.4^o.

b.) With *m* equal to 30 kg, the amount of chemical energy converted to kinetic energy is equal to the increase of kinetic energy (i.e., ΣKE). This is:

$$\Delta KE = KE_f - KE_o$$
$$= [(1/2)(2/3)mv_1^2 + (1/2)(1/3)mv_2^2] - [(1/2)mv_o^2]$$
$$= .5[.67(30 \text{ kg})(260 \text{ m/s})^2 + .33(30 \text{ kg})(643.7 \text{ m/s})^2 - (30 \text{ kg})(240 \text{ m/s})^2]$$
$$= 1.87\text{x}10^6 \text{ joules.}$$

This may be an unreasonable figure for a typical explosion, but what do you want from an off-the-wall problem?

7.7) This is a one-dimensional collision problem in which momentum is conserved "through the collision." That means:

$$\Sigma p_{before} = \Sigma p_{after}$$
$$m_8 v_8 + m_{10}(0) = (m_8 + m_{10})v_{aft}$$
$$\Rightarrow \quad v_{aft} = m_8 v_8/(m_8 + m_{10})$$

$$= (880/1880)v_8$$
$$\Rightarrow \quad v_{aft} = .468v_8.$$

We need a second expression that has v_{aft} in it. We know something about what happens to the energy in the system after the collision, so using the modified conservation of energy approach for the time interval *after the collision up to the complete standstill point*, we get:

$$KE_1 + \Sigma U_1 + \Sigma W_{ex} = KE_2 + \Sigma U_2$$
$$(1/2)(m_8 + m_{10})v_{aft}^2 + (0) - f_k d = 0 + 0.$$

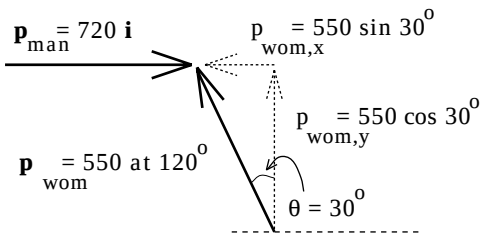
The frictional force f_k is due to m_{10}*'s* brakes locking (m_8*'s* brakes are assumed to remain unlocked). N.S.L. suggests that the normal force on m_{10} in this case is $m_{10}g$ and that the frictional force is $\mu_k N_{10} = \mu_k m_{10} g$. Substituting this into the above expression and solving for v_{aft} yields:

$$(1/2)(m_8 + m_{10})v_{aft}^2 - f_k d = 0$$
$$\Rightarrow \quad (1/2)(m_8 + m_{10})v_{aft}^2 = \mu_k m_{10} g d$$
$$v_{aft} = [2\mu_k m_{10} g d/(m_8 + m_{10})]^{1/2}.$$

Substituting $v_{aft} = .468v_8$ from above into this expression yields:

$$v_{aft} = [2\mu_k m_{10} g d/(m_8 + m_{10})]^{1/2}$$
$$.468v_8 = [2\mu_k m_{10} g d/(m_8 + m_{10})]^{1/2}$$
$$\Rightarrow \quad v_8 = [2(.6)(1000 \text{ kg})(9.8 \text{ m/s}^2)(1.2 \text{ m})/(1880 \text{ kg})]^{1/2}/(.468)$$
$$\Rightarrow \quad v_8 = 5.85 \text{ m/s}.$$

7.8) The man's initial momentum is in the *x direction*. The woman's initial momentum is in both the *x* and *y direction*. During the collision, the only forces acting are internal to the *two-person system*. This means momentum will be conserved in both the *x* and *y directions* through the collision. Remembering that the two individuals stick together (it is a *perfectly inelastic collision*), we can approach the problem by looking independently at what has happened to the system's momentum in the *x direction*, then *y direction*. Assuming the



final velocity of the two is v_x in the *x direction* and v_y in the *y direction*, we can write:

x direction:

$$\Sigma p_{before,x} = \Sigma p_{after,x}$$
$$p_{man\ before,x} + p_{woman\ before,x} = p_{man\ after,x} + p_{woman\ after,x}$$
$$m_m(v_{man\ before,x}) + m_w(v_{woman\ before,x}) = m_m(v_x) + m_w(v_x)$$
$$(90\ kg)(8\ m/s) + (55\ kg)[(-10\ m/s)(\sin 30^o)] = (90\ kg)v_x + (55\ kg)v_x$$
$$\Rightarrow \quad 445 = 145v_x$$
$$\Rightarrow \quad v_x = 3.07\ m/s.$$

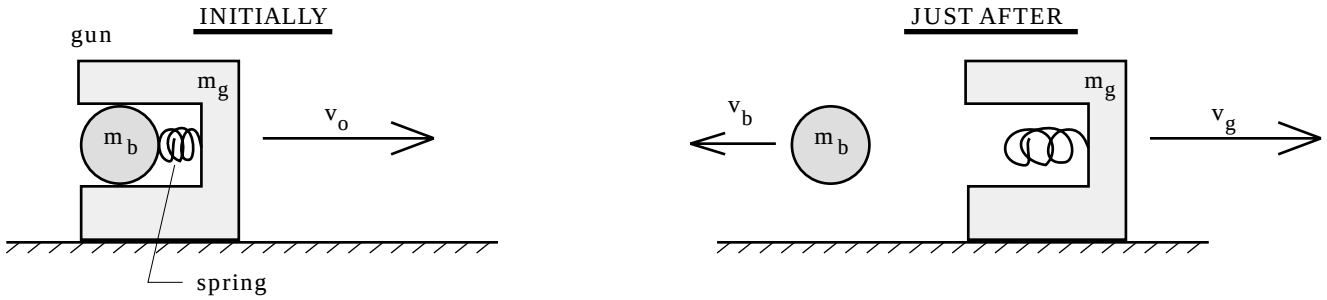
y direction:

$$\Sigma p_{before,y} = \Sigma p_{after,y}$$
$$p_{man\ before,y} + p_{woman\ before,y} = p_{man\ after,y} + p_{woman\ after,y}$$
$$m_m(v_{man\ before,x}) + m_w(v_{woman\ before,x}) = m_m(v_y) + m_w(v_y)$$
$$(90\ kg)(0\ m/s) + (55\ kg)[(10\ m/s)(\cos 30^o)] = (90\ kg)v_y + (55\ kg)v_y$$
$$\Rightarrow \quad 476 = 145v_y$$
$$\Rightarrow \quad v_y = 3.28\ m/s.$$

The final velocity of the two as a vector will be:

$$\mathbf{v}_{fin} = (3.07\mathbf{i} + 3.28\mathbf{j})\ m/s.$$

7.9) Momentum is conserved "through the one-dimensional firing of the gun" (see sketch on the previous page). As such, we can write:



$$\Sigma p_{before,x} = \Sigma p_{after,x}$$
$$p_{both} = p_{gun} + p_{ball}$$
$$(m_g + m_b)v_o = m_g v_g - m_b v_b$$
$$(2.04 \text{ kg})(5 \text{ m/s}) = (2 \text{ kg})v_g - (.04 \text{ kg})v_b$$
$$\Rightarrow \quad v_g = (.04v_b + 10.2)/2$$
$$= .02v_b + 5.1 \qquad \text{(Equation A).}$$

The spring is ideal so no energy is lost in the gun's firing. Using *conservation of energy* through the firing yields:

$$(1/2)m_g v_o^2 + (1/2)m_b v_o^2 + (1/2)kx^2 = (1/2)m_g v_g^2 + (1/2)m_b v_b^2.$$

Dividing out the *1/2's*:

$$(2 \text{ kg})(5 \text{ m/s})^2 + (.04 \text{ kg})(5 \text{ m/s})^2 + (120 \text{ nt/m})(.15 \text{ m})^2 = (2 \text{ kg})v_g^2 + (.04 \text{ kg})v_b^2$$
$$\Rightarrow \quad 53.7 = 2v_g^2 + .04v_b^2.$$

Substituting *Equation A* in for v_g, we get:

$$53.7 = 2(.02v_b + 5.1)^2 + .04v_b^2.$$

Expanding yields:

$$.0408v_b^2 + .408v_b - 1.68 = 0.$$

The Quadratic Formula yields:

$$v_b = [-.408 \pm [(-.408)^2 - 4(.0408)(-1.68)]^{1/2}]/[2(.0408)]$$
$$= 3.13 \text{ m/s or } -13.13 \text{ m/s.}$$

Assuming for the moment that the solution is 3.13 m/s, *Equation A* will give us the gun's velocity:

$$v_g = .02v_b + 5.1$$
$$= .02(3.13) + 5.1$$
$$= 5.16 \text{ m/s.}$$

Assuming for the moment that the solution is -13.13 m/s, *Equation A* will give us the gun's velocity:

$$\begin{aligned} v_g &= .02v_b + 5.1 \\ &= .02(-13.13) + 5.1 \\ &= 4.84 \text{ m/s.} \end{aligned}$$

The physical significance of a velocity calculated to be negative, given that we have unembedded the signs on all the velocity terms (hence making them magnitudes), is that the direction of motion has been assumed incorrectly. V_b was assumed to move to the left relative to the ground (i.e., in the negative *x direction*). It is possible we could have been wrong. That is, if the spring had been weak, it would have ejected the ball out the back of the gun, but the ball could have trailed the gun moving slower than the gun but nevertheless to the right in the POSITIVE *x direction*. If that had been the case, we would have computed a negative value for v_b and the negative sign would have told us we had assumed the WRONG DIRECTION for v_b in the first place. That is why we had to at least try the negative velocity value for v_b in *Equation A*.

With v_b negative, the final velocity for the GUN was in the correct direction--to the right--but with LESS VELOCITY MAGNITUDE than it had to start with (it started with 5 m/s velocity--v_g calculates to 4.84 m/s if $v_b = -13.13$ *m/s*). Intuition tells us that this is clearly wrong. Conclusion? $V_b = 3.13$ *m/s* making $v_g = 5.16$ *m/s*.

7.10)

a.) To stop the cart, Tarzan's momentum must be the same as the cart's but opposite in direction. Momentum will be conserved through the collision. If the system is to be brought to absolute rest (i.e., zero momentum) after the collision, we can write:

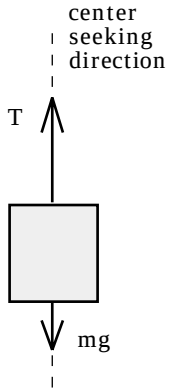
$$\begin{aligned} \Sigma p_{\text{before,x}} &= \Sigma p_{\text{after,x}} \\ (p_j + p_c) + p_T &= 0 \\ (m_J + m_c)v_c - m_T v_{bot} &= 0 \\ (40 \text{ kg} + 190 \text{ kg})(11 \text{ m/s}) - 90v_{bot} &= 0 \\ \Rightarrow \quad v_{bot} &= (230 \text{ kg})(11 \text{ m/s})/(90 \text{ kg}) \\ &= 28.1 \text{ m/s.} \end{aligned}$$

This is the velocity at which Tarzan must move to stop Jane and the cart dead in their tracks.

With this velocity, we can calculate how much energy Tarzan needs at the bottom of his arc. Using *conservation of energy*, we can determine how much of that energy will come from the freefall and how much must come from the run. Using that approach, we write:

$$KE_{top} + \Sigma U_{top} + \Sigma W_{ext} = KE_{bot} + \Sigma U_{bot}$$
$$(1/2)m_T v_{top}^2 + m_T g h_{top} + (0) = (1/2)m_T v_{bot}^2 + (0)$$
$$\Rightarrow \quad v_{top}^2 = v_{bot}^2 - 2gh_{top}$$
$$= (28.1 \text{ m/s})^2 - 2(9.8 \text{ m/s}^2)(38 \text{ m})$$
$$\Rightarrow \quad v_{top} = 6.7 \text{ m/s}.$$

b.) The f.b.d. to the right shows tension *up* and weight (i.e., *mg*) *down*. Using N.S.L. to sum the forces in the CENTER-SEEKING DIRECTION, we get:



$\underline{\Sigma F_c}$:

$$T - m_T g = m_T(v^2/r)$$
$$T = m_T g + m_T v_{bot}^2/R$$
$$= (90 \text{ kg})(9.8 \text{ m/s}^2) + (90 \text{ kg})(28.1 \text{ m/s})^2/(19 \text{ m})$$
$$= 4622 \text{ nts}.$$

This is over five times Tarzan's weight of 882 newtons.

7.11) Because this is essentially a collision problem, and because the only force acting (gravitational attraction between the two bodies) is internal to the system, momentum will be conserved in this problem. The difficulty lies in the fact that the planet is huge in comparison to the satellite. That is, the momentum of the planet will only change minusculely due to its size. In short, using *conservation of momentum* really won't work here.

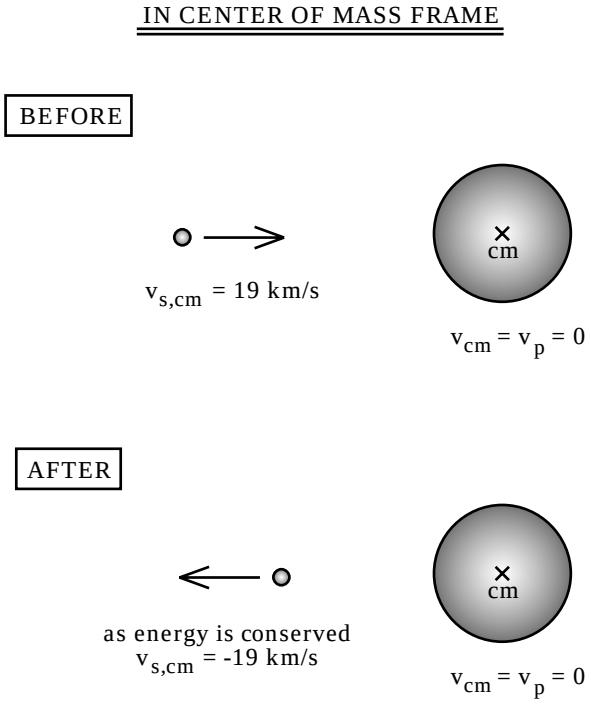
There is a clever way to approach the problem, though. Consider it from a *center of mass frame of reference.*

In the free-space frame (i.e., a frame that is stationary relative to both the planet and satellite), the motion of the center of mass and the motion of the planet will, for all intents and purposes, be exactly the same (almost all of the mass in the system is in the planet). That means that in the *center of mass frame*, the planet will appear stationary. It additionally means that the *satellite's incoming velocity* in that frame (i.e., in the center of mass frame) will be $v_{s,cm}$ *= 7 km/s + 12 km/s = 19 km/s*.

If the so-called collision is elastic, energy will be conserved. That means the satellite will leave the collision with the same amount of energy, hence same velocity, as it entered with . . . FROM THE PERSPECTIVE OF THE *CENTER OF MASS FRAME*.

Relative to space (i.e., the lab frame), the velocity of the center of mass and the velocity of the planet are both 12 km/s. That means the velocity of the satellite, relative to space, will be:

$$\begin{aligned} v_s &= v_{cm} + v_{sat.rel.to\ cm} \\ &= 12\ km/s + 19\ km/s \\ &= 31\ km/s. \end{aligned}$$



In other words, the satellite will come into the situation moving with velocity 7 km/s and will leave after interacting with the planet with velocity 31 km/s. This slingshot was used by NASA to boost the speed of both Voyager spacecrafts as they passed by Jupiter.

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CHAPTER 8 --ROTATIONAL MOTION I

8.1) Do not spend a lot of time on this problem.

a.) The integral $\int r^2 dm$ essentially tells you to identify all the mass (dm) a distance r units from the axis of interest (in this case $r = y$), multiply that dm by r^2, then sum up all possible $r^2 dm$ quantities using an integral.

In this case, we can define a differential strip of thickness dy and mass dm a distance y units from the x *axis* (i.e., the axis of rotation). Noting that the width of the strip at any particular y is $2x$ (see Figure), that the Pythagorean relationship yields $x = (R^2 - y^2)^{1/2}$, and that the surface area density function is $\sigma = ky$, we can write:

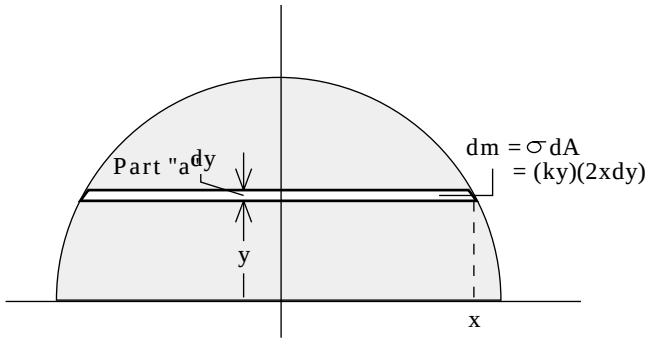


FIGURE I

$$\begin{aligned} dm &= \sigma dA \\ &= (ky)[(2x)dy] \\ &= (ky)[2(R^2 - y^2)^{1/2}dy]. \end{aligned}$$

Using this expression for dm in our *moment of inertia* expression, we get:

$$\begin{aligned} I &= \int y^2 dm \\ &= \int_{y=0}^{R} y^2 \left[(ky)\left[2\left(R^2 - y^2\right)^{1/2} dy \right] \right] \\ &= 2k\int_{y=0}^{R} y^3 \left(R^2 - y^2\right)^{1/2} dy \\ &= 2k\left[\left(\frac{\left(R^2 - y^2\right)^{5/2}}{5} - \frac{R^2\left(R^2 - y^2\right)^{3/2}}{3} \right) \right]_{y=0}^{R} \\ &= 2k\left[\left(\frac{\left(R^2 - R^2\right)^{5/2}}{5} - \frac{R^2\left(R^2 - R^2\right)^{3/2}}{3} \right) - \left(\frac{\left(R^2 - (0)^2\right)^{5/2}}{5} - \frac{R^2\left(R^2 - (0)^2\right)^{3/2}}{3} \right) \right] \\ &= .26kR^5. \end{aligned}$$

Note: How did we solve this integral? There is certainly a nifty way to do it using the approaches taught in your Calculus class, but not remembering the technique off hand, I simply looked it up in a *book* of integrals.

Bottom line: If you get a problem like this on a test, *setting up* the integral will be worth 95% of the problem.

b.) This geometry is more difficult than that in the previous problem because the given density is a function of *y* while the symmetry is radial. That is, if the body was homogenous we could define a differential half-hoop of radius *r* and thickness *dr* (note that the mass in the hoop is all the same distance from the *z axis*), determine the amount of mass *dm* in that section (see Figure II), and multiply that mass by the square of the distance *r* from the *z-axis*.

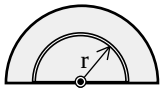


FIGURE II

Unfortunately, the mass distribution is not a function of *r*, it is a function of *y*.

To circumvent this problem, we must take an arbitrary *slice* of the hemi-sphere, *section* the slice into pieces, pick one piece whose height is defined as *dr* and whose width is $r\,d\theta$ (see Figure IIIa), determine the amount of mass in the section, sum over all of the possible sections in the slice, then sum over all of the slices.

$dm = \sigma\ dA$
$= [k\ y\ ][(rd\theta)dr]$
$= [k(r \sin\theta)][rd\theta dr]$

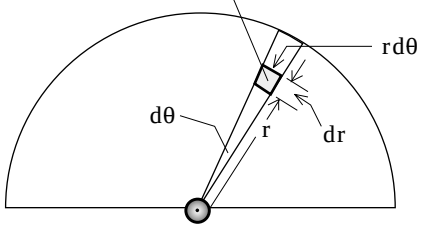


FIGURE IIIa

Noting that $y = r\ sin\ \theta$ (see Figure IIIb) and using the expression for *dm* shown in the sketch, we can use our *moment of inertia* expression to write:

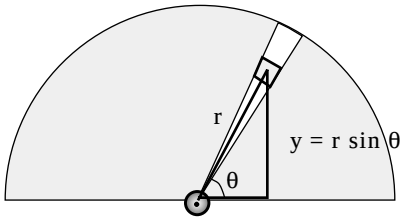


FIGURE IIIb

$$I = \int r^2 dm$$
$$= \int r^2 \left[k(r^2 \sin\theta)drd\theta\right].$$
$$= \int k(r^4 \sin\theta)drd\theta.$$

This is a double integral. Fortunately, the radial variable *r* is not dependent upon the angular variable θ, so the integral can be re-written as shown below (in doing so, the evaluation becomes nowhere near as difficult as one might expect at first glance).

The integral, after variable separation, is:

$$I = k\int r^4 dr \int \sin\theta d\theta.$$

Doing the integral yields:

$$\begin{aligned} I &= k\int_{r=0}^{R} r^4 dr \int_{\theta=0}^{\pi} \sin\theta d\theta \\ &= \left[k\left[\frac{r^5}{5} \right]_{r=0}^{R} \right] \int_{\theta=0}^{\pi} \sin\theta d\theta \\ &= \frac{kR^5}{5}\left[-\cos\theta\right]_{\theta=0}^{\pi} \\ &= \frac{kR^5}{5}\left[(-\cos\pi)-(-\cos 0)\right] \\ &= \frac{kR^5}{5}\left[(-(-1))-(-1)\right] \\ &= \frac{2kR^5}{5}. \end{aligned}$$

The moral of the story? If you understand what the *moment of inertia* expression is asking you to do, using it in a problem like this isn't that hard. Setting up the integral may take a little thinking, but its underlying tenets are fairly transparent. Find all the mass a distance *r* from the axis of interest, call that mass *dm*, then multiply *dm* by that distance *r* squared.

8.2) This problem has been included to give you a look at the basic kinematic and rotation/translational relationships. Don't spend a lot of time on it; it's more to get you familiar with the ideas than anything else.

a.) Using dimensional analysis, going from *km*/*hr* to *m*/*s* requires converting kilometers to meters and hours to seconds. The best way to do this is by simply placing the units first, then by plugging in the numbers. That is, the relationship has the units of *hours* in the denominator. That means you want to multiply by a quantity that has *hours* in the numerator and the desired *seconds* in the denominator, or *hours*/*second*. In that way the *hour* terms cancel out and we are left with seconds in the denominator. Similar reasoning follows the *kilometer* variable in the numerator. Doing the calculation yields:

(km/hr)(hr/sec)(m/km).

Putting in the numbers, we get:

(54 km/hr)[(1 hr)/(3600 sec)][(1000 m)/(1 km)] = 15 m/s.

b.) The relationship between the velocity of the *center of mass* of a rolling object (i.e., one that both translates and rotates) is:

$$v_{cm} = R\omega.$$

Minor side point: The temptation is to assume that the R term is simply the radius of the round object. Although the value for R is equal to the radius of the object in this case, R in this expression is really telling us how many meters there are on the arc of a one radian angle (remember, we are relating a linear measure v_{cm} to an angular measure ω). Using the relationship $v_{cm} = R\omega$, rearranging, then plugging in the numbers, we get:

$$\begin{aligned} \omega &= v_{cm}/R \\ &= (15 \text{ m/s})/(.3 \text{ m/rad}) \\ &= 50 \text{ rad/sec.} \end{aligned}$$

Note: Because we have been careful with our units, we get the correct units for the *angular velocity*.

As for *4 m/s* corresponding to *13.33 rad/sec*, we can use $v_{cm} = R\omega$ to write:

$$\begin{aligned} 4 \text{ m/s} &= (.3 \text{ m})\ \omega \\ \Rightarrow\ \omega &= 13.33 \text{ rad/sec.} \end{aligned}$$

c.) If the *angular displacement* had been one rotation--2π radians--the distance traveled by a point on the wheel's edge would have been the arclength of an angular displacement $\Delta\theta$ equal to 2π radians, or $\Delta s = 2\pi R$ meters (remember, the relationship between arclength Δs, radius-to-the-point-in-question r, and angular displacement $\Delta\theta$ is $\Delta s = r\,\Delta\theta$). If the *angular displacement* had been two rotations--4π radians--the distance traveled would have been 4πR meters

If we lay the arclength Δs out flat, we will get the *total linear distance* the wheel traveled. We know that distance. If we additionally know the radius of the wheel, we can write:

$$\begin{aligned} \Delta s &= R\,\Delta\theta \\ \Rightarrow \quad \Delta\theta &= \Delta s/R \\ &= (50 \text{ m})/(.3 \text{ m/rad}) \\ &= 166.7 \text{ radians.} \end{aligned}$$

d.) Angular acceleration:

$$(\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1)$$
$$(13.33 \text{ rad/s})^2 = (50 \text{ rad/s})^2 + 2a(166.7 \text{ rad})$$
$$\Rightarrow \quad \alpha = -6.97 \text{ rad/s}^2.$$

e.) We know the relationship between *angular acceleration* of a body and the magnitude of the *translational acceleration* of a point a distance *r* units from the axis of rotation. Using it we get:

$$a = r\alpha$$
$$= (.3 \text{ rad/m})(-6.97 \text{ rad/s}^2)$$
$$= -2.09 \text{ m/s}^2.$$

f.) For elapsed time using $(\theta_2 - \theta_1) = \Delta\theta$, we get:

$$\Delta\theta = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$(166.7 \text{ rad}) = (50 \text{ rad/s})t + .5(-6.97 \text{ rad/s}^2)t^2.$$

Using the quadratic formula, we get:

$$t = 5.27 \text{ seconds.}$$

g.) For elapsed time using ω_2 and not $\Delta\theta$:

$$\alpha = (\omega_2 - \omega_1)/\Delta t$$
$$\Rightarrow \quad \Delta t = [(13.33 \text{ rad/s} - 50 \text{ rad/s})]/(-6.97 \text{ rad/s}^2)$$
$$= 5.26 \text{ sec} \qquad \text{(yes, } \textit{Parts } f \text{ and } g \text{ match).}$$

h.) For the average angular velocity:

$$\omega_{avg} = (\omega_2 + \omega_1) / 2$$
$$= [(13.33 \text{ rad/s}) + (50 \text{ rad/s})]/2$$
$$= 31.67 \text{ rad/s.}$$

i.) Determine angular displacement $\Delta\theta$:

$$\Delta\theta = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$= (50 \text{ rad/s})(.5 \text{ s}) + .5(-6.97 \text{ rad/s}^2)(.5 \text{ s})^2$$
$$= 24.13 \text{ rad.}$$

j.) *How far* translates as, "What was the arclength of the wheel's motion?"

$$\Delta s = R\Delta\theta$$
$$= (.3 \text{ m/rad})(24.13 \text{ rad})$$
$$= 7.24 \text{ m.}$$

k.) Angular velocity calculated <u>without</u> using *time* variable:

$$(\omega_3)^2 = (\omega_1)^2 + 2\alpha\Delta\theta$$
$$= (50 \text{ rad/s})^2 + 2(-6.97 \text{ rad/s}^2)(24.13 \text{ rad})$$
$$= 46.51 \text{ rad/s.}$$

l.) Angular displacement $\Delta\theta$:

$$\Delta\theta = \omega_3 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$= (46.51 \text{ rad/s})[(.7 \text{ s}) - (.5 \text{ s})] + .5(-6.97 \text{ rad/s}^2)(.2 \text{ s})^2$$
$$= 9.16 \text{ rad.}$$

m.) From the information given in *Part b*, we know that the angular velocity of our wheel when moving at 4 m/s is 13.33 rad/sec. We can solve for $\Delta s = R\Delta\theta$ if we know $\Delta\theta$ during the motion (R is the radius of the wheel, or .3 m). We can use $\Delta\theta = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2$ if we know α. We start:

$$\alpha = (\omega_4 - \omega_2) / \Delta t$$
$$= [(20 \text{ rad/s}) - (13.33 \text{ rad/s})]/(3 \text{ s})$$
$$= 2.22 \text{ rad/s}^2.$$

$$\Delta\theta = \omega_2 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$= (13.33 \text{ rad/s})(3 \text{ s}) + .5(2.22 \text{ rad/s}^2)(3 \text{ s})^2$$
$$= 49.98 \text{ radians.}$$

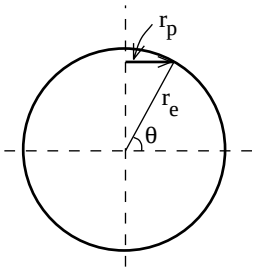
$$\Delta s = R\Delta\theta$$
$$= (.3 \text{ m/rad})(49.98 \text{ rad})$$

= 14.99 meters.

8.3) We are given the earth's mass at $m_e = 5.98x10^{24}\ kg$, the earth's period $T = 24\ hours$ ($8.64x10^4$ seconds--use dimensional analysis to get this if you don't believe me), and its radius at $r_e = 6.37x10^6\ meters$.

a.) The earth rotates through 2π radians in 24 hours ($8.64x10^4$ seconds). Its angular displacement per unit time (i.e., its angular velocity) is, therefore:

$$\omega = \Delta\theta/\Delta t$$
$$= (2\pi \text{ rad})/(8.64x10^4 \text{ s})$$
$$= 7.27x10^{-5} \text{ rad/s}.$$



b.) The earth's equatorial velocity (magnitude) is equal to the linear distance a point on the equator travels per unit time. That is:

$$v_{eq} = (2\pi \text{ rad})(6.37x10^6 \text{ m/rad})/(8.64x10^4 \text{ s})$$
$$= 463.2 \text{ m/s} \qquad \text{(this is about 1000 mph).}$$

Note: According to theory, v_{equ} should equal $R\omega_e$. Putting the numbers in yields:

$$R\,\omega_e = (6.37x10^6 \text{ m})(7.27x10^{-5} \text{ rad/s})$$
$$= 463.1 \text{m/s}.$$

Given round-off error, this is close enough for government work.

c.) Looking at the sketch above, the radius of the circle upon which that particle will be traveling will be:

$$r_p = r_e \cos 60^o$$
$$= (6.37x10^6 \text{ m})\ (.5).$$

It takes the same amount of time T for the particle at 60^o to travel through one rotation as it does for a particle on the equator, so:

$$v_p = 2\pi r_p/T$$
$$= (2\pi)(3.185\text{x}10^6 \text{ m})/(8.64\text{x}10^4 \text{ s})$$
$$= 231.6 \text{ m/s}.$$

Does this make sense? Sure it does. As you approach the geographic north pole the travel velocity should go to zero.

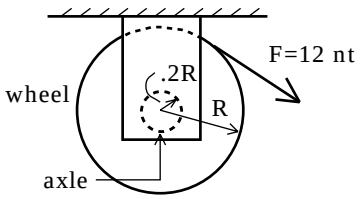
d.) Using the table at the end of Chapter 8 in your text, the *moment of inertia* of a solid sphere I_{ss} is:

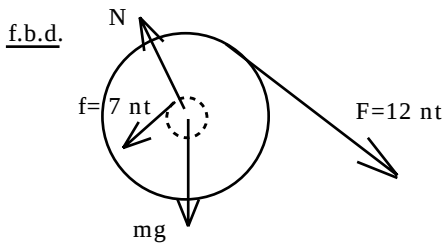
$$I_{ss} = (2/5)MR^2$$
$$= .4(5.98\text{x}10^{24} \text{ kg})(6.37\text{x}10^6 \text{ m}))^2$$
$$= 9.7\text{x}10^{37} \text{ kg}\cdot\text{m}^2.$$

Bottom line: It's going to take a mighty stiff cosmic breeze to change the earth's rotational motion!

CHAPTER 9 -- ROTATIONAL MOTION II

9.1) A sketch for this set-up is shown to the right, complete with f.b.d. for the forces acting on the system. Notice that although the frictional force acts everywhere on the axle, it provides nothing more than a torque on the system. For that reason, we can assume that the net frictional force acts *at one place only* (it doesn't matter how it's oriented--*x* and *y components* aren't relevant when determining torques). Also, the normal force *N* at the pin, which does not provide a torque to the system, cannot be vertical as *F* has *x* and *y components* that *N* must counter.





a.) The request for the magnitude of the *angular acceleration* should bring to mind N.S.L. Using the rotational counterpart of that law and assuming the pulley is a disk whose $I_{cm} = (1/2)mR^2$, we get:

$$\underline{\textstyle\sum \Gamma_{pin}}:$$

$$\Gamma_f + \Gamma_F = I \alpha$$

$$f(.2R) - FR = -[(1/2)MR^2]\,\alpha.$$

Dividing out an *R* yields:

$$(7\text{ nt})(.2) - (12\text{ nt}) = -.5(8\text{ kg})(.6\text{ m})\,\alpha$$

$$\Rightarrow \quad \alpha = 4.42\text{ rad/sec}^2.$$

b.) The relationship between the rotating object's *angular acceleration* α and the *instantaneous translational acceleration a* of a point *(2/3)R* from the axis of rotation is $a = r\alpha$. Noting that *r* is numerically equal to *(2/3)R*, but that *r's* units are *meters/radian*, we get:

$$\begin{aligned} a &= r\alpha \\ &= (2R/3)\,\alpha \\ &= .67(.6\text{ m/rad})(4.42\text{ rad/s}^2) \\ &= 1.78\text{ m/s}^2. \end{aligned}$$

c.) The rotational counterpart to Newton's Second Law, written in terms of angular momentum, is "the *sum of the torques* equal to the *change of the angular momentum with time*," or $\Gamma_{net} = \Delta L / \Delta t$. Using this:

$$\underline{\Sigma \Gamma_{pin}}:$$

$$\Gamma_f + \Gamma_F = \Delta L / \Delta t$$

$$f(.2R) - FR = (L_5 - L_0) / t.$$

As the initial *angular momentum* is zero (the wheel is initially at rest), multiplying both sides by *t* gives us:

$$[(7 \text{ nt})(.2)(.6 \text{ m}) - (12 \text{ nt})(.6 \text{ m})]t = L_5$$

$$\Rightarrow \quad L_5 = [(7 \text{ nt})(.12 \text{ m}) - (7.2 \text{ nt·m})][5 \text{ sec}]$$

$$= -31.8 \text{ kg·m}^2/\text{s}.$$

Note: The negative sign means the rotation is clockwise.

d.) Using the rotational form to *angular momentum*, we know that $L = I\omega$. Using that relationship to determine the angular velocity at *t = 5 seconds*, we find:

$$L_5 = I\omega_5$$

$$\Rightarrow \quad \omega_5 = L_5/(.5MR^2)$$

$$= (-31.8 \text{ kg·m}^2/\text{s})/[.5(8 \text{ kg})(.6 \text{ m})^2]$$

$$= -22.08 \text{ rad/sec}.$$

e.) Whenever you are asked to determine an *angular VELOCITY*, think conservation of energy (at least to start). Note that whereas the work done by a force *F* as a body displaces a distance *d* is $\boldsymbol{F}\cdot\boldsymbol{d}$ (equal to *Fd* if the force and displacement are in the same direction), the work done by a torque Γ applied through an *angular displacement* $\Delta\theta$ will be $\Gamma\Delta\theta$ (remember, planar rotation is one-dimensional). Friction-produced torques will always do *negative* work. Torques that make a body's angular velocity *increase* will do *positive* work; those that make a body's angular speed *decrease* will do *negative* work. Keeping this in mind, the *conservation of energy* implies:

$$\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} = \Sigma KE_2 + \Sigma U_2$$

$$0 + 0 + [-\Gamma_f\Delta\theta + \Gamma_F\Delta\theta] = (1/2)I\omega_5^2 + 0$$

$$[[-(.2R)f]\Delta\theta + FR\Delta\theta] = .5(.5MR^2)\omega_5^2$$

$$[-(.2)(.6\text{ m})(7\text{ nt})(55.15\text{ rad}) + (12\text{nt})(.6\text{ m})(55.15\text{ rad}) = .25(8\text{ kg})(.6\text{ m})^2\,\omega_5{}^2$$
$$\Rightarrow \quad \omega_5{}^2 = 487.16\text{ rad}^2/\text{sec}^2$$
$$\Rightarrow \quad \omega_5 = 22.07\text{ rad/sec.}$$

Note: Remember, the conservation of energy yields *magnitudes only*.

f.) Using rotational kinematics while knowing that the initial *angular velocity* is zero and the *angular acceleration* is 4.4 rad/sec^2, we get:

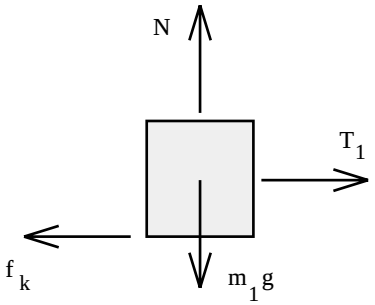
$$\Delta\theta = \omega_1 t + (1/2)\alpha t^2$$
$$= 0 + .5(4.42\text{ rad/s}^2)(5\text{ s})^2$$
$$= 55.25\text{ rad.} \qquad (\ldots\text{ close enough).}$$

g.) Using kinematics:

$$\omega_5 = \omega_1 + \alpha t$$
$$= 0 + (4.42\text{ rad/s}^2)(5\text{ s})$$
$$= 22\text{ rad/sec} \qquad (\ldots\text{ close enough).}$$

9.2) The thing to remember whenever you have a massive pulley is that the tension on either side of the pulley will be different.

a.) This is a Newton's Second Law problem. F.b.d.'s are shown to the right and on the next page, and N.S.L. is presented below (notice that you end up needing FIVE independent equations to solve this problem):



<u>for mass m_1:</u>

<u>$\sum F_y$:</u>

$$N - m_1 g = m_1 a_y = 0 \qquad (\text{as } a_y = 0)$$
$$\Rightarrow \quad N = m_1 g.$$

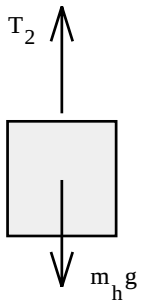
<u>$\sum F_x$:</u>

$$-\mu_k N + T_1 = m_1 a$$
$$\Rightarrow \quad T_1 = m_1 a + \mu_k m_1 g.$$

Note: This presents a problem. We don't want the acceleration a of the mass m_1, we want the angular acceleration α of the pulley. We need a relationship between those two quantities. Noticing that the string's acceleration a is that of m_1 and the string's acceleration is also the acceleration *of a point on the pulley's circumference*, we can use the relationship $a = R\alpha$ for the job. Doing so yields:

$$T_1 = m_1 \; a \; + \mu_k m_1 g$$
$$T_1 = m_1(R\alpha) + \mu_k m_1 g.$$

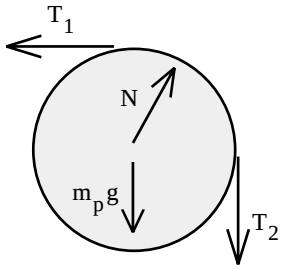
<u>for mass m_h:</u>



<u>$\sum F_y$:</u>

$$T_2 - m_h g = -m_h a$$
$$\Rightarrow \quad T_2 = -m_h \; a \; + m_h g$$
$$\Rightarrow \quad T_2 = -m_h(R\alpha) + m_h g.$$

<u>for the pulley:</u>



<u>$\sum \Gamma_{\text{pulley axis}}$:</u>

$$\Gamma_{T_1} \; - \; \Gamma_{T_2} = \; - \; I_p \; \alpha$$
$$T_1 R \; - T_2 R = - \; [(1/2)M_p R^2]\,\alpha.$$

Note that the negative sign in front of the $I_p \alpha$ term denotes that the α term is a magnitude and the unembedded sign is designating a *clockwise angular acceleration*. Dividing out an R yields:

$$T_1 \; - T_2 = - \; [(1/2)M_p R]\,\alpha.$$

Substituting the expressions for T_1 and T_2 we get:

$$T_1 \qquad - \qquad T_2 \qquad = -[(1/2)M_p R]\,\alpha$$
$$[m_1(R\alpha) + \mu_k m_1 g \,] \; - [-m_h(R\alpha) + m_h g \,] \; = -[(1/2)M_p R]\,\alpha$$
$$\Rightarrow \quad \alpha = [-\mu_k m_1 g \; + m_h g] \,/\, [m_1 R + m_h R + .5 M_p R].$$

Put in the numbers and you should come out with 29.32 rad/sec^2.

b.) The acceleration of a point on the edge of the pulley (this is also the acceleration of the string which, in turn, is the acceleration of the hanging mass), is $a = r\alpha$, where r in this case is the radius R of the pulley and α is provided from *Part a* above. Doing the work yields:

$$\begin{aligned} a &= R\alpha \\ &= (.1875 \text{ m/rad})(29.32 \text{ rad/sec}^2) \\ &= 5.5 \text{ m/s}^2. \end{aligned}$$

c.) The total work done by all the *tension* forces in the system equals zero. Noting that, the *conservation of energy* yields:

$$\begin{aligned} \Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} &= \Sigma KE_2 + \Sigma U_2 \\ 0 + m_h gh + (-fh) &= [(1/2)m_1 v^2 + (1/2)m_h v^2 + (1/2)I_{cm}\omega^2] + 0. \end{aligned}$$

Noting that the velocity of a point on the pulley's circumference will equal v, and that $v = R\omega$, and the frictional force is $\mu_k N = \mu_k m_1 g$, we can write:

$$m_h gh = [(\mu_k m_1 gh) + (1/2)m_1 (R\omega)^2 + (1/2)m_h (R\omega)^2 + (1/2)[I_{cm}\omega^2].$$

Rewriting this, eliminating the units (for space):

$$(1.2)(9.8)(1.5) = (.7)(.4)(9.8)(1.5) + .5(.4)(.1875)^2\omega^2 + .5(1.2)(.1875)^2\omega^2 + .5[1.4\text{x}10^{-3}]\omega^2$$

$$\Rightarrow \quad \omega = 21.66 \text{ rad/sec.}$$

d.) Noting that the string will have a velocity equal to that of both m_1 and m_h AND to a point on the circumference of the pulley, we can relate the angular velocity of the pulley and the string's velocity by $v = R\omega$. Remembering that the units for R in this usage are "meters/radian," we get:

$$\begin{aligned} v &= R\omega \\ &= (.1875 \text{ m/rad})(21.66 \text{ rad/sec}) \\ &= 4.06 \text{ m/s}. \end{aligned}$$

e.) This is a trick question. The acceleration is going to be the same no matter how far the hanging mass has fallen. The answer was derived in *Part b*.

f.) Angular momentum is defined rotationally as $L = I\omega$. With that we get:

$$\begin{aligned} L &= I\,\omega \\ &= [(1/2)m_p R^2]\,\omega \\ &= .5(.08\text{ kg})(.1875\text{ m})^2(21.66\text{ rad/sec}) \\ &= .03\text{ kg}\cdot\text{m}^2/\text{s}. \end{aligned}$$

9.3) An f.b.d. for the beam is shown below.

a.) This is a rigid body (i.e., equilibrium) problem. The easiest way to get the tension in the line is to sum the torques about the pin (that will eliminate the need to deal with H and V and will additionally give you an equation that has only one unknown--the T variable you are looking for).

Note:

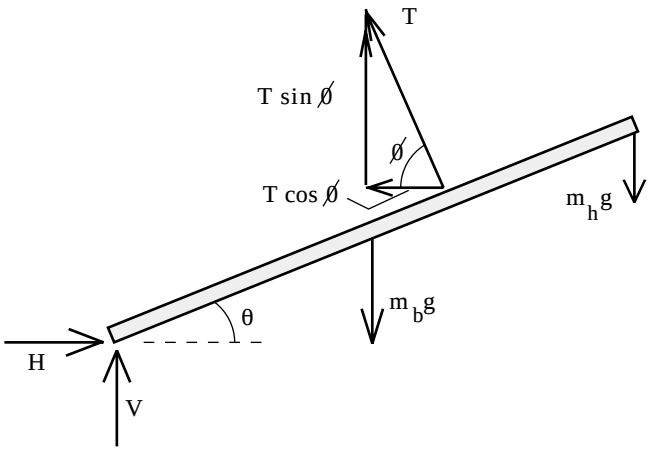
--Call the distance between the pin and the cable's connection to the beam $d = (2/3)L$, where L is the beam's length.

--The angle between T and d is 90^o which means the torque due to T about the pin will be $Td \sin 90^o = T(2/3)L$.

--The distance between the pin and the hanging mass's connection to the beam is L.

--The component of L perpendicular to the *line of the hanging mass* (i.e., *r-perpendicular*) will be $L \cos 30^o$ (this is like determining the shortest distance between the pin and the *line of the force*). That means the torque provided by the hanging mass will be $(m_h g)L\cos 30^o$. As this torque will try to make the beam rotate clockwise, it is a *negative* torque.

--The *beam's mass* can be assumed to be located at the beam's *center of mass* at $L/2$.



Putting all this together, N.S.L. yields:

$\underline{\sum \Gamma_{pin}}$:

$$\Gamma_T + \Gamma_{m_h} + \Gamma_{m_b} = 0 \quad (\text{as } \alpha = 0)$$

$$T[(2/3)L] - (m_h g)L\cos 30^o - (m_b g)(L/2)\cos 30^o = 0.$$

Canceling the *L's* and solving for *T*, we get:

$$\begin{aligned} T &= [(m_h g)\cos 30^o + (m_b g)(.5)\cos 30^o]/(2/3) \\ &= [(3\text{ kg})(9.8\text{ m/s}^2)(.87) + (7\text{ kg})(9.8\text{ m/s}^2)(.5)(.87)]/[.67] \\ &= 82.7\text{ nts.} \end{aligned}$$

Knowing *T*, we can sum the forces in the *x* and *y directions* to determine *H* and *V*. Doing so yields:

$\underline{\sum F_x}$:

$$\begin{aligned} -T\cos 60^o + H &= ma_x \\ &= 0 \quad \text{as } a_x = 0. \\ \Rightarrow \quad H &= T\cos 60^o \\ &= (82.7\text{ nt})(.5) \\ &= 41.35\text{ nts.} \end{aligned}$$

$\underline{\sum F_y}$:

$$\begin{aligned} T\sin 60^o + V - m_h g - m_b g &= ma_y \\ &= 0 \quad (\text{as } a_y = 0) \\ \Rightarrow \quad V &= -T\sin 60^o + m_h g + m_b g \\ &= -(82.7\text{ nt})(.87) + (3\text{ kg})(9.8\text{ m/s}^2) + (7\text{ kg})(9.8\text{ m/s}^2) \\ &= 26.05\text{ nts.} \end{aligned}$$

Note: If there had been a negative sign in front of the 26.05 newtons, it would have meant that we had assumed the wrong direction for the force *V*. The *magnitude* would nevertheless have been correct.

b-i.) The *moment of inertia* about an axis other than one through the *center of mass* but parallel to an axis of known *moment of inertia* that is through the *center of mass* is determined using the *Parallel Axis Theorem*. The *moment of inertia* about the beam's pin is:

$$\begin{aligned} I_p &= I_{cm} + mh^2 \\ &= (1/12)m_b L^2 + m_b(L/2)^2 \\ &= (1/3)m_b L^2 \\ &= (1/3)(7\text{ kg})(1.7\text{ m})^2 \\ &= 6.74\text{ kg}\cdot\text{m}^2. \end{aligned}$$

b-ii.) The *moment of inertia* of a point mass (i.e., the hanging mass) r units from a reference axis (note that $r = L$ in this problem) is:

$$\begin{aligned} I_{hm} &= mr^2 \\ &= m_h L^2 \\ &= (3\text{ kg})(1.7\text{ m})^2 \\ &= 8.67\text{ kg}\cdot\text{m}^2. \end{aligned}$$

b-iii.) The total *moment of inertia* about the pin is the beam's *moment of inertia* about the pin added to the hanging mass's *moment of inertia* about the pin, or:

$$\begin{aligned} I_{tot,pin} &= I_p + I_{hm} \\ &= (6.74\text{ kg}\cdot\text{m}^2) + (8.67\text{ kg}\cdot\text{m}^2) \\ &= 15.4\text{ kg}\cdot\text{m}^2. \end{aligned}$$

c.) Angular acceleration--think N.S.L. Summing up the torques *about the pin* (the beam is executing a pure rotation about the pin--no reason to sum torques about any other point) and putting that equal to the *moment of inertia* ABOUT THE PIN times the *angular acceleration* about the pin yields:

$\underline{\sum \Gamma_{pin}}$:

$$\Gamma_T + \Gamma_{m_{hm}} + \Gamma_{m_b} = I_{tot,pin}\,\alpha$$

$$T[(2/3)L] - (m_h g)L\cos 30^o - (m_b g)(L/2)\cos 30^o = I_{tot,pin}\,\alpha.$$

The line has been cut (which means $T = 0$) and we know the total *moment of inertia* about the pin from above. Using this we get:

$$\Gamma_T + \Gamma_{m_h} + \Gamma_{m_b} = I_{tot,pin}\,\alpha$$

$$0 - (m_h g)L\cos 30^o - (m_b g)(L/2)\cos 30^o = -(15.4\text{ kg}\cdot\text{m}^2)\,\alpha.$$

Solving for α we get:

$$\begin{aligned} \alpha &= [(m_h g)L\cos 30^o + (m_b g)(L/2)\cos 30^o)]/(15.4\text{ kg}\cdot\text{m}^2) \\ &= [(3\text{ kg})(9.8\text{ m/s}^2)(1.7\text{ m})(.87) + (7\text{ kg})(9.8\text{ m/s}^2)(.85\text{ m})(.87)]/(15.4\text{ kg}\cdot\text{m}^2) \\ &= 6.11\text{ rad/sec}^2. \end{aligned}$$

d.) Knowing the *angular acceleration* of the beam just after it lets loose allows us to determine the *translational acceleration* of any point on the beam using $a = r\alpha$, where r is the distance between the pin and the point-in-question. In this case, that distance is $L/2$. Using this information, we get:

$$\begin{aligned} a &= r\alpha \\ &= [(1.7 \text{ m})/2]\ (6.11 \text{ rad/sec}^2) \\ &= 5.19 \text{ m/s}^2. \end{aligned}$$

e.) This is a *conservation of energy* problem. The beam has *potential energy* wrapped up in the fact that its *center of mass* will have fallen a vertical distance equal to $(L/2) \sin 30^o$ during the freefall. That means that its initial *potential energy* will be $m_b g(L/2) \sin 30^o$ while its final U will be zero. The hanging mass also falls a vertical distance equal to $L \sin 30^o$.

The *potential energy* changes of both bodies (i.e., the beam and hanging mass) must be taken into account if we are going to use *conservation of energy*--the approach of choice whenever we are looking for a velocity-type variable.

Assuming we approach the beam as though it were executing a pure rotation about its pin, we can relate the change in the potential energy of the system before the snip to the *rotational kinetic energy* of the beam about the pin plus the *translational kinetic energy* of the hanging mass just as the beam becomes horizontal. Using that approach:

$$\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} = \Sigma KE_2 + \Sigma U_2$$

$$0 + [m_b g((L/2) \sin 30^o) + m_h g(L \sin 30^o)] + 0 = [(1/2)m_h v^2 + (1/2)I_p \omega^2] + 0.$$

Noting that the velocity of a point on the beam will equal $v = R\omega$, we can write the velocity of the hanging mass as $v_h = (L)\omega$:

$$m_b g((L/2) \sin 30^o + m_h g(L \sin 30^o) = (1/2)m_h\ v_h^2 + (1/2)\ I_p\ \omega^2$$

$$m_b g((L/2) \sin 30^o + m_h g(L \sin 30^o) = (1/2)m_h[L\omega]^2 + (1/2)\left[(1/3)m_b L^2\right]\omega^2.$$

Putting in the numbers while eliminating the units (for space), we get:

$$(7)(9.8)(.85)(.5) + (3)(9.8)(1.7)(.5) = .5(3)(1.7^2)\omega^2 + .5(.33)(7)(1.7)^2\omega^2$$

$$\Rightarrow \quad \omega = 2.62 \text{ rad/sec}.$$

f.) The *translational velocity* of the *center of mass* will be:

$$\begin{aligned} v &= (L/2)\ \omega \\ &= [(1.7\text{ m})/2](2.62\text{ rad/sec}) \\ &= 2.23\text{ m/s.} \end{aligned}$$

g.) *Angular momentum* is defined rotationally as $L = I\omega$, where I is the TOTAL *moment of inertia* for the beam and hanging mass about the pin, and the fact that the length of the beam has been defined as L turns out to be a really, really bad choice of variables (it's the same as *angular momentum*). Remembering that $I_{tot,pin} = 15.4\ kg{\cdot}m^2/s$, we get:

$$\begin{aligned} L &= I\,\omega \\ &= [15.4](2.62\text{ rad/sec}) \\ &= 40.35\text{ kg}{\cdot}\text{m}^2/\text{s.} \end{aligned}$$

h.) *Angular momentum* is conserved *if and only if* all the torques acting on the system are internal (i.e., are the consequence of the interaction of the various parts of the system). Gravity is an external force which means that any torque it produces on the beam will be an *external* torque. In short, *angular momentum* should not be conserved.

(This should be obvious given the fact that the beam is accelerating angularly).

9.4) The merry-go-round's mass is $m_m = 225\ kg$ while each child's mass is $m_c = 35\ kg$. The radius of the merry-go-round is $R = 2.5\ meters$ and its *angular velocity* is $\omega_1 = .8\ radians/second$ when the kids climb on.

a.) An *angular velocity* of $\omega_1 = .8\ rad/sec$ produces a *translational velocity* of $r\omega = (2.5\text{ m})(.8\text{ rad/sec}) = 2\ m/s$ at the edge of the merry-go-round (i.e., where the children are when they first jump on). Also, the *moment of inertia* of one child when he or she first jumps onto the merry-go-round is $I_{c,1} = m_c R^2 = (35\text{ kg})(2.5\text{ m})^2 = 218.75 kg{\cdot}m^2$ while the *moment of inertia* of the merry-go-round itself is assumed to be that of a disk and is equal to $I_m = (1/2)m_m R^2 = .5(225\text{ kg})(2.5\text{ m})^2 = 703\ kg{\cdot}m^2$. Using *conservation of energy,* we can write:

$$\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} = \Sigma KE_2 + \Sigma U_2$$
$$0 + 0 + 3[(RF)\Delta\theta] = [3(1/2)I_c\omega_2{}^2 + (1/2)I_m\omega_2{}^2] + 0.$$

Note: $\Gamma_{one\ kid\ on\ mgr} = RF\ sin\ 90^o$. The counterpart to $\mathbf{F}\cdot\mathbf{d}$ for rotational motion is $\boldsymbol{\Gamma}\cdot\Delta\boldsymbol{\theta}$, so the three kids do $3[(RF)\Delta\theta]$ worth of extra work on the m.g.r.

Solving, we get:

$$3(2.5\text{ m})(15\text{ nt})\Delta\theta = [3(.5)(218.75\text{ kg}\cdot\text{m}^2)(.8\text{ rad/s})^2 + .5(703\text{ kg}\cdot\text{m}^2)(.8\text{ r/s})^2]$$
$$\Rightarrow \quad \Delta\theta = 3.87\text{ radians.}$$

Note that the relationship between angular displacement and linear displacement is:

$$\Delta s = R\Delta\theta$$
$$= (3.87\text{ rad})(2.5\text{ m/rad})$$
$$= 9.68\text{ meters.}$$

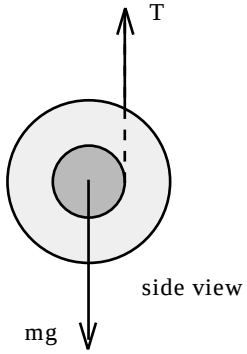
b.) Once the kids have climbed aboard, the torques acting on the system are all internal. That is, they are due to the interaction of the system's parts (the kids act on the merry-go-round while the merry-go-round acts on the kids). That means we can use the *conservation of <u>angular momentum</u>:*

$$L_{1,tot} = L_{2,tot}$$
$$L_{m,1} + L_{c,1} = L_{m,2} + L_{c,2}$$
$$I_m\ \omega_1 + (\ I_{c,1}\ \omega_1\ ) = I_m\ \omega_2 + [\ I_{c,2}\ \omega_2]$$
$$(703\text{ kg}\cdot\text{m}^2)(.8\text{ r/s}) + (656\text{ kg}\cdot\text{m}^2)(.8\text{ r/s}) = (703\text{ kg}\cdot\text{m}^2)(\omega_2) + [3(35\text{ kg})(1\text{ m})^2\ \omega_2]$$
$$\Rightarrow \quad \omega_2 = 1.35\text{ rad/sec and}$$
$$\Rightarrow \quad v_2 = r\omega$$
$$= (1\text{ m/rad})(1.346\text{ rad/sec})$$
$$= 1.35\text{ m/s.}$$

c.) As there are no external torques, the *angular momentum* should be conserved.

Also, the forces acting on the system are *internal* so the *total momentum* of the system will be conserved as the children move toward the merry-go-round's center. It should be noted, though, that this isn't a very

useful fact as far as problem-solving goes. The addition of the momentum components of the children's motion at the beginning *and at any time thereafter* will be zero.



d.) The forces being exerted on the system are not conservative (the kids burn chemical energy as they move on the merry-go-round), so energy is not conserved.

e.) We have already deduced that energy should not be conserved. Let's see if we were right by calculating the energy in the system just after the kids jumped onto the merry-go-round and the energy once they reached the *r = 1 meter* point:

$$\begin{aligned} KE_1 &= KE_{c,1} + KE_{m,1} \\ &= (1/2)m_{c,1}v_1^2 + (1/2)I_m\omega_1^2 \\ &= .5[3(35\text{ kg})](2\text{ m/s})^2 + .5(703\text{ kg}\cdot\text{m}^2)(.8\text{ r/s})^2 \\ &= 435\text{ joules.} \end{aligned}$$

$$\begin{aligned} KE_2 &= KE_{c,2} + KE_{m,2} \\ &= (1/2)m_{c,2}v_2^2 + (1/2)I_m\omega_2^2 \\ &= .5[3(35\text{ kg})](1.35\text{ m/s})^2 + .5(703\text{ kg}\cdot\text{m}^2)(1.35\text{ r/s})^2 \\ &= 736.3\text{ joules.} \end{aligned}$$

Energy is obviously not conserved.

9.5) Among other reasons, this problem was designed to allow you to see that either the *pure rotation* or the *rotation about the center of mass plus translation of the center of mass* approaches will work when dealing with either a Newton's Second Law or *conservation of energy* problem. An f.b.d. for the problem is shown to the right.

a.) Analyzing the wheel problem from a *rotation about the center of mass plus translation of the center of mass* approach:

This is a N.S.L. problem. We will begin by summing the torques about the center of mass:

$\underline{\sum\Gamma_{cm}}$:

$$Tr_a = I_{cm}\alpha \qquad \text{(Equation A).}$$

We have two unknowns here; we need another equation. Consider the sum of the forces in the vertical:

$$\underline{\textstyle\sum F_v}:$$

$$T - mg = -ma$$
$$= -m(r_a \alpha)$$

$$\Rightarrow \quad T = mg - mr_a \alpha \qquad \text{(Equation B).}$$

Putting *Equations A* and *B* together:

$$Tr_a = I_{cm} \alpha$$
$$\Rightarrow \quad (mg - mr_a \alpha) r_a = I_{cm} \alpha$$
$$\Rightarrow \quad \alpha = (mgr_a)/(I_{cm} + mr_a^2)$$
$$= [(.6 \text{ kg})(9.8 \text{ m/s}^2)(.015 \text{ m})]/[(1.2\text{x}10^{-4} \text{ kg}\cdot\text{m}^2) + (.6 \text{ kg})(.015)^2]$$
$$= 346 \text{ rad/sec}^2.$$

b.) We have to use the Parallel Axis Theorem to determine the *moment of inertia* about the new axis. Doing so yields:

$$I_a = I_{cm} + m h^2$$
$$= (1.2\text{x}10^{-4} \text{ kg}\cdot\text{m}^2) + (.6 \text{ kg})(.015 \text{ m})^2$$
$$= 2.55\text{x}10^{-4} \text{ kg}\cdot\text{m}^2.$$

c.) Analyzing the wheel problem from a *pure rotation* approach:

This is a N.S.L. problem. We will begin by summing the torques about the axis .015 meters from the *center of mass* (call this *Point P*).

$$\underline{\textstyle\sum \Gamma_a}:$$

$$(mg)r_a = I_a \alpha.$$

We determined I_a in *Part b*, so all we have to do is solve for α:

$$\alpha = (mgr_a)/I_a$$
$$= [(.6 \text{ kg})(9.8 \text{ m/s}^2)(.015 \text{ m})]/(2.55\text{x}10^{-4} \text{ kg}\cdot\text{m}^2)$$
$$= 346 \text{ rad/sec}^2.$$

Great jumping Huzzahs! *Parts a* and *c* match. Both approaches work.

d.) This is a *conservation of energy* problem. Major note: *Work* requires that a force act *through a distance*. Tension acts at the one point on the disk that *isn't moving*--it acts at the instantaneously stationary point about which the disk rotates. As such, tension acts through *no distance* and does *no work*.

--from the *pure rotation* approach:

$$\begin{array}{ccccccc} \Sigma KE_1 & + & \Sigma U_1 & + & \Sigma W_{ext} & = & \Sigma KE_2 + \Sigma U_2 \\ 0 & + & mgd & + & 0 & = & (1/2)I_a\omega^2 + 0 \end{array}$$

$$0 + (.6\text{ kg})(9.8\text{ m/s}^2)(.18\text{ m}) + 0 = .5(2.55\text{x}10^{-4}\text{ kg}\cdot\text{m}^2)\omega^2 + 0$$

$$\Rightarrow \quad \omega = 91.1\text{ rad/sec.}$$

--from the *rotation about the center of mass plus translation of the center of mass* approach, noting that $v_{cm} = r_a\omega$:

$$\begin{array}{ccccccccc} \Sigma KE_1 & + & \Sigma U_1 & + & \Sigma W_{ext} & = & \Sigma KE_2 & + & \Sigma U_2 \\ 0 & + & mgd & + & 0 & = & [\ (1/2)I_{cm}\omega^2 + (1/2)mv_{cm}^2\ ] & + & 0 \end{array}$$

$$0 + (.6\text{ kg})(9.8\text{ m/s}^2)(.18\text{ m}) + 0 = [.5(1.2\text{x}10^{-4}\text{ kg}\cdot\text{m}^2)\omega^2 + .5(.6\text{ kg})[(.015\text{ m})\omega]^2] + 0$$

$$\Rightarrow \quad \omega = 91.1\text{ rad/sec.}$$

Again, we get the same answer no matter which approach we use.

e.) This is trivial, given the fact that you know the *angular velocity* calculated in *Part d*.

$$\begin{aligned} v_{cm} &= r_a\omega \\ &= (.015\text{ m/rad})(91.1\text{ rad/sec}) \\ &= 1.37\text{ m/s.} \end{aligned}$$

9.6)

a.) As the *force per unit length* function is $\xi = kx$, it makes sense that k times ξ must have the units *newtons per meter*. This will be the case if k has the units nt/m^2.

b.) This is a N.S.L. problem. The moment of inertia of a rod about one end is $(1/3)mL^2$, where m is the rod's mass and L is its length. Since a beam is simply a squared off rod, we will take the moment of inertia of the

beam about its pin to be $(1/3)mL^2$. There are two torques acting about the pin. To do the problem we need to determine those torques.

--The magnitude of the torque about the pin due to the beam's weight acting at the beam's center of mass (i.e., at $L/2$) is:

$$\Gamma_{mg} = (mg)(L/2) \sin 90^o.$$

--The magnitude of the torque about the pin due to the differential bits of force acting downward on the beam along its length is more difficult to determine. To do so, follow along while considering the figure below.

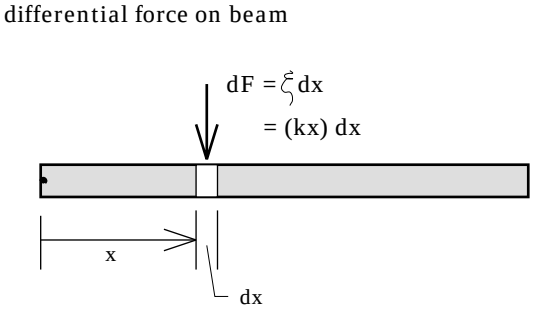
--The magnitude of the differential force dF acting over a differential length dx applied a distance x meters from the pin will be:

$$\begin{aligned} dF &= \xi\, dx \\ &= (kx)\, dx. \end{aligned}$$

--The magnitude of the differential torque $d\Gamma$ about the pin applied by the differential force dF is:

$$\begin{aligned} d\Gamma &= |\mathbf{r}| \quad |\mathbf{F}| \quad \sin 90^o \\ &= x \ (kx\, dx) \quad (1) \\ &= kx^2\, dx. \end{aligned}$$

--The net torque about the pin will be the sum of the differential torques (i.e., those applied due to the presence of the force distributed along the beam's length) and the torque due to gravity. Remembering that the moment of inertia about the beam's pin is $I_{pin} = (1/3)mL^2$:



$\underline{\Sigma\Gamma_{pin}}$:

$$-(mg)(L/2) - \int_{x=0}^{L}(kx^2)dx = -I_{pin}\alpha$$

$$\Rightarrow \alpha = \frac{(mg)(L/2) + k\left(\frac{L^3}{3}\right)}{\left(\frac{1}{3}\right)mL^2}$$

$$= \frac{3g}{2L} + \frac{kL}{m}.$$

9.7)

a.) This is a collision problem. *Energy* is not conserved through the collision because non-conservative forces act during the collision and because the collision was not close enough to being lossless to be approximated as *elastic*. *Momentum* is not conserved because the pin applies an external force to the rod. *Angular momentum* is conserved as there are no external torques (the external force at the pin applies no torque to the system because the pin force is applied at the axis of rotation).

As the rod is massless, we can treat all of the masses in the system as point masses. We know that there are two ways to calculate angular momentum (i.e., by using the magnitude of $\boldsymbol{rx(m\upsilon)}$ or by using $I\omega$), and we know that the *moment of inertia* of a point mass is mr^2 (this means the total moment of inertia of the system is $m_1(d/2)^2 + 2m_2(d/2)^2 = (m_1 + 2m_2)(d/2)^2$). With all of this, we can write:

$$\begin{aligned} \Sigma L_o &= \Sigma L_f \\ L_{wad,o} + 2L_{mass,o} &= I_{tot}\,\omega_1 \\ m_1 v_o (d/2)\sin 90^o + 0 &= [(m_1 + 2m_2)(d/2)^2]\,\omega_1. \end{aligned}$$

Plugging in the numbers, we get:

$$(.9\text{ kg})(2.8\text{ m/s})(1.2/2\text{ m})\sin 90^o = [(.9\text{ kg} + 2(2\text{ kg}))(1.2/2\text{ m})^2]\,\omega_1$$

$$\Rightarrow \quad \omega_1 = .857\text{ rad/sec.}$$

$$\Rightarrow \quad v_{\text{at ends}} = (d/2)\,\omega_1$$

$$= (1.2/2\text{ m})(.857\text{ rad/sec})$$

$$= .514\text{ m/s.}$$

b.) As there is essentially no *potential energy* change through the collision, the *total energy change* will be wrapped up in the *kinetic energy*

difference. The initial *kinetic energy* is all translational, being associated with the wad. The final *kinetic energy* can be treated as purely rotational. Doing so yields an energy difference of:

$$\begin{aligned}\Delta KE &= KE_{after} - KE_{bef} \\ &= (1/2)\, I_{tot}\, \omega_1^2 - (1/2)(m_1)v_o^2 \\ &= .5\,[(m_1 + 2m_2)(d/2)^2]\ \omega_1^2 - .5(m_1)v_o^2.\end{aligned}$$

Putting in the numbers we get:

$$\begin{aligned}\Delta KE &= .5[(.9\text{ kg} + 2(2\text{ kg}))(1.2/2\text{ m})^2](.857\text{ rad/sec})^2 - .5(.9\text{ kg})(2.8\text{ m/s})^2 \\ &= -2.884\text{ joules} \quad \text{(energy is lost).}\end{aligned}$$

c.) The net angular displacement of the system can be determined using *conservation of energy* for the motion *after the collision*. In fact, the symmetry of the problem makes the calculation particularly easy as the *potential energy* picked up by the *right mass* in its transit is equal to the potential energy lost by the *left mass*. As such, the only *potential energy* gain will be associated with the position-change of the wad. That means:

$$\begin{aligned}\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} &= \Sigma KE_2 + \Sigma U_2 \\ (1/2)I_{tot}\omega_1^2 + 0 + 0 &= 0 + m_1 gh.\end{aligned}$$

where *h* is the vertical distance above the wad's initial position. Solving for *h* yields:

$$\begin{aligned}h &= (1/2)I_{tot}\omega_1^2/(m_1 g) \\ &= (1/2)[(.9\text{ kg} + 2(2\text{ kg}))(1.2/2\text{ m})^2](.857\text{ rad/sec})^2 / [(.9\text{ kg})(9.8\text{ m/s}^2] \\ &= .073\text{ meters.}\end{aligned}$$

With *h*, we can form a right triangle whose hypotenuse is *d/2 = .6 meters* and whose *opposite side* is *h = .073 meters* (see the triangle formed in Figure IV on the previous page). Taking the *inverse sine* yields an angle of .122 radians. That means the rod's net angular displacement

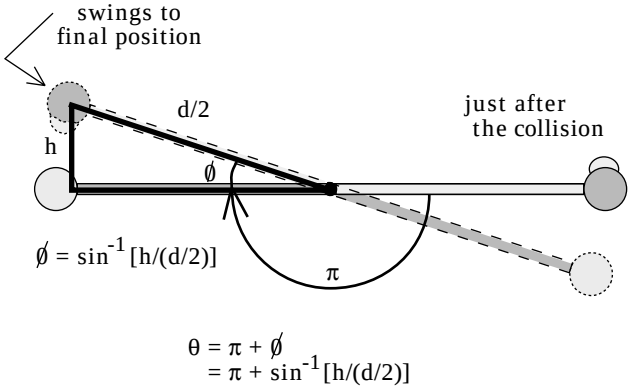


FIGURE IV

will be π radians plus .122 radians, or 3.26 radians.

Note: As we know the energy lost from *Part b*, we could have used the *modified conservation of energy* equation *through the collision.* Doing so would have yielded:

$$\begin{aligned} \Sigma KE_o \quad &+ \quad \Sigma U_o + \Sigma W_{ext} \quad = \Sigma KE_1 + \Sigma U_1 \\ (1/2)m_1 v_o^2 \quad &+ \quad 0 \quad + (-2.884 \text{ j}) = \quad 0 \quad + m_1 gh \\ \Rightarrow \quad (1/2)(.9 \text{ kg})(2.8 \text{ m/s})^2 &+ \quad (-2.884 \text{ j}) \quad = \quad (.9 \text{ kg})(9.8 \text{ m/s}^2)h \\ \Rightarrow \quad h &= .073 \text{ meters.} \end{aligned}$$

9.8) This obviously has within it a collision problem. Through the collision, *energy* is not conserved because the forces involved in the collision are undoubtedly non-conservative and because the problem's author has not deemed it necessary to imbue the collision with the magical label of *elastic*. *Momentum* is not conserved due to the fact that an external force is provided by the pin. *Angular momentum* is conserved because the external force at the pin produces no torque on the system.

If we can determine the amount of energy there is in the system just after the collision, we can use conservation of energy to determine the rise of the stick's center of mass after the collision and, from that, the final angle of the stick. As the *conservation of angular momentum* will allow us to determine the angular velocity of the system just after the collision and, with that, the energy in the system just after the collision, we will begin there.

a.) Assume the before-collision velocity of the block (i.e., after free falling to the bottom of the incline) is v_1 and the after-collision angular velocity of the stick is ω_2. Also, assume the block stays stationary just after the collision. With all that, the angular momentum about the pin is:

$$\begin{aligned} L_{before} \quad &= \quad L_{after} \\ L_{block,bef} + L_{rod,bef} \quad &= \quad L_{rod,aft} + L_{block,aft} \\ mv_1 d \sin 90^o + \quad 0 \quad &= \quad I_{p,rod}\ \omega_2 + \quad 0 \\ &= [(1/3)(5m)d^2]\,\omega_2 \\ \Rightarrow \quad \omega_2 &= (3/5)v_1/d. \end{aligned}$$

The problem here? We don't know what v_1 is. To determine that quantity, use *conservation of energy* during the block's slide down the frictionless incline. Doing so yields:

$$\begin{aligned} \sum KE_o + \sum U_o + \sum W_{ext} &= \sum KE_1 + \sum U_1 \\ 0 + mg(.4d) + 0 &= (1/2)mv_1^2 + 0 \\ \Rightarrow \quad v_1 &= [2g(.4d)]^{1/2} \\ &= 2.8(d)^{1/2}. \end{aligned}$$

With v_1, the angular velocity of the stick just after the collision becomes:

$$\begin{aligned} \omega_2 &= (3/5)v_1/d \\ &= (3/5)[2.8(d)^{1/2}/d] \\ &= 1.68/(d)^{1/2}. \end{aligned}$$

We are now in a position to use *conservation of energy* from the time just after the collision to the time when the stick gets to the top of its motion (see Figure V). Defining the position of both the center of mass of the stick and the position of the block *just after the collision* to be the *potential energy equals zero* level for each object, we can write:

$$\begin{aligned} \sum KE_2 + \sum U_2 + \sum W_{ext} &= \sum KE_3 + \sum U_3 \\ [KE_{2,bl} + KE_{2,st}] + [U_{2,bl} + U_{2,st}] + \sum W_{ext} &= [KE_{3,bl} + KE_{3,st}] + [U_{3,bl} + U_{3,st}] \\ [0 + (1/2)I_p\omega_2^2] + [0 + 0] + 0 &= [0 + 0] + [0 + (5m)gh]. \end{aligned}$$

Solving:

$$\begin{aligned} \Rightarrow \quad .5 \; I_{p,st} \; \omega_2^2 &= (5m)\, g \, h \\ .5[(1/3)(5m)d^2]\,[1.68/(d)^{1/2}]^2 &= (5m)\, g\, [(d/2)(1 - \cos\theta)] \\ \Rightarrow \quad \theta &= .44 \text{ radians or } 25.3^o. \end{aligned}$$

b.) Assume the block adheres to the stick after the collision. Assume also that the before-collision velocity of the block (i.e., after free falling to the bottom of the incline) is v_1, the block's *moment of inertia* about the pin is md^2, and the after-collision angular velocity of both the block and stick is ω_2. With all this, we can write:

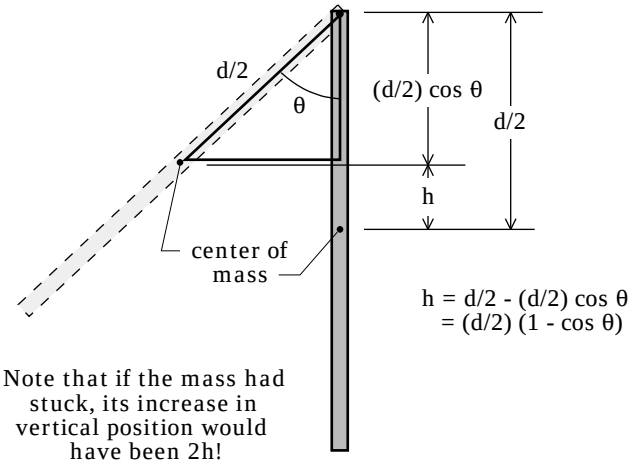

d/2
θ
(d/2) cos θ
d/2
h
center of mass
h = d/2 - (d/2) cos θ
= (d/2) (1 - cos θ)
Note that if the mass had stuck, its increase in vertical position would have been 2h!

FIGURE V

$$
\begin{aligned}
L_{before} &= L_{after} \\
L_{block,bef} + L_{rod,bef} &= L_{rod,aft} + L_{block,aft} \\
mv_1 d \sin 90^o + 0 &= I_{pin,rod}\,\omega_2 + I_{p,bl}\,\omega_2 \\
\Rightarrow \quad mv_1 d &= [(1/3)(5m)d^2 + md^2]\,\omega_2.
\end{aligned}
$$

Canceling *d's* and *m's* and simplifying, we get:

$$\omega_2 = .375 v_1/d.$$

What is v_1? As derived in *Part a*:

$$v_1 = 2.8d^{1/2}.$$

With v_1, the angular velocity of the stick just after the collision becomes:

$$
\begin{aligned}
\omega_2 &= .375 \quad v_1 \quad /d \\
&= .375[2.8(d)^{1/2}]/d \\
&= 1.05/d^{1/2}.
\end{aligned}
$$

We are now in a position to use *conservation of energy* from the time just after the collision to the time when the stick gets to the top of its motion (Figure V still captures the spirit of this calculation). Defining the position of the center of mass of the meter stick and the position of the block *just after the collision* to be the *potential energy equals zero* levels for each object, remembering that the moment of inertia of the block is md^2, and noting that if the stick's center of mass rises a distance h, the block rises a distance $2h$ (see Figure V), we can write:

$$
\begin{aligned}
\Sigma KE_2 + \Sigma U_2 + \Sigma W_{ext} &= \Sigma KE_3 + \Sigma U_3 \\
[KE_{2,bl} + KE_{2,st}] + 0 + 0 &= [KE_{3,bl} + KE_{3,st}] + [U_{3,bl} + U_{3,st}] \\
[.5(md^2)\omega_2{}^2 + .5I_p\omega_2{}^2] + 0 + 0 &= [\,0 + 0\,] + [mg(2h) + (5m)gh].
\end{aligned}
$$

Noting that the stick's *moment of inertia* about the pin is $I_p = (1/3)(5m)d^2 = (5/3)md^2$, we can simplify to get:

$$
\begin{aligned}
.5md^2\omega_2{}^2 + .5[(5/3)md^2]\omega_2{}^2 &= mg[2(d/2)(1 - \cos\theta)] + (5m)g[(d/2)(1 - \cos\theta)] \\
\Rightarrow \quad [.5md^2 + (5/6)md^2]\,\omega_2{}^2 &= [2mg + (5m)g]\,[(d/2)(1 - \cos\,\theta)] \\
\Rightarrow \quad 1.33md^2\omega_2{}^2 &= (7/2)mgd(1 - \cos\,\theta)].
\end{aligned}
$$

Using $\omega_2 = 1.05/d^{1/2}$, we find:

$$1.33md^2[1.05/d^{1/2}]^2 = (7/2)mgd - (7/2)mgd \cos\theta.$$

Canceling *m's* and the *d's*, we get:

$$1.33[1.05^2] = (7/2)g - (7/2)g\cos\theta$$
$$\Rightarrow \quad \theta = .293 \text{ radians or } 16.8^o.$$

9.9) This is a slippage problem in which nothing is conserved. Approach such problems with N.S.L. Treat the frictional force involved as an average, and define the time it takes f_{avg} to bring the two bodies into a pure roll as Δt.

The f.b.d. for the situation is shown in Figure VI to the right.

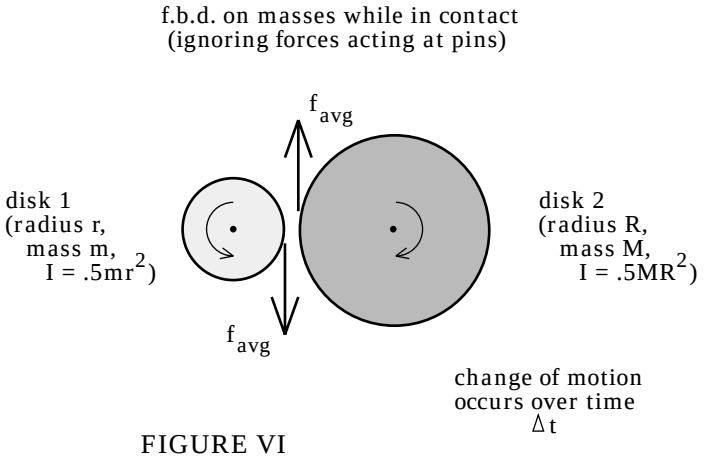


FIGURE VI

Summing the torques acting on the small mass, we get:

$$\underline{\Sigma\Gamma_{cm,1}}:$$
$$-f_{avg}r = I_{1,m}\left(\frac{\omega_3 - \omega_0}{\Delta t}\right)$$
$$\Rightarrow \quad -f_{avg}\Delta t = \left(.5mr^2\right)\left(\frac{\omega_3 - \omega_0}{r}\right)$$
$$\Rightarrow \quad -f_{avg}\Delta t = \left(.5mr\right)\left(\omega_3 - \omega_0\right) \quad \text{(Equ.A).}$$

In doing a similar summation on the large disk, we will have to unembed the required negative sign inherent within the angular velocity term (we didn't have to do that for *m* because its angular velocities were both positive and the required negative sign was provided by the fact that $\omega_o > \omega_3$). That calculation yields:

$\underline{\Sigma \Gamma_{cm,2}}$:

$$-f_{avg}R = -I_{1,M}\left(\frac{\omega_4 - 0}{\Delta t}\right)$$

$$\Rightarrow \quad f_{avg}\Delta t = \left(.5MR^2\right)\left(\frac{\omega_4}{R}\right)$$

$$\Rightarrow \quad f_{avg}\Delta t = \left(.5MR\right)\omega_4 \qquad \text{(Equ.B).}$$

Adding *Equations A* and *B* yields:

$$0 = .5mr\,\omega_3 - .5mr\,\omega_o + .5MR\,\omega_4.$$

We need a way to relate ω_3 to ω_4. This can be done by noting that when the disks are finally rolling without slippage, the translational velocity of each disk *at the point of contact* must be the same (otherwise, there would be slippage). That means:

$$v_{3,edge} = r\omega_3 = V_{4,edge} = R\omega_4$$
$$\Rightarrow \quad \omega_4 = (r/R)\,\omega_3.$$

Using this, we can write:

$$0 = .5mr\,\omega_3 - .5mr\,\omega_o + .5MR\,\omega_4$$
$$\Rightarrow \quad 0 = .5mr\,\omega_3 - .5mr\,\omega_o + .5MR(r/R)\,\omega_3$$
$$\Rightarrow \quad \omega_3 = \frac{m}{(m+M)}w_o.$$

CHAPTER 10 -- GRAVITATION

10.1) According to Newton, the magnitude of the gravitational force between any two bodies will always be equal to Gm_1m_2/r^2.

a.) The gravitation force *you* exert on your friend will be:

$$\begin{aligned} F_{\text{on friend}} &= G\, m_{\text{you}}\, m_{\text{fr}} / r^2 \\ &= (6.67\text{x}10^{-11}\ \text{nt}\cdot\text{m}^2/\text{kg}^2)(80\ \text{kg})(60\ \text{kg})/(5\ \text{m})^2 \\ &= 1.28\text{x}10^{-8}\ \text{nts.} \end{aligned}$$

Note: If you cut a ping pong ball into a million and a half or so pieces, this force would approximately equal the weight of one piece.

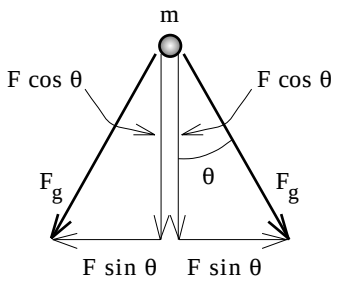
b.) As for the acceleration, Newton's Second Law yields:

$$\begin{aligned} &\underline{\textstyle\sum F_x}: \\ &\quad F_{fr} = m_{fr}a \\ \Rightarrow\quad &a = F_{fr} / m_{fr} \\ &\quad = (1.28\text{x}10^{-8}\ \text{nt})/(60\ \text{kg}) \\ \Rightarrow\quad &\quad = 2.13\text{x}10^{-10}\ \text{m/s}^2. \end{aligned}$$

Note: You can now see why you don't feel a gravitational force when you brush past a friend on the street. The Universal Gravitational Constant G is so small that at least one of the masses has to be enormous before gravitational effects become noticeable.

10.2) Figure I below shows an f.b.d. for the forces acting on the top mass, complete with components. Figure II below shows how the radius r of the orbits can be determined.

FIGURE I



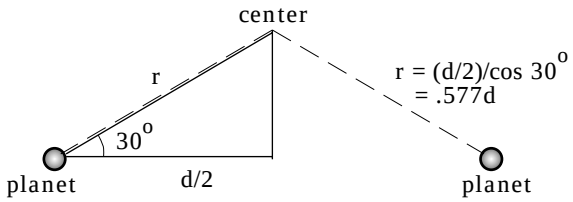


FIGURE II

Due to symmetry, the horizontal components will add to zero leaving only the vertical components with which to contend. Noting that the masses are all the same and the radius of the circular motion is *.577d,* we can use N.S.L. in the *center-seeking direction* to write:

$$\underline{\textstyle\sum F_{\text{center-seeking}}}:$$

$$-F_g\cos\theta - F_g\cos\theta = -ma_c$$

$$\Rightarrow \quad 2[Gm^2/d^2]\cos 30^o = m(v^2/r)$$

$$= mv^2/[.577d]$$

$$\Rightarrow \quad v = (Gm/d)^{1/2}.$$

10.3) This is a straight freefall problem (i.e., although we will be using Newton's gravitational force, the force is not acting centripetally).

a.) The *earth to moon distance* (*center of mass* to *center of mass*) is $3.82\text{x}10^8$ meters plus $1.74\text{x}10^6$ meters plus $6.37\text{x}10^6$ meters, or approximately $3.9\text{x}10^8$ meters. Summing the forces in the direction of freefall, we can write:

$$\underline{\textstyle\sum F_r}:$$

$$-Gm_m m_e/r^2 = -m_m a$$

$$\Rightarrow \quad a = Gm_e/r^2$$

$$= (6.67\text{x}10^{-11}\ \text{nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24}\ \text{kg})/(3.9\text{x}10^8\ \text{m})^2$$

$$= 2.62\text{x}10^{-3}\ \text{m/s}^2.$$

b.) The distance between the moon's center of mass and the earth's center of mass just before they strike one another will be the sum of their radii. Using that observation and repeating the calculation done in *Part a*, we get:

$$\underline{\textstyle\sum F_r}:$$

$$-Gm_m m_e/r_{e,m}^{\ 2} = -m_m a$$

$$\Rightarrow \quad a = Gm_e/r_{e,m}^{\ 2}$$

$$= (6.67\text{x}10^{-11}\ \text{nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24}\ \text{kg})/(8.11\text{x}10^6\ \text{m})^2$$

$$= 6.06\ \text{m/s}^2.$$

c.) The acceleration is not a constant (obviously), so kinematics is out. The best approach is using *conservation of energy*. Remembering that the *potential energy function* for a varying gravitational field is $U = -Gm_1m_2/r$ (a scalar), we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$(0) + (-Gm_m m_e/r) + (0) = (1/2)m_m v^2 + [-Gm_m m_e/(r_m + r_e)]$$
$$\Rightarrow \quad v = [2Gm_e[-1/r + 1/(r_m + r_e)]]^{1/2}$$
$$= [2(6.67\text{x}10^{-11}\ \text{nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24}\ \text{kg})[-1/(3.9\text{x}10^8\ \text{m}) + 1/(1.74\text{x}10^6 + 6.37\text{x}10^6\ \text{m})]]^{1/2}$$
$$\Rightarrow \quad v = 9814\ \text{m/s}.$$

d.) If the earth is allowed to move, we have to take into consideration its *kinetic energy* at the end of the freefall (its initial kinetic energy is zero). This means we now have a second unknown--the earth's velocity--to deal with. To get an expression relating the earth's velocity v_e to the moon's velocity v_m, note that the only force acting in the system--gravity--is internal to the system (that is, the moon exerts a gravitational force on the earth and the earth exerts an equal and opposite gravitational force on the moon). That means momentum is conserved.

With everything starting from rest, the system's initial net momentum (and the system's subsequent net momentum) is zero. Using that bit of information, we can write:

$$p_o = p_{\text{just before impact}}$$
$$\Rightarrow \quad 0 = -m_m v_m + m_e v_e$$
$$\Rightarrow \quad v_e/v_m = m_m/m_e.$$

Knowing the mass of the earth and moon, we now have the relationship we need between the earth's velocity and the moon's velocity (i.e., our second equation).

10.4)

a.) The magnitude of the force applied to m_1 a distance r from the planet's center is:

$$F = G(m_{\text{inside sphere}})m_1/r^2,$$

where $m_{\text{inside sphere}}$ is the mass inside the sphere upon which the body sits. To determine that mass, define a differential sphere of radius a and thickness da. Its differential volume will be its *surface area times its thickness*, or $dV = (4\pi a^2)da$. Defining the *volume density function* to be ρ, we can write:

$$
\begin{aligned}
m_{\text{insidesph}} &= \int \rho dv \\
&= \int_{a=0}^{r} \left[\frac{m_p}{r_p^{\,4}} a \right] \left[4\pi a^2 da \right] \\
&= \frac{4\pi m_p}{r_p^{\,4}} \int_{a=0}^{r} a^3 da \\
&= \frac{4\pi m_p}{r_p^{\,4}} \left[\frac{a^4}{4} \right]_{a=0}^{r} \\
&= \frac{\pi m_p}{r_p^{\,4}} r^4 .
\end{aligned}
$$

Using this, we can write:

$$
\begin{aligned}
F &= G(m_{\text{inside sphere}}) m_1 / r^2 \\
&= G(\pi m_p r^4 / r_p^{\,4}) m_1 / r^2 \\
&= G m_p m_1 (\pi / r_p^{\,4}) r^2 .
\end{aligned}
$$

Note that at $r = 0$, we don't find a zero in the denominator as before.

b.) The *potential-energy-equals-zero-point* for a variable force function is ALWAYS placed where the force is zero. In this case, that is at the planet's center. If we assume the body is sitting at some position $y = r$ up the vertical axis (i.e., we are assuming the tunnel is in the vertical), then the force on the mass m_1 when at an arbitrary point y will be $-Gm_p m_1(\pi/r_p^{\,4})y^2$ (from *Part a*) Setting $U(y = 0) = 0$, we can write:

$$
\begin{aligned}
U(r) - U(y = 0) &= -\int \mathbf{F} \bullet d\mathbf{r} \\
\Rightarrow U(r) &= -\int_{y=0}^{r} \left[-\frac{G m_p m_1 \pi}{r_p^{\,4}} y^2 \mathbf{j} \right] \bullet (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\
&= \frac{G m_p m_1 \pi}{r_p^{\,4}} \int_{y=0}^{r} y^2 dy \\
&= \frac{G m_p m_1 \pi}{r_p^{\,4}} \left[\frac{y^3}{3} \right]_{y=0}^{r} \\
&= \frac{G m_p m_1 \pi}{3 r_p^{\,4}} r^3 .
\end{aligned}
$$

Again, this makes sense. At $r = 0$, the *potential energy* is zero.

10.5) We know the *orbital distance* from the earth's *center of mass* is $r = 1.3x10^6\ m + 6.3x10^6\ m = 7.6x10^6$ *meters*, and we know that this is additionally the distance between the *centers of mass* of the two bodies (the two quantities are the same as the satellite's mass is minuscule in comparison to the earth's mass).

a.) One of the things that makes orbital problems tricky is that there are a number of ways one can determine a velocity. *Conservation of energy* has velocities in it, but so does *N.S.L.* coupled with *centripetal acceleration*. Knowing which approach to use in a given situation is something that comes only with experience. In this particular case, we want to look at the *forces* acting on the satellite.

$\underline{\sum F_c}$:

$$-F_g = -ma_c$$

$$\Rightarrow \quad Gm_sm_e/r^2 = m_s(v^2/r)$$

$$\Rightarrow \quad v = (Gm_e/r)^{1/2}$$

$$= [(6.67x10^{-11}\ nt{\cdot}m^2/kg^2)(5.98x10^{24}\ kg)/(7.6x10^6\ m)]^{1/2}$$

$$= 7244\ m/s.$$

b.) If it takes time T for the satellite to travel the circumference of the circle upon which it moves, then the ratio of those two quantities (i.e., distance/time) will yield the magnitude of its velocity. As T is defined as the period of the motion (i.e., the time for one complete orbit), we can write:

$$v = (\text{circumference})/(\text{period})$$

$$= 2\pi r/T$$

$$\Rightarrow \quad T = 2\pi r/v$$

$$= 2\pi(7.6x10^6\ m)/(7244\ m/s)$$

$$= 6592\ \text{seconds} \qquad (\text{approx. 1 hr, 50 min.}).$$

c.) The work *we* have to do in getting the satellite *up to speed* is equal to the body's *final kinetic energy*. The work *we* have to do in *lifting the satellite* into the appropriate orbit is equal to *minus* the work gravity does on the body as the body rises, or ΔU_g (remember, the work gravity *itself* does as the body rises is $-\Delta U$). If we add these two work quantities together, we have the amount of energy *we* have to provide to the system to get the satellite into orbit.

. . . Or, we could simply use the *modified conservation of energy equation*. In that expression, the $\sum W_{ext}$ term denotes the extra energy needed from *us* to put the satellite in orbit. Using that expression, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$(0) + (-Gm_s m_e/r_e) + E_{you} = (1/2)m_s v^2 + (-Gm_s m_e/r),$$

where v is the orbital velocity of the satellite and r is the orbital radius from the earth's center.

Solving yields:

$$E_{you} = (1/2)m_s v^2 + Gm_s m_e[1/r_e - 1/r]$$
$$= .5(400 \text{ kg})(7244 \text{ m/s})^2 + (6.67\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})(400 \text{ kg})[1/(6.3\text{x}10^6 \text{ m}) - 1/(7.6\text{x}10^6 \text{ m})]$$
$$\Rightarrow \quad E_{you} = 1.48\text{x}10^{10} \text{ joules.}$$

Note: If we had used the *orbital energy* expression $E = -Gm_s m_e/2r$ from the chapter, we would have ended up with $-1.05\text{x}10^{10}$ joules (this, if you will remember, is $KE_r + U_r$). So why doesn't the *orbital energy* expression equal the work required to put the satellite into orbit?

The answer is simple. The body started out with potential energy. Sitting on the earth's surface with $-Gm_e m_s/r_e = -2.53\text{x}10^{10}$ *joules* of potential energy (put in the numbers--that's what you get). Add the energy needed to get it into orbit (i.e., the $1.48\text{x}10^{10}$ joules) and you end up with the satellite's net final energy equaling $-1.05\text{x}10^{10}$ joules.

d.) The earth's angular velocity is 2π radians per 24 hours (give or take a bit). That calculates out to $\omega = 7.27\text{x}10^{-5}$ *radians/second.* At the equator, the magnitude of the earth's translational velocity is $v = r_{equ}\,\omega = (6.3\text{x}10^6 \text{ m})(7.272\text{x}10^{-5} \text{ rad/sec}) = 458 \text{ m/s}$.

A satellite launched in the direction of the earth's rotation at the equator will begin its trip with initial kinetic energy equal to $(1/2)mv_e^2$, where v_e is the velocity of the rotating earth.

In the case of our satellite, it would start out with kinetic energy equal to $(1/2)(400 \text{ kg})(458 \text{ m/s})^2 = 4.2\text{x}10^7$ *joules* worth of energy *we* wouldn't have to supply to the system to get it into its orbit.

It should now be obvious why launching pads in the U.S. (Cape Canaveral, for instance) are as close to the equator as possible.

e.) In 1800 revolutions, the energy lost to the satellite will be:

$$E_{lost} = (1800 \text{ rev})(2\text{x}10^5 \text{ joules/rev})$$
$$= 3.6\text{x}10^8 \text{ joules.}$$

The amount of energy it starts out with as it traveled in its orbit is:

$$E_1 = -Gm_em_s/2r_1$$
$$= -(6.6\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})(400 \text{ kg})/[2(7.6\text{x}10^6 \text{ m})]$$
$$= -1.049\text{x}10^{10} \text{ joules.}$$

After losing $3.6\text{x}10^8$ joules of energy, the satellite has:

$$E_2 = E_1 - E_{lost}$$
$$= (-1.049\text{x}10^{10} \text{ joules}) - (3.6\text{x}10^8 \text{ joules})$$
$$= -1.085\text{x}10^{10} \text{ joules.}$$

An orbit with that amount of energy must be such that:

$$E_2 = -Gm_em_s/2r_2$$
$$\Rightarrow \quad r_2 = -Gm_em_s/2E_2$$
$$= -(6.67\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})(400 \text{ kg})/[2(-1.085\text{x}10^{10} \text{ j})]$$
$$= 7.35\text{x}10^6 \text{ meters.}$$

This is the distance from the earth's center. From the earth's surface, the height will be *$7.35\text{x}10^6$ meters minus $6.3\text{x}10^6$ meters equals $1.05\text{x}10^6$ meters*.

Although the problem did not ask for the velocity of the satellite when in this orbit, we will need it later. Using N.S.L., we can write:

$\sum F_r$:

$$F_g = ma_c$$
$$\Rightarrow \quad Gm_sm_e/r_2^2 = m_s(v_2^2/r_2)$$
$$\Rightarrow \quad v_2 = (Gm_e/r_2)^{1/2}$$
$$= [(6.67\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})/(7.35\text{x}10^6 \text{ m})]^{1/2}$$
$$= 7366 \text{ m/s.}$$

f.) The retarding force does $2\text{x}10^5$ joules *per revolution* of work on the satellite as the satellite moves along in its path. On average, the work done by the retarding force in *one orbit* will equal the average force *times* the average circumference of motion *times* cos 180^o (the retarding force and the direction of

motion will be opposite one another, hence the 180^o angle). Approximating the *average radius* to be $(r_1 + r_2)/2$, we can determine the work calculation for one average orbit. Doing so yields:

$$\begin{aligned} 2\text{x}10^5 &= F_{avg}(2\pi r_{avg}) \cos 180^o \\ &= -F_{avg}[2\pi(r_2 + r_1)/2] \\ &= -F_{avg}[2\pi(7.35\text{x}10^6 + 7.6\text{x}10^6)/2] \\ &= -(4.7\text{x}10^7)F_{avg} \\ \Rightarrow \quad F_{avg} &= -.0043 \text{ newtons} \end{aligned}$$

g.) The *modified conservation of angular momentum* expression is:

$$\sum L_1 + \sum \Gamma_{ext}\Delta t = \sum L_2,$$

where the L terms are angular momentum quantities and the $\Gamma\Delta t$ is a torque driven, *impulse-related* quantity (remember, the *modified conservation of linear momentum* equation was $\sum p_1 + \sum F_{ext}\Delta t = \sum p_2$, where the $F_{ext}\Delta t$ term was the impulse being delivered by external forces over the time interval Δt).

The average external torque about the planet's orbital axis is:

$$\Gamma_{ext} = \mathbf{r}_{s,avg}\text{x}\mathbf{F}_{avg}.$$

Noting that the frictional force $\boldsymbol{F}_{avg}$ is perpendicular to the radius vector $\boldsymbol{r}_{s,avg}$, the magnitude of the torque will simply be the product of the radius and force vectors ($\sin 90^o = 1$). Assuming the direction of the satellite's angular velocity is *positive* and noting that the torque slows the motion, our friction-induced torque will be negative. As such, we can write:

$$\Gamma_{ext} = -r_{s,avg}F_{avg}.$$

The angular momentum of the satellite will equal:

$$L = \mathbf{r}_{s,avg}\text{x}\mathbf{p},$$

where again the angle between the vectors is 90^o and the sine of the angle is *1*. As the magnitude of the momentum is $m_s v_s$, we can write in general:

$$L = r_{s,avg} m_s v_s.$$

Using all this information in our *modified conservation of angular momentum* equation, we can solve for Δt:

$$\sum L_1 + \sum \Gamma_{ext} \Delta t = \sum L_2$$

$$r_{s,1} m_s v_{s,1} - r_{s,avg} F_{avg} \Delta t = r_{s,2} m_s v_{s,2}$$

$$(7.6\text{x}10^6 \text{ m})(400 \text{ kg})(7244 \text{ m/s}) - (7.475\text{x}10^6 \text{ m})(.004 \text{ nt})\Delta t = (7.35\text{x}10^6 \text{ m})(400 \text{ kg})(7366 \text{ m/s})$$

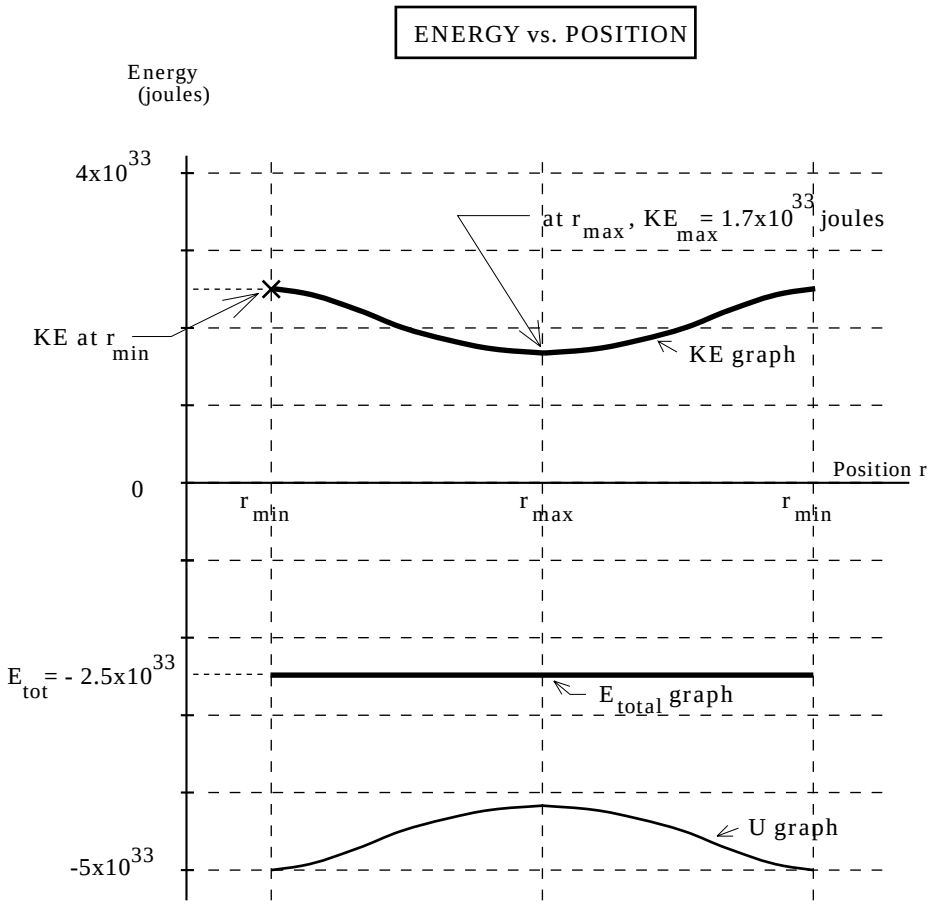
$$\Rightarrow \quad \Delta t = 1.223\text{x}10^7 \text{ seconds.}$$

This rounded-off value is approximately 141 days, 13.5 hours.

10.6)

a.) The conserved quantities for planetary motion are *angular momentum* (there are no external torques acting on the planet), and *energy* (there is practically no frictional effect in space, and there are generally no appreciable non-conservative forces acting on planets--asteroid collisions excepted).

b.) At r_{min}, the star has $U_1 = -5\text{x}10^{33}$ *joules* (see graph) and kinetic energy $KE_1 = 2.5\text{x}10^{33}$ *joules*. That means the total energy of the system is $KE_1 + U_1 = -2.5\text{x}10^{33}$ *joules*. In addition, the sum of the potential and kinetic energies must *always* equal that number (energy *is* conserved). Because we know the *total energy* in



the system, and because we have a graph of the *potential energy* as a function of position, we can determine the *kinetic energy* as a function of position for any point in the orbit. Doing that for key points (example: at r_{max}, the graph tells us that the *potential energy* is approximately U_{max} = $-4.2x10^{33}$ *joules* . . . that means KE_{max} - $4.2x10^{33}$ *joules = total energy =* $-2.5x10^{33}$ *joules*, or KE_{max} = $1.7x10^{33}$ *joules* at that point), and remembering that the *kinetic energy function* mirrors the *potential energy function* when the *total energy* is constant, we can draw the *kinetic energy* graph as shown below.

CHAPTER 11 -- VIBRATORY MOTION

11.1) *A = .5 meters*; *T = .3 sec/cycle*; and *m = 1.2 kg*.

a.) Oscillatory frequency:

$$\begin{aligned} \nu &= 1/T \\ &= 1/(.3 \text{ sec/cycle}) \\ &= 3.33 \text{ cycles/sec} \quad (\text{or } 3.33 \text{ hz}). \end{aligned}$$

b.) Angular frequency (the number of *radians per second* the motion sweeps through):

$$\begin{aligned} \omega &= 2\pi\nu \\ &= 2(3.14)(3.33 \text{ hz}) \\ &= 20.94 \text{ rad/sec}. \end{aligned}$$

c.) Spring constant:

$$\begin{aligned} \omega &= (k/m)^{1/2} \\ \Rightarrow \quad k &= \omega^2 m \\ &= (20.94 \text{ rad/sec})^2(1.2 \text{ kg}) \\ &= 526.4 \text{ nt/m}. \end{aligned}$$

d.) Maximum velocity (occurs where acceleration is zero, or while mass passes through equilibrium position):

$$\begin{aligned} v_{max} &= \omega A \\ &= (20.94 \text{ rad/sec})(.5 \text{ m}) \\ &= 10.47 \text{ m/s}. \end{aligned}$$

e.) Magnitude of the maximum acceleration (occurs where force is greatest, or at extremes of motion):

$$\begin{aligned} a_{max} &= \omega^2 A \\ &= (20.94 \text{ rad/sec})^2(.5 \text{ m}) \\ &= 219.2 \text{ m/s}^2. \end{aligned}$$

f.) Energy in system:

$$E = (1/2)kA^2$$
$$= .5\ (526.4 \text{ nt/m})(.5 \text{ m})^2$$
$$= 65.8 \text{ joules.}$$

11.2) *$m = .25$ kg; $k = 500$ nt/m; $v_{max} = 3$ m/s.*

a.) Angular frequency:

$$\omega = (k/m)^{1/2}$$
$$= [(500 \text{ nt/m})/(.25 \text{ kg})]^{1/2}$$
$$= 44.72 \text{ rad/sec.}$$

b.) Frequency:

$$\omega = 2\pi\nu$$
$$\Rightarrow \quad \nu = \omega/2\pi$$
$$= (44.72 \text{ rad/sec})/2\pi$$
$$= 7.12 \text{ hz.}$$

c.) Period:

$$T = 1/\nu$$
$$= 1/(7.12 \text{ hz})$$
$$= .14 \text{ sec/cycle.}$$

d.) Amplitude:

$$v_{max} = \omega A$$
$$\Rightarrow \quad A = v_{max}/\omega$$
$$= (3 \text{ m/s})/(44.72 \text{ rad/sec})$$
$$= .067 \text{ m.}$$

e.) Total energy:

$$E_t = (1/2)kA^2$$
$$= .5(500 \text{ nt/m})(.067 \text{ m})^2$$
$$= 1.125 \text{ joules.}$$

f.) Magnitude of maximum force (this will be applied when the acceleration is maximum--i.e., at the extremes):

$$a_{max} = \omega^2 A$$
$$= (44.72 \text{ rad/sec})^2(.067 \text{ m})$$
$$= 134 \text{ m/s}^2.$$

N.S.L. $\Rightarrow$ $F_{max.} = ma_{max}$
$= (.25 \text{ kg})(134 \text{ m/s}^2)$
$= 33.5$ nts.

11.3) The most general *<u>position</u> as a function of <u>time</u>* expression is:

$$x = A \sin (\omega t \pm \phi) \quad \text{or} \quad x = A \sin (2\pi \nu t \pm \phi).$$

In our case, we know that $x = .7 \sin (14t - .35)$. Matching the appropriate variables leads to:

a.) Amplitude: A = .7 meters (by inspection).

b.) Angular frequency: $\omega = 14$ rad/sec (by inspection).

c.) Frequency:

$$\omega = 2\pi\nu$$
$$\Rightarrow \quad \nu = \omega/2\pi$$
$$= (14 \text{ rad/sec})/2\pi$$
$$= 2.23 \text{ hz.}$$

d.) Position at *t = 3 seconds*:

$$x = .7 \sin (14t - .35)$$
$$= .7 \sin (14(3 \text{ sec}) - .35)$$
$$= -.51 \text{ m.}$$

e.) Position at *t = 3.4 seconds*:

$$x = .7 \sin (14t - .35)$$
$$= .7 \sin (14(3.4 \text{ sec}) - .35)$$
$$= -.088 \text{ m.}$$

f.) Velocity at $t = 0$:

$$\begin{aligned} v &= \omega A \cos(\omega t + \phi) \\ &= (14 \text{ rad/sec})(.7 \text{ m}) \cos(14t - .35) \\ &= (14 \text{ rad/sec})(.7 \text{ m}) \cos(14(0) - .35) \\ &= 9.2 \text{ m/s}. \end{aligned}$$

g.) Acceleration at $t = 0$:

$$\begin{aligned} a &= -\omega^2 A \sin(\omega t + \phi) \\ &= -(14 \text{ rad/sec})^2(.7 \text{ m}) \sin(14t - .35) \\ &= -(14 \text{ rad/sec})^2(.7 \text{ m}) \sin(14(0) - .35) \\ &= 47 \text{ m/s}^2. \end{aligned}$$

11.4) The period of oscillation T is related to the motion's frequency ν by the relationship $\nu = 1/T$. The frequency ν is related to the *angular frequency* ω by the relationship $\omega = 2\pi\nu$. In simple harmonic motion, the angular frequency is a function of the pendulum arm length L and the acceleration of gravity g. That is:

$$\begin{aligned} \omega &= (g/L)^{1/2} \\ \Rightarrow \quad g &= \omega^2 L \\ &= (2\pi\nu)^2 L \\ &= (2\pi/T)^2 L \\ &= (2\pi/(2 \text{ sec}))^2(1.75 \text{ m}) \\ &= 17.27 \text{ m/s}^2. \end{aligned}$$

Notice that the mass has nothing to do with anything here (just as the mass would have nothing to do with the rate at which velocity changes as a body falls near the planet's surface).

11.5)

a.) The equation $\alpha + (12g/7L)\,\theta = 0$ is of the form "acceleration plus constant-times-displacement equals zero" (even though the acceleration and displacement terms are angular ones). That means the motion is, by definition, *simple harmonic* in nature.

b.) For *simple harmonic motion*, we know that the *square of the angular frequency* of the motion is equal to the constant in front of the *displacement* term. In this case:

$$\omega^2 = 12g/7L$$
$$\Rightarrow \quad \omega = (12g/7L)^{1/2}.$$

We also know that $w = 2\pi\nu$, or:

$$\begin{aligned}\nu &= \omega/2\pi \\ &= [(12g)/(7L)]^{1/2}/(2\pi) \\ &= (1.7g/L)^{1/2}/(2\pi) \\ &= [(1.7)(9.8\ m/s^2)/(1.3\ m)]^{1/2}/[(2(3.14)] \\ &= .57\ \text{cycles/second.}\end{aligned}$$

11.6) Note that when the 3 kg mass is attached to the spring, gravity acting on the mass applies a force such that the spring elongates .7 meters. Also, at *t = 0, y = -.15 meters* moving *away from* equilibrium:

a.) Spring constant: The spring constant tells you how much force *F* is required to elongate the spring. In this case, GRAVITY is used to elongate the spring a distance *d*, or:

$$\begin{aligned}k &= (mg)/d \\ &= (3\ kg)(9.8\ m/s^2)/(.7\ m) \\ &= 42\ nt/m.\end{aligned}$$

b.) Angular frequency:

$$\begin{aligned}\omega &= (k/m)^{1/2} \\ &= [(42\ nt/m)/(3\ kg)]^{1/2} \\ &= 3.74\ rad/sec.\end{aligned}$$

c.) Amplitude: *A = .4 meters* (stated in problem: "once at equilibrium, the system is displaced an additional .4 meters and released").

d.) Frequency:

$$\omega = 2\pi\nu$$
$$\Rightarrow \quad \nu = \omega/2\pi$$
$$= (3.74\ rad/sec)/2\pi$$
$$= .6\ hz.$$

e.) Period:

$$\begin{aligned} T &= 1/\nu \\ &= 1/(.6 \text{ cycles/sec}) \\ &= 1.67 \text{ sec/cycle}. \end{aligned}$$

f.) Total energy:

$$\begin{aligned} E_t &= (1/2)kA^2 \\ &= .5(42 \text{ nt/m})(.4 \text{ m})^2 \\ &= 3.36 \text{ joules}. \end{aligned}$$

g.) Maximum velocity:

$$\begin{aligned} v_{max} &= \omega A \\ &= (3.74 \text{ rad/sec})(.4 \text{ m}) \\ &= 1.5 \text{ m/s}. \end{aligned}$$

h.) The velocity is maximum at equilibrium.

i.) Maximum acceleration:

$$\begin{aligned} a_{max} &= \omega^2 A \\ &= (3.74 \text{ rad/sec})^2(.4 \text{ m}) \\ &= 5.6 \text{ m/s}^2. \end{aligned}$$

j.) The acceleration is maximum at either extreme (i.e., at $x = \pm A$).

k.) To determine the general algebraic expression (using y as the position variable), we start with the standard form:

$$y = A \sin(\omega t + \phi) \quad \text{or} \quad y = A \sin(2\pi \nu t + \phi).$$

Knowing the *amplitude* and the *angular frequency*, we can immediately write:

$$y = .4 \sin(3.74t + \phi).$$

The only variable we need to determine anew is the phase shift ϕ. We know that at $t = 0$, $y = -.15$ moving away from equilibrium. The sine wave shown on the next page highlights this situation. To determine the phase shift ϕ:

We know that in general,

$$y = .4 \sin (3.74t + \phi).$$

Substituting in $t = 0$ and $y = -.15$, we can write:

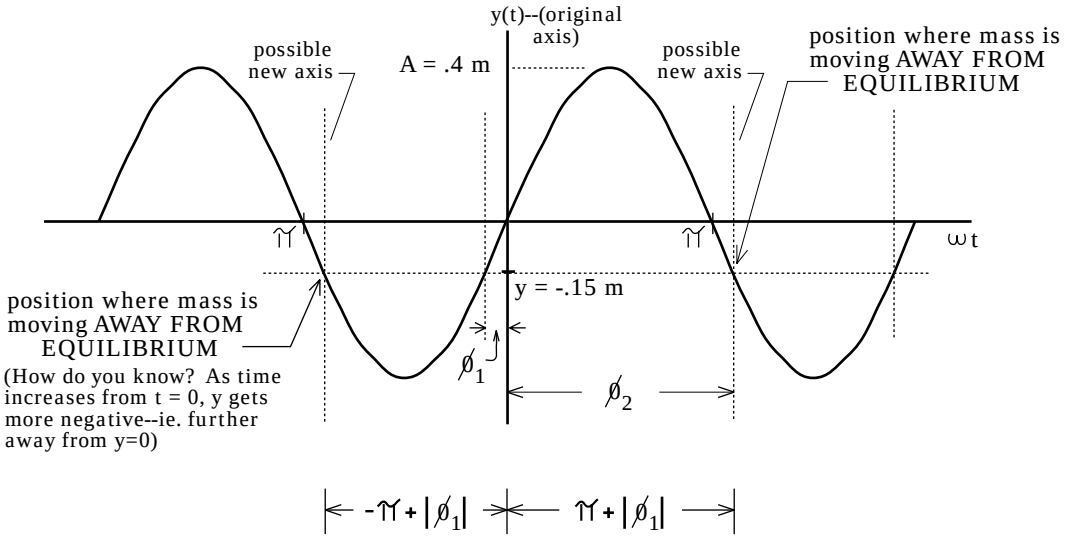
$$-.15 = .4 \sin (3.74(0) + \phi)$$
$$-.15 = .4 \sin (\phi).$$

Solving for ϕ yields:

$$\phi = \sin^{-1} [(-.15)/(.4)]$$
$$= -.384 \text{ rad.}$$

The question is, "What does that mean?" That is, there are two angles that could possibly satisfy the math in this situation, one in the fourth quadrant (ϕ_1 in the sketch below) and one in the third quadrant (ϕ_2). We have been given a value for the fourth quadrant angle (that is, $\phi_1 = -.384$ *radians*). We can find the other using math and a bit of logic.

To tell which of the two we really want, look at the sketch below (both possible axes are shown).


y(t)--(original axis)
possible new axis
A = .4 m
possible new axis
position where mass is moving AWAY FROM EQUILIBRIUM
ωt
y = -.15 m
position where mass is moving AWAY FROM EQUILIBRIUM
(How do you know? As time increases from t = 0, y gets more negative--ie. further away from y=0)
ϕ1
ϕ2
-π + |ϕ1|
π + |ϕ1|

From observation, it is evident that the object is moving *away from* equilibrium when in the *third* quadrant. There is no formula to get the angle required to move the axis to the appropriate third-quadrant position

on the sine wave (we can move left or right to get there--either will do), so we will have to use our heads.

Doing so, the sketch allows us to see that ϕ_2 is equal to the addition of the *magnitude* of ϕ_1 and $-\pi$ (i.e., *-3.14 + .384 = -2.76 rad*). Another possibility would be to add the magnitude of ϕ_1 to $+\pi$ (i.e., *3.14 + .384 = 3.53 rad*). Both cases are shown in the figure; both cases designate an appropriate position for the new axis.

Bottom line: The general algebraic expression that defines the *position* of the body as a function of *time* is:

$$y = .4 \sin (3.74t - 2.76)$$

or

$$y = .4 \sin (3.74t + 3.53).$$

Either is acceptable.

11.7) Although this probably appears to be a completely off-the-wall problem, it is characteristic of a class of problems that have a common hallmark. Specifically, they all give information about the force acting on a body moving in periodic motion (or they give enough information for you to derive a force relationship--in this case, we derived the required expression in the previous chapter), and they always ask something about the *period* of the motion.

a.) If we can determine Jack's period, we will have the period of the satellite (the two have to be the same if the satellite is going to take a picture of Jack head every time it appears out the top (at the bottom, it will be his feet that will appear). The key to finding Jack's period is found in the gravitational force that keeps him oscillating between poles.

Assuming the motion is in the *y* direction and leaving the sign of the acceleration embedded in the a_y term, we can write:

$$\sum F_y: \quad -\left[\frac{G m_e m_J}{r_e^3}\right] y = m_J a_y.$$

Rearranging, this implies that:

$$a_y + \left[\frac{G m_e}{r_e^3}\right] y = 0.$$

This is the characteristic equation for *simple harmonic motion.* Knowing that, we also know that the *angular frequency* of Jack's oscillatory motion must be equal to the *square root of the constant in front of the displacement term*. Using our relationships between *angular frequency*, *frequency*, and *period*, we can write:

$$\omega = (Gm_e/r_e^{\,3})^{1/2}.$$

As

$$\omega = 2\pi\nu$$

and

$$T = 1/\nu,$$

we can write:

$$T = 2\pi/(Gm_e/r_e^{\,3})^{1/2}.$$

b.) As the note in the problem said, this really isn't a vibratory motion problem--it's more of a gravitation problem in review. No matter. The practice will do you good.

Knowing the period of motion, we can generate an expression for the velocity of the mass in terms of its radius. That is:

$$\begin{aligned} v_s &= (\text{circumference of orbit})/T \\ &= [2\pi r_{satellite}]/[2\pi/(Gm_e/r_e^{\,3})^{1/2}] \\ &= (Gm_e/r_e^{\,3})^{1/2} r_s. \end{aligned}$$

Knowing the velocity, we can use N.S.L. and the fact that the satellite's motion is centripetal (we'll assume a circular path for simplicity). Using that approach, we get:

$$\underline{\textstyle\sum F_c}:$$

$$\begin{aligned} Gm_e m_s/r_s^{\,2} &= m_s(v_s^{\,2}/r_s) \\ \Rightarrow \quad r_s &= (Gm_e/v_s^{\,2}). \end{aligned}$$

Plugging this back into our velocity expression above, we get:

$$\begin{aligned} v_s &= (Gm_e/r_e^{\,3})^{1/2} r_s \\ &= (Gm_e/r_e^{\,3})^{1/2}(Gm_e/v_s^{\,2}) \\ \Rightarrow \quad v_s &= [(Gm_e/r_e^{\,3})^{1/2} Gm_e]^{1/3} \\ &= (Gm_e/r_e)^{1/2}. \end{aligned}$$

We derived an expression for r_s in terms of v_s. Substituting v_s into that expression yields:

$$\begin{aligned} r_s &= (Gm_e/v_s^2) \\ &= Gm_e/[(Gm_e/r_e)^{1/2}]^2 \\ &= r_e. \end{aligned}$$

Amazing! The satellite would have to skim the earth's surface to have the velocity associated with the correct period.

Wasn't that fun!

CHAPTER 12 -- WAVE MOTION

12.1) The relationship between a wave's *frequency* ν, its *wavelength* λ, and its *wave velocity* v is $v = \lambda \nu$. For sound in air, the wave velocity is approximately $v = 330$ *m/s*. To get the wavelength:

a.) For $\nu = 20$ *hz*:

$$\lambda = v/\nu$$
$$= (330 \text{ m/s})/(20 \text{ hz})$$
$$= 16.5 \text{ meters} \qquad \text{(around 50 feet).}$$

Note: Technically, the units of *frequency* are $seconds^{-1}$ and of *wavelength* are *meters*. The *cycles* term in the frequency units is a label, being the same for MKS, CGS, and the English system of units. This means that by dividing frequency into velocity we get the units *(m/s)/(1/s) = meters*. If you had included the cycles label, that division would have yielded units of *(m/s)/(cycles/s) = meters/cycle*. There is really nothing wrong with this--it is a literal description of what the wavelength is (the number of meters there is in one wave--one cycle), but using it could potentially get you in trouble later. Best go with $seconds^{-1}$ for simplicity.

b.) For $\nu = 20{,}000$ *hz*:

$$\lambda = v/\nu$$
$$= (330 \text{ m/s})/(20{,}000 \text{ hz})$$
$$= .0165 \text{ meters} \quad \text{(a little over half an inch).}$$

12.2)

a.) The sketch is shown on the next page.

b.) If the first wave has an amplitude of $A_1 = 1$ *meter*, the second largest amplitude wave will have an amplitude of approximately $A_2 = .33$ *meters* and the third approximately $A_3 = .2$ *meters*.

Note that the largest wave (the *first wave* as defined above) has one half-wavelength in the same space that the *second wave* has 3 half-

wavelengths and the *third wave* has 5 half-wavelengths. That means that if the first wave has a frequency of $\nu_1 = 1\nu$, the second wave will have a frequency of $\nu_2 = 3\nu$ and the third wave a frequency of $\nu_3 = 5\nu$.

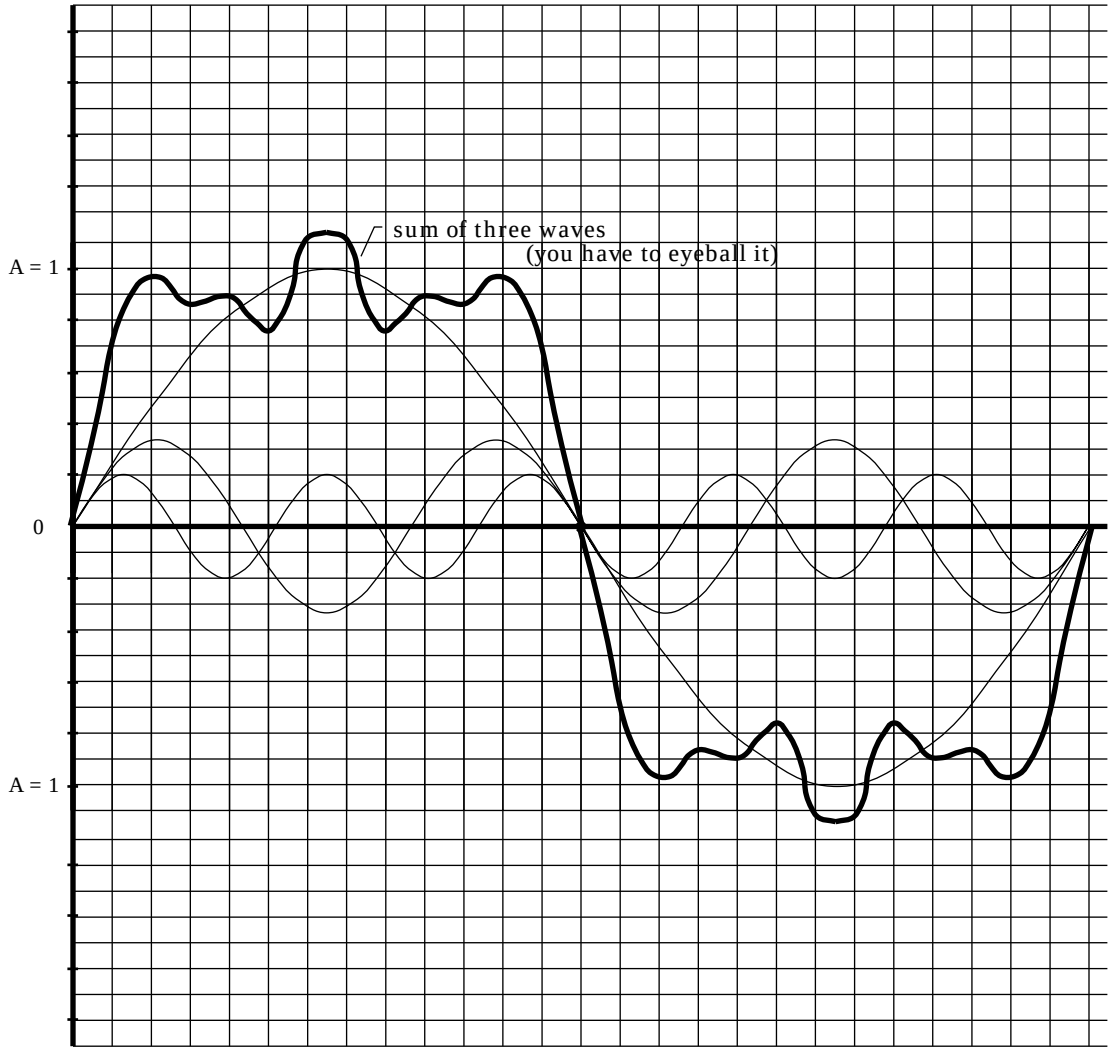


FIGURE I

Frequency is proportional to the *angular frequency*. That means that if the *angular frequency* of the first wave is $\omega_1 = 1$ *rad/sec*, the second wave's angular frequency will be $\omega_2 = 3$ *rad/sec* and the third wave's angular frequency will be $\omega_3 = 5$ *rad/sec*.

Putting it all together, remembering that the general algebraic expression for a sine wave with no phase shift is $A \sin \omega t$, we get:

$$\begin{aligned} y_{tot} &= A_1 \sin \omega_1 t + A_2 \sin \omega_2 t + A_3 \sin \omega_3 t \\ &= 1 \sin t + .33 \sin 3t + .2 \sin 5t. \\ &= (1/1) \sin t + (1/3) \sin 3t + (1/5) \sin 5t. \end{aligned}$$

c.) The first six terms of the series are:

$$y_t = 1\sin 1t + (1/3)\sin 3t + (1/5)\sin 5t + (1/7)\sin 7t + (1/9)\sin 9t + (1/11)\sin 11t.$$

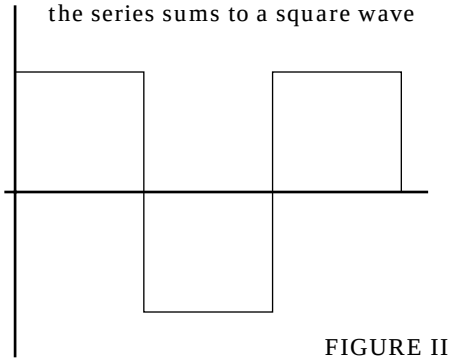


FIGURE II

d.) The waveform is shown to the right. It is called a *square wave*.

12.3)

a.) At $t = 0$:

$$\begin{aligned} y &= 12 \sin (25x - .67(0)) \\ &= 12 \sin 25x. \end{aligned}$$

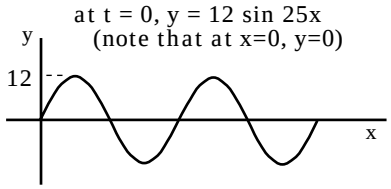


FIGURE III

This function is graphed in Figure III to the right.

At $t = 1$ *second*:

$$\begin{aligned} y &= 12 \sin (25x - .67(1)) \\ &= 12 \sin (25x - .67). \end{aligned}$$

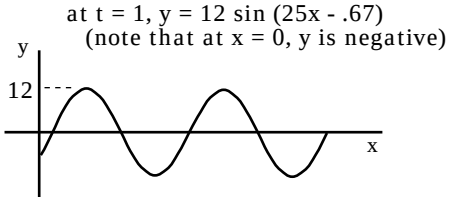


FIGURE IV

This function is graphed in Figure IV to the right.

b.) The wave is moving to the right (note that its peaks are further to the right at $t = 1$ *second* than they are at $t = 0$ *seconds*).

c.) *Positive* values for the time dependent part of this equation yield wave motion (in time) to the left (in the negative *x direction*); *negative* values for the time dependent part of this equation yield wave motion to the right (i.e., in the +*x direction*), as was pointed out in *Part b*.

d.) The *angular frequency* is .67 rad/sec. As:

$$\omega = 2\pi\nu$$
$$\Rightarrow \quad \nu = \omega/2\pi$$
$$= (.67 \text{ rad/sec})/2\pi$$
$$= .107 \text{ Hz.}$$

e.) The period is:

$$T = 1/\nu$$
$$= 1/(.107 \text{ Hz})$$
$$= 9.35 \text{ sec/cycle.}$$

f.) We know that the wave number is $k = 25\ m^{-1}$. The wavelength is related to the wave number by:

$$k = 2\pi/\lambda$$
$$\Rightarrow \quad \lambda = 2\pi/k$$
$$= 2\pi/(25\ m^{-1})$$
$$= .25 \text{ meters.}$$

Note: Just as *angular frequency* tells you how many radians the wave sweeps through per unit time at a given point, the *wave number* tells you how many radians of wave there are per meter of wave.

Look at the units if this isn't clear. The equation states that there are (*2π radians*/*wavelength*) divided by (*λ meters* of wave per wavelength), or *2π / λ radians per meter* of wave. Put another way, if *k = 2π rad*/*m*, we are being told that one full cycle of wave (i.e., 2π radians worth) spans *one meter*.

g.) Wave velocity:

$$v = \lambda\nu.$$
$$= (.25 \text{ m})(.107 \text{ Hz})$$
$$= .02675 \text{ m/s.}$$

h.) The amplitude is 12 meters (by inspection).

12.4) We know that the frequency is 225 Hz, the amplitude is .7 meters, and the wave velocity is 140 m/s. Knowing the wave velocity, we can write:

$$\begin{aligned} v &= \lambda \nu \\ \Rightarrow \quad \lambda &= v/\nu \\ &= (140 \text{ m/s})/(225 \text{ Hz}) \\ &= .622 \text{ meters.} \end{aligned}$$

Traveling waves have a general algebraic expression of:

$$\begin{aligned} y &= A \sin (kx + \omega t) \\ &= A \sin [(2\pi/\lambda)x + 2\pi \nu t] \\ &= .7 \sin [[2\pi/(.622 \text{ m})]x + 2\pi(225 \text{ Hz})t] \\ &= .7 \sin (10.1x + 1413.7t). \end{aligned}$$

12.5) Dividing out the coefficient of the α term to get this equation in the right form (i.e., the standard *simple harmonic motion* equation), we get:

$$\alpha + (3g/2L)\ \theta = 0.$$

a.) The angular frequency for this motion is:

$$\begin{aligned} \omega &= (3g/2L)^{1/2} \\ &= [3(9.8 \text{ m/s}^2)/2(.8 \text{ m})]^{1/2} \\ &= 4.29 \text{ rad/sec.} \end{aligned}$$

Knowing this, we can find the natural frequency-of-oscillation for this system:

$$\begin{aligned} \nu &= \omega/2\pi \\ &= (4.29 \text{ rad/sec})/2\pi \\ &= .683 \text{ Hz.} \end{aligned}$$

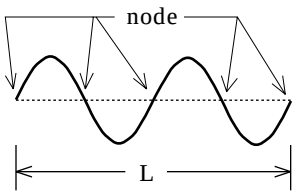
b.) The period is:

$$\begin{aligned} T &= 1/\nu \\ &= 1/(.683 \text{ Hz}) \\ &= 1.46 \text{ sec/cycle.} \end{aligned}$$

If the frequency (hence period) of the applied force is close to the natural frequency (hence period) of the system, resonance will occur and the amplitude of the motion will grow immensely. If not, the applied force

will fight the natural motion and the net effect will be small amplitude motion. The period in *Part i* (1.31 seconds) fits into the latter category; the period in *Part ii* (1.47 seconds) fits into the former category.

12.6) We know $\nu = 800\ Hz;\ L = .3\ meters$; and there are 5 nodes with one at each end (that is, the string is split into four sections by the three remaining nodes). A sketch of the system is shown to the right.



To get the velocity, we will use $v = \lambda\ \nu$. We know ν; we need λ. To get it, notice that there are TWO full wavelengths in the length L. Mathematically:

$$\begin{aligned} 2\lambda &= L \\ &= .3\ m \\ \Rightarrow \quad \lambda &= .15\ m. \end{aligned}$$

Putting it all together, we get:

$$\begin{aligned} v &= \lambda\ \nu \\ &= (.15\ m)(800\ Hz) \\ &= 120\ m/s. \end{aligned}$$

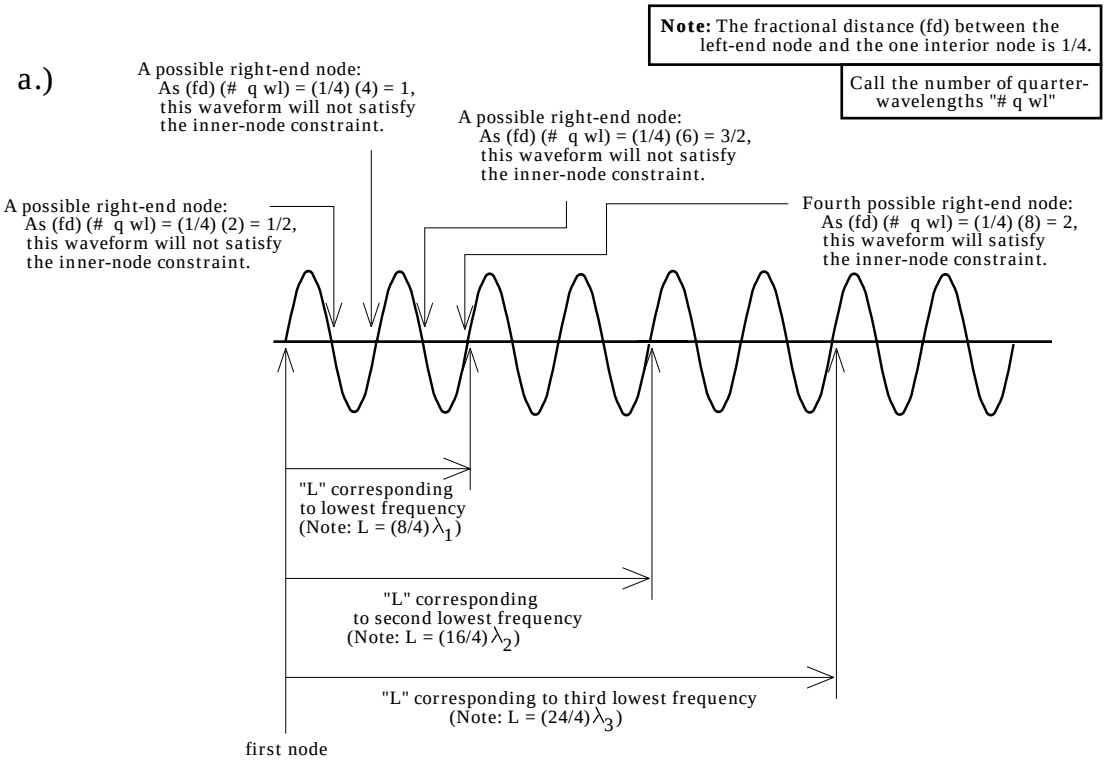
12.7) Calculations for all parts follow the sketches on the next few pages.

a.) We need a node at both ends and one *L/4* units from the left end. The *sine wave* on the next page depicts the various possibilities.

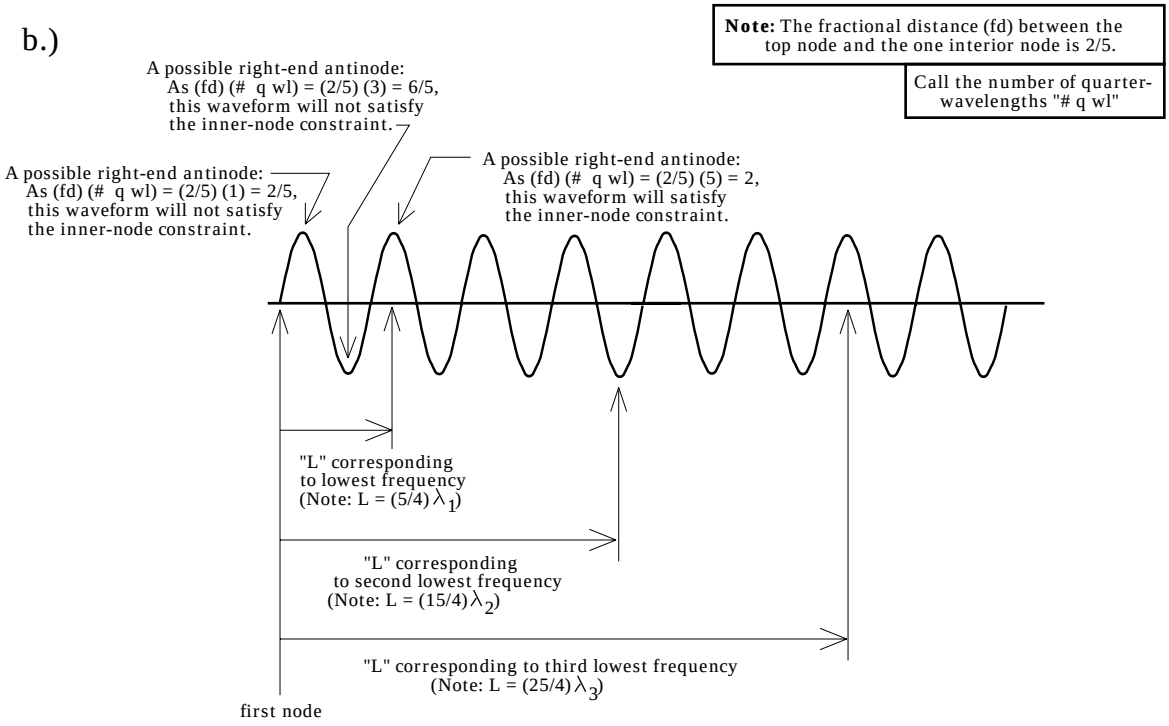
b.) We need a node at the ceiling, an antinode at the free end, and a node *(2/5)L* of the way down from the ceiling. The *sine wave* on the next page depicts the various possibilities.

c.) We need an antinode at the top, a node at the bottom, and a node at *L/3* from the top. The *sine waves* on the next two pages depict the various possibilities (I've pictured the sine wave horizontally for simplicity).

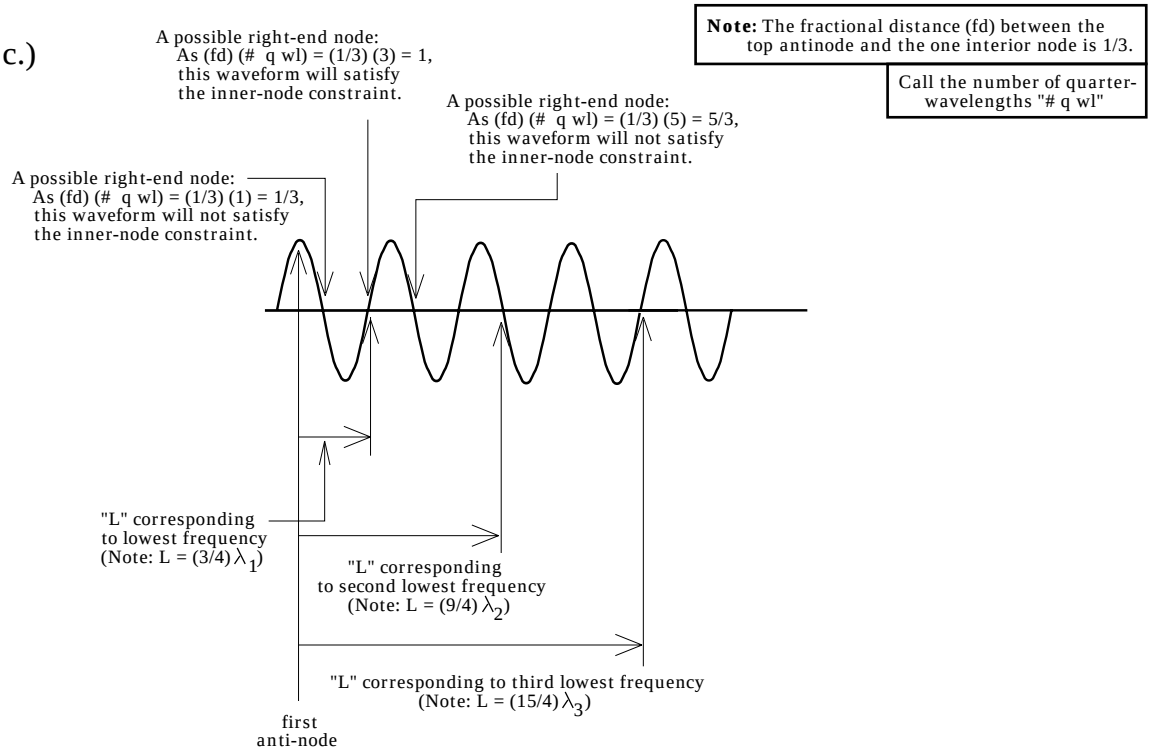
a.)


Note: The fractional distance (fd) between the left-end node and the one interior node is 1/4.
Call the number of quarter-wavelengths "# q wl"
A possible right-end node: As (fd) (# q wl) = (1/4) (4) = 1, this waveform will not satisfy the inner-node constraint.
A possible right-end node: As (fd) (# q wl) = (1/4) (6) = 3/2, this waveform will not satisfy the inner-node constraint.
A possible right-end node: As (fd) (# q wl) = (1/4) (2) = 1/2, this waveform will not satisfy the inner-node constraint.
Fourth possible right-end node: As (fd) (# q wl) = (1/4) (8) = 2, this waveform will satisfy the inner-node constraint.
"L" corresponding to lowest frequency (Note: L = (8/4) λ1)
"L" corresponding to second lowest frequency (Note: L = (16/4) λ2)
"L" corresponding to third lowest frequency (Note: L = (24/4) λ3)
first node

b.)


Note: The fractional distance (fd) between the top node and the one interior node is 2/5.
Call the number of quarter-wavelengths "# q wl"
A possible right-end antinode: As (fd) (# q wl) = (2/5) (3) = 6/5, this waveform will not satisfy the inner-node constraint.
A possible right-end antinode: As (fd) (# q wl) = (2/5) (1) = 2/5, this waveform will not satisfy the inner-node constraint.
A possible right-end antinode: As (fd) (# q wl) = (2/5) (5) = 2, this waveform will satisfy the inner-node constraint.
"L" corresponding to lowest frequency (Note: L = (5/4) λ1)
"L" corresponding to second lowest frequency (Note: L = (15/4) λ2)
"L" corresponding to third lowest frequency (Note: L = (25/4) λ3)
first node

c.)


Note: The fractional distance (fd) between the top antinode and the one interior node is 1/3.
Call the number of quarter-wavelengths "# q wl"
A possible right-end node: As (fd) (# q wl) = (1/3) (3) = 1, this waveform will satisfy the inner-node constraint.
A possible right-end node: As (fd) (# q wl) = (1/3) (5) = 5/3, this waveform will not satisfy the inner-node constraint.
A possible right-end node: As (fd) (# q wl) = (1/3) (1) = 1/3, this waveform will not satisfy the inner-node constraint.
"L" corresponding to lowest frequency (Note: L = (3/4) λ1)
"L" corresponding to second lowest frequency (Note: L = (9/4) λ2)
"L" corresponding to third lowest frequency (Note: L = (15/4) λ3)
first anti-node

AS FOR THE NUMBERS:

a.) From the sketch it can be seen that the *third lowest frequency* corresponds to a wavelength/beam length ratio that leaves:

$$L = 6\lambda_3$$
$$\Rightarrow \quad \lambda_3 = L/6.$$

Using this with $v_{beam} = \lambda\, \nu$, we get:

$$\nu_3 = v_{beam}/\lambda_3$$
$$= v_{beam}/(L/6)$$
$$= 6v_{beam}/L.$$

b.) From the sketch it can be seen that the *third lowest frequency* corresponds to a wavelength/string length ratio that leaves:

$$L = (25/4)\lambda_3$$
$$\Rightarrow \quad \lambda_3 = 4L/25.$$

Using this with $v_{str} = \lambda\ \nu$, we get:

$$\begin{aligned}\nu_3 &= v_{str}/\lambda_3 \\ &= v_{str}/(4L/25) \\ &= 25v_{str}/4L.\end{aligned}$$

c.) From the sketch it can be seen that the *third lowest frequency* corresponds to a wavelength/air-column-length ratio that leaves:

$$L = (15/4)\lambda_3$$
$$\Rightarrow \quad \lambda_3 = 4L/15.$$

Using this with $v_{air} = \lambda\ \nu$, we get:

$$\begin{aligned}\nu_3 &= v_{air}/\lambda_3 \\ &= v_{air}/(4L/15) \\ &= 15v_{air}/4L.\end{aligned}$$

We can go a little further with this problem because we know that the velocity of sound in air is approximately 330 m/s. Putting that in yields:

$$\begin{aligned}\nu_3 &= 15v_{air}/4L \\ &= 15(330\text{ m/s})/4L \\ &= 1237.5/L.\end{aligned}$$

NOTE: The hard part of these problems is finding the appropriate *piece of sine-wave* (relating its wavelength to *L* isn't hard at all). Make sure you understand how to do this. If you are confused, come see me!

PREAMBLE

SOLUTIONS TO MULTIPLE CHOICE TESTS

As was pointed out earlier, the multiple choice tests in this book are designed to be a collective study guide the questions of which are presented in no particular order in an effort to make you think broadly, versus focusing on the material of just one chapter at a time. The hope is that you are not seeing anything in the tests that you haven't already grappled with during the year. As the commentary in the Solutions was designed to remind you of material already mastered, there is nothing wrong with skipping the tests proper and jumping straight to the Solutions themselves. In short, THIS IS FOR REVIEW. Use it well.

MULTIPLE CHOICE SOLUTIONS--MECHANICS

TEST I

1.) The position function for an oscillating body is $x = 20 \sin(.6t - \pi/2)$. At $t = 0$, the magnitude of the body's acceleration is:

a.) 20 m/s^2. [The acceleration function is $a = -\omega^2 A \sin(\omega t + \delta)$. Putting in the numbers sans´ units yields $a = -(.6)^2(20)\sin[(.6(0)-\pi/2)$, or -7.2 m/s^2. The magnitude is 7.2 m/s^2, and this response is false.]

b.) 12 m/s^2. [Nope.]

c.) 7.2 m/s^2. [This is the one.]

d.) None of the above. [Nope.]

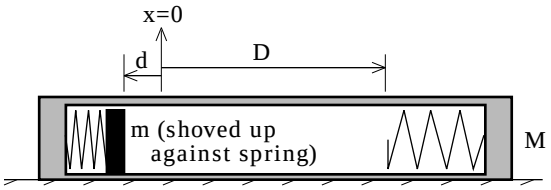
2.) A satellite following an elliptical path around a planet has an angular velocity ω_{far} when at its maximum distance d units from the planet's center. At its closest point, the distance between the satellite and planet's center is $d/3$. The satellite's angular velocity at that closest point is:

a.) $\omega_{far}/3$. [As there are no external torques acting on the system, angular momentum is conserved. There are two ways to do this: the hard way and the short way. Both are educational, so we will try each one. Looking at the body as an object moving in a circular motion, we can write the conservation of angular momentum in angular terms as: $I_{far}\omega_{far} = I_{near}\omega_{near}$, or $(mR_{far}^2)\omega_{far} = (mR_{near}^2)\omega_{near}$. Setting $R_{near} = d/3$ and $R_{far} = d$, we get $\omega_{near} = 9\omega_{far}$. An alternative is to look at the body's translational motion. We can write the conservation of angular momentum as $\boldsymbol{R}_{far}x\boldsymbol{p}_{far} = \boldsymbol{R}_{near}x\boldsymbol{p}_{near}$. At the near and far points, the angle between $\boldsymbol{R}$ and $\boldsymbol{p}$ is 90^o, so the cross products become (p)(R)sin 90^o terms and we can write $(mv_{far})d = (mv_{near})(d/3)$, or $v_{near} = 3v_{far}$. Noting that $v_{far} = d\omega_{far}$ and $v_{near} = (d/3)\omega_{near}$, we can rewrite $v_{near} = 3v_{far}$ as $(d/3)\omega_{near} = 3(d)\omega_{far}$, or $\omega_{near} = 9\omega_{far}$. By the way, what is really interesting to note here is that even though the velocity goes up by a factor of *3*, the angular velocity goes up by a factor of *9*--not something you might have expected.]

b.) ω_{far}. [From above, this is not true.]

c.) $3\omega_{far}$. [Nope.]

d.) $9\omega_{far}$. [Yup.]



3.) Assume M is *not* fixed to the ground and can move frictionlessly. Assume also that the frictional force f is removed from the system (i.e., the slide area across D is now frictionless). The spring is released accelerating m.

a.) Momentum should be conserved in the system because the spring applies a conservative force. [Nope. *Conservative forces* are associated with *energy* considerations. Conservation of momentum is associated with *external impulses* (read this *external forces*) acting on the system. An

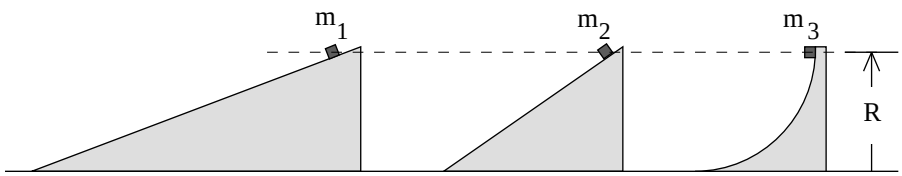
EXTERNAL FORCE *can be* a CONSERVATIVE FORCE (gravity is often external to a system, but it is a conservative force) as well as non-conservative (friction between two bodies within a system can be an internal force, but friction is the quintessential *non-conservative force*). This response is false.]

b.) Just after it leaves the spring, *m's* velocity will be $(kx^2)/(m+m^2/M)$. [In typical A.P. fashion, this question was designed to be tricky. Note that although the general expression for the potential energy function for a spring is $.5kx^2$, the spring's initial displacement <u>in this particular problem</u> is not *x* but rather *d*. In other words, there should be no *x* term in the answer. If you noticed this, you realized that this response must be false. If you didn't, you undoubtedly went through the process of solving. If you solved you most probably proceeded as follows: Both energy and momentum are conserved in this situation. The initial momentum was zero, so we can write: $0 = mv_1 - Mv_2$, or $v_2 = (mv_1/M)$. Energy is conserved, so you might be nudged to write: $.5kx^2 = .5mv_1^2 + .5Mv_2^2$. Plugging in for v_2 yields: $.5kx^2 = .5mv_1^2 + .5M[mv_1/M]^2$. Solving yields: $v_1 = (kx^2)/(m+m^2/M)$. Again, the only thing wrong with this analysis is the fact that the problem does not deal with *x's*, it deals with *d's*.]

c.) When *m* reaches the second spring, it will depress it a distance *d* before coming to rest relative to the spring. Also, at that point *M* will have come to rest. [Energy is conserved in this situation. When *m* is shoved up against the first spring, there is spring potential energy in the system. All of that energy will be reclaimed as potential energy when *m* has fully depressed the second spring. Momentum is conserved in the system and is equal to the system's initial momentum, or zero. That means that when *m* is stationary with respect to *M* (i.e., when *m* has fully depressed the second spring), the velocity of both *m* and *M* must also be zero. This response is true.]

d.) All of the above. [Nope.]

4.) All three masses are released at once. Mass m_2 travels the same distance as does mass m_3. The mass with the largest velocity at the bottom will be:



a.) Mass m_1. [From energy considerations and no friction in the system, the kinetic energy at the bottom of the incline must equal the potential energy at the top. Because the three bodies have different masses, it is tempting to assume that the velocities will be different. Using the math, though, yields another story. Specifically: $mgh = .5mv^2$, or $v = (2gh)^{1/2}$. The velocity has nothing to do with the mass (this shouldn't be terribly surprising, considering the fact that two different masses allowed to fall from the same height should accelerate at the same rate--at *g*-- reaching the ground at the same time). In other words, if all three masses start from rest at the same height in a frictionless situation, they should all have the same velocity at the bottom. This response is false.]

b.) Mass m_2. [Nope.]

c.) Mass m_3. [Nope.]

d.) Masses m_2 and m_3. [Nope.]

e.) They will all have the same velocity at the bottom. [This statement is true.]

5.) Looking at the sketch in #4, friction is introduced into the system such that the coefficient of friction is the same for all three inclines. The mass with the largest velocity at the bottom will be:

a.) Mass m_1. [In this case, the body with the largest velocity will be the one that has had the least amount of energy removed from it by friction. That will be the one that travels the shortest distance AND has the smallest frictional force applied to it. As the frictional force is proportional to the normal force, that means we want the incline that, on average, applies the least normal force on the body (see note at the end of this *Solutions* section for more about this). That combination of requirements is met by m_3. This response is false.]

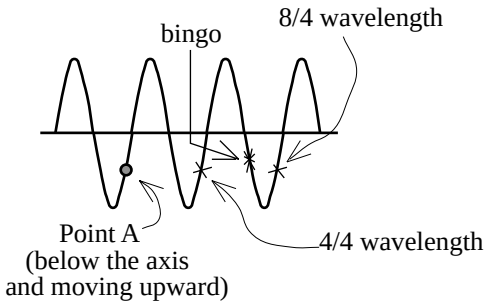
b.) Mass m_2. [Nope.]

c.) Mass m_3. [This statement is true.]

d.) Masses m_2 and m_3. [Nope.]

e.) They will all have the same velocity at the bottom. [Nope.]

6.) *Point A* and *Point B* on a wave are observed to be 7/4 wavelengths apart. At a particular instant, *Point A* is below the axis and moving upward toward equilibrium. At that time, *Point B* is:



a.) Below the axis and moving upward toward equilibrium. [The best way to determine what is actually happening in this situation is to draw a sine wave, place a point below the axis and moving upward (toward equilibrium-- see *Point A* in the sketch), and see what is happening 7/4 wavelengths away. The sketch does this, with the exception that two wavelengths (8/4 wavelengths) has been identified for ease. From there it should be obvious that the 7/4 wavelengths will put a point that is below the axis moving away from equilibrium. From the diagram, this response is false.]

b.) Below the axis and moving downward away from equilibrium. [From above, true.]

c.) Above the axis and moving upward away from equilibrium. [Nope.]

d.) Above the axis and moving downward toward equilibrium. [Nope.]

7.) A single, constant force is applied to a body. After the force does 20 joules of work, the body's velocity has changed from zero to 6 m/s. The work required to change the body's velocity from 6 m/s to 12 m/s is:

a.) 20 joules. [There is really no clever way to do this problem but longhand. Using the information you know, you can write: $W_{net} = KE_2 - KE_1 = .5m(v_2^2 - v_1^2)$, or $(20\ joules) = .5m(6^2 - 0^2)$, or $m = 1.111\ kg$. Using this information to analyze the second situation, we get: $W_{net} = KE_2 - KE_1 = .5(1.111\ kg)(12^2 - 6^2) = 60\ joules$. This statement is false.]

b.) 40 joules. [From above, this statement is false.]

c.) 60 joules. [From above, this statement is true.]

d.) 400 joules. [From above, this statement is false.]

8.) A man on a unicycle pedaling northward begins to slow. The direction of the cycle's angular acceleration vector will be:

a.) North. [By definition, the direction of any angular acceleration vector will always coincide with the direction of the axis about which the rotation occurs. The axis of rotation for a bike moving northward will be east/west. But which is it, east or west? Think about the direction of the angular velocity. If you make the fingers of your right hand curl to mimic the wheel's motion (i.e., curl the fingers to follow the rotation of the wheel), your right-hand thumb will point west. That is the direction of the angular velocity of the wheel. If the angular acceleration's direction were also to the west, the sign of the angular velocity and angular acceleration would be the same and the bike would be speeding up. But the bike is slowing down, so the direction of the angular acceleration must be opposite west, or east. This response is false.]

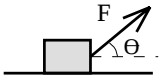
b.) South. [Nope.]

c.) East. [Yup.]

d.) West. [Nope.]

e.) Upward, relative to the ground. [Nope.]

9.) A mass m has a force F applied to it as shown to the right:



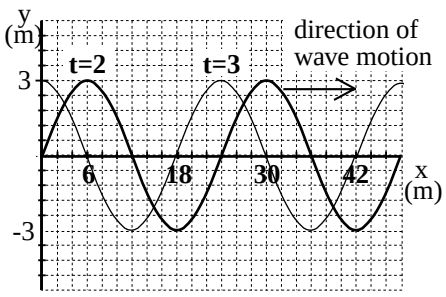
a.) The normal force on the block will be mg. [There is a component of F in the normal direction. N.S.L. yields $N - mg + F\sin\theta = ma_y = 0$. This statement is false.]

b.) The acceleration of the block will be a function of F only. [The *component of F* that accelerates the block is a function of the angle θ. This statement is false.]

c.) If there was kinetic friction acting on the block, it would be directed toward the left. [This is tricky. Because F is to the right, one might assume that the motion was to the right. *If* that were the case, this statement would be true. Unfortunately, nothing was said about the direction of motion (just because F is to the right doesn't mean the body is moving to the right--it could be slowing down while moving to the left). Because the statement could be false, it can-not be true.]

d.) None of the above. [Yup.]

10.) A graph of a traveling wave as seen at $t = 2$ *seconds* and $t = 3$ *seconds* is shown to the right. The period of the wave is:



a.) (3/4) seconds/cycle. [The period tells you how long it takes for a wave to pass by. The wave we are dealing with here moves 18 meters in 1 second (look at the graph and the distance between the position at $t = 2$ *seconds* and $t = 3$ *seconds*) for a wave velocity of 18 m/s. The wavelength is 24 meters (again, look at the graph--the distance between two peaks is 24 meters). Using the relationship $v = \lambda\nu$, we find that the frequency is 3/4 cycles/second. The inverse of that (i.e., the period) is 4/3 seconds per cycle. This response is false.]

b.) (4/3) second/cycle. [This is the one.]

c.) 4 seconds/cycle. [Nope]

d.) Can't tell with the information given. [Nope.]

e.) None of the above. [Nope.]

11.) A mass m sits on a frictionless incline plane of angle θ.

a.) The acceleration of m will be g. [It isn't gravity that accelerates the body, it's the *component* of *gravity along the line of the incline* that accelerates the body. This statement is false.]

b.) The acceleration of m will be dependent upon the size of the mass. [If you do the math, the mass terms cancel out. If you simply think about the interplay between inertial mass and gravitational mass, it should be obvious that the amount of mass will have no effect on the outcome. This statement is false.]

c.) The acceleration of m will be dependent upon θ. [The component of gravity that accelerates the mass will be a function of the incline's angle. This statement is true.]

d.) All of the above. [Obviously not true.]

12.) It is known that a single, conservative force that runs along the *x axis* does -25 joules of work on a body moving in the *negative x-direction*.

a.) The force is in the negative direction, and the body's change of potential energy is -25 joules. [Negative work implies that energy is being pulled *out of* the system and the body is slowing down. If the body is moving in the negative direction and additionally slowing down, the force must be in the positive direction. From that observation, this statement is false.]

b.) The force is in the negative direction, and the body's change of potential energy is 25 joules. [This is false for the same reason stated in *Response a*.]

c.) The force is in the positive direction, and the body's change of potential energy is -25 joules. [This has the correct force direction. As for the potential energy change, we know that $W_{consforce} = -\Delta U$. In this case, then, -25 joules = $-\Delta U$, or the potential energy change is +25 joules. Wrong sign for this response. This statement is false.]

d.) The force is in the positive direction, and the body's change of potential energy is 25 joules. [This statement is true, given what has been said in *Responses a* and *c*.]

e.) None of the responses make any sense as gravity is not oriented along the *x axis*. [No, gravity is NOT the only possible conservative force with a potential energy function associated with it. This statement is false.]

13.) A satellite following an elliptical path around a planet has a velocity v_{far} when at its maximum distance d units from the planet's center. At its closest point, the distance between the satellite and planet's center is $d/3$. The satellite's velocity at that closest point is:

a.) $v_{far}/3$. [As there are no external torques acting in the system, we can use conservation of angular momentum here. There are two ways to do this: the hard way and the short way. Each is educational, so we will try both. We will write the conservation of angular momentum in angular terms as: $I_{far}\omega_{far} = I_{near}\omega_{near}$, or $(mR_{far}^2)\omega_{far} = (mR_{near}^2)\omega_{near}$. Setting $R_{near} = d/3$ and $R_{far} = d$, we get $\omega_{near} = 9\omega_{far}$. Noting that $\omega_{far} = v_{far}/d$ and $\omega_{near} = v_{near}/(d/3)$, we can rewrite $\omega_{near} = 9\omega_{far}$ as $v_{near}/(d/3) = 9[v_{far}/d]$, or $v_{near} = 3v_{far}$. This is the hard way. Looking at the body as a point mass, we can write the conservation of angular momentum as $\boldsymbol{R}_{far}x\boldsymbol{p}_{far} = \boldsymbol{R}_{near}x\boldsymbol{p}_{near}$. At the near and far points, the angle between $\boldsymbol{R}$ and $\boldsymbol{p}$ is 90^o, so the cross products take the form (p)(R)sin 90^o. The angular momentum expression becomes $(mv_{far})d = (mv_{near})(d/3)$, or $v_{near} = 3v_{far}$. As a side note, what is really interesting to note here is that even though the velocity goes up by a factor of *3*, the angular velocity goes up by a factor of *9*--not something you might have expected.]

b.) v_{far}. [From above, this is not true.]

c.) $3v_{far}$. [This is the one.]

d.) $9v_{far}$. [Nope.]

14.) A turntable has an angular velocity of .8 rad/sec. It is accelerated at a rate of -4 rad/sec^2.

a.) After 5 seconds, its angular velocity is -19.2 rad/sec, and it will have rotated through a net angular displacement of -46 radians. [For the first part of the response, rotational kinematics maintains: $\alpha = \frac{\omega_2 - \omega_1}{t} \Rightarrow \omega_2 = \omega_1 + \alpha t = (.8\ rad/sec) + (-4\ rad/sec^2)(5\ sec) = -19.2\ radians$. The first part is OK. For the second part of the response, kinematics again maintains: $\Delta\theta = \omega_1 t + .5\alpha t^2 = (.8\ rad/sec)(5\ sec) + .5(-4\ rad/sec^2)(5\ sec)^2 = 46\ radians$. This response is true, but are there others?]

b.) It will take half as long to go from .8 rad/sec to .4 rad/sec as it takes to go from .8 rad/sec to 0 rad/sec. [Think about the relationship between *translational* velocity and time. With constant, negative acceleration, doubling the time will double the velocity change. Keeping to our rotation/translation parallel, the same must be true of the relationship between time and *angular* velocity. If you don't believe, do the problem: $\alpha = (\omega_2 - \omega_1)/t, \Rightarrow t = (\omega_2 - \omega_1)/\alpha$. Going from .8 rad/sec to .4 rad/sec yields $t = (.4\ rad/sec - .8\ rad/sec)/(-4\ rad/sec^2) = .1\ second$. Going from .8 rad/sec to 0 rad/sec yields $t = (0\ rad/sec - .8\ rad/sec)/(-4\ rad/sec^2) = .2\ seconds$. The time is doubled and this response is true.]

c.) It will have moved through a greater net angular displacement after the first .2 seconds than after the first .4 seconds. [The body is slowing down (as was the case with translational systems, if the signs of the velocity and acceleration terms differ, the body must be slowing down). According to the calculation in *Response b*, the time it takes for the body to slow to zero will be *.2 seconds*. During that period, the body will have traveled some net angular distance $\Delta\theta$. In .4 seconds, the body will have rotated to a stop after covering a net angular distance $\Delta\theta$ in the first .2 seconds, then will have retraced its steps back to its original angular position during the second .2 seconds. In other words, the body's net angular displacement after .4 seconds will be zero, and this statement is true.]

d.) Both *a* and *b*. [This is false.]

e.) All of the above. [This is the one.]

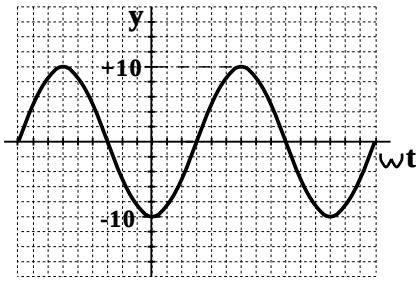
15.) A resting wheel has a constant torque applied to it. After the first 10 seconds, the wheel has turned through an angular displacement of 1.2 radians. After the first 20 seconds, the angular displacement will be:

a.) 2.4 radians. [The relationship we want to use here is $\Delta\theta = \omega_1 t + .5\alpha t^2$. With the initial angular velocity equal to zero, the angular displacement is a function of t^2. This means that doubling the time should quadruple the angular displacement. 1.2 radians quadrupled is 4.8 radians. This response is false.]

b.) 4.8 radians. [This is the one.]

c.) 7.2 radians. [Nope.]

d.) None of the above. [Nope.]



16.) A graph of the position function for a body oscillating with a frequency of 1/(4π) radians per second in *simple harmonic motion* is shown to the right. The equation that best describes the motion is:

a.) y = 20 sin (.5t - π/2). [The amplitude of this graph is 10 meters, which eliminates *Responses a* and *d*.]

b.) y = 10 sin ([1/(4π)]t - 5). [This has the correct amplitude. The angular frequency will be 2π times the frequency, or (2π)(1/(4π) = .5 cycles/second. This eliminates *Response b*.]

c.) y = 10 sin (.5t + 3π/2). [This has the correct amplitude and angular frequency. The easiest way to determine the phase shift in this case is to eyeball it. If you shift a sine wave by π/2 radians to the right, you will have a graph defined by $x = A$ at $t = 0$. If you shift a sine wave by π radians to the right, you will have a graph defined by $x = 0$ going negative at $t = 0$. If you shift a sine wave by 3π/2 radians to the right, (this is the phase shift for the function we are testing), you will have a graph defined by $x = -A$ at $t = 0$. In fact, that is exactly what our graph is doing at $t = 0$, so this function must be O.K.]

d.) y = 20 sin ([1/(4π)]t - π/2). [Nope.]

e.) None of the above. [Nope.]

17.) A sound wave moving at 330 m/s has a frequency of 220 Hz. Its wavelength is:

a.) 2/3 meter. [Using the relationship $v = \lambda\nu$, the wavelength is found to be 1.5 meters. The incorrect value of this response is the inverse of that value.]

b.) 1.5 meters. [This is the one.]

c.) 66,000 meters. [Not likely, but quite a number (it is the product of 220 and 330). This is a false response.]

d.) None of the above. [Nope.]

18.) A mass m_1 sits on top of a second mass m_2 which sits on a frictionless surface. The coefficient of static friction between the two is .7. A force F is applied to the top mass (m_1).

a.) The maximum force F that m_1 can experience without slipping over m_2 is $\mu_s m_1 g$. [This is tricky. The normal force acting on m_1 is $m_1 g$, so the static frictional force will be $\mu_s m_1 g$. This means that if m_2 had been fixed to the ground (i.e., held stationary), the largest non-slip value F could attain would be $\mu_s m_1 g$. Unfortunately, that's not the situation--m_2 isn't fixed to the ground, it resides on a frictionless surface. Therefore, this statement is false.]

b.) The maximum force m_1 can experience without slipping over m_2 is $2\mu_s m_1 g$. [Summing the forces on m_1 yields $F - \mu_s m_1 g = m_1 a$, or $F = m_1 a + \mu_s m_1 g$. To get a, noting that "no slippage" means that m_1's acceleration equals m_2's acceleration, we can use N.S.L. on m_2. Doing so yields the following: $\mu_s m_1 g = m_2 a$, or $a = (\mu_s m_1 g)/m_2$. Substituting a into our first equations yields: $F = m_1[\mu_s m_1 g)/m_2] + \mu_s m_1 g = 2\mu_s m_1 g$. This statement is true . . . but are there other correct responses?]

c.) The maximum force m_2 can experience without slipping relative to m_1 is $\mu_s m_1 g$. [The only force acting on m_2 is friction. The maximum static frictional force between m_1 and m_2 is, indeed, $\mu_s m_1 g$. This response is true.]

d.) Both *b* and *c*. [Yup.]

e.) None of the above. [Nope.]

19.) A spinning skater's angular speed is 2 rad/sec when her kinetic energy is 20 joules. If the skater's moment of inertia changes by half during the spin, what is her new angular speed?

a.) .5 rad/sec. [Assuming there are no external torques being applied to the system, which is the case with a spinning body like this, conservation of angular momentum maintains that $I\omega$ must remain the same during the spin no matter how the moment of inertia might change. With a constant $I\omega$, halving I would necessitate the doubling of ω. In short, the solution should be 4 rad/sec. This response is false.]

b.) 2 rad/sec. [Nope.]

c.) $(8)^{1/2}$ rad/sec. [This is an example of one of the ways that A.P. tests can be tricky. Energy was placed in this problem specifically to nudge you toward using conservation of energy in a situation where it is not appropriate (energy is *not* conserved here). The kinetic energy information can allow you to determine the moment of inertia for the system, but as *Response a* points out, you don't need that information to think through the problem. If you went for the deception and used conservation of energy on the problem, you undoubtedly came up with an angular velocity of $(8)^{1/2}$ rad/sec. That response is false.]

d.) 4 rad/sec. [This is the one.]

20.) A mass m is attached to a string of length L that is, itself, attached to the ceiling. The mass is drawn to the side and released so that it swings back and forth. At the bottom of its arc, the mass's velocity is v_o.

a.) The acceleration of the mass when at the bottom of the arc is *zero*. [If the acceleration were *zero*, the mass would continue moving in a straight line with velocity v_o. As that isn't what happens, this statement is false.]

b.) The acceleration of the body when at the bottom of the arc is a function of the body's mass. [The relationship between the inertial mass and gravitational mass of a body holds here. The mass of the object makes no difference in determining the body's acceleration. This statement is false.]

c.) The force on the mass when at the bottom of the arc is mg. [In terms of force variables, the force at the bottom of the arc will be upward and equal to $T - mg$. This is also equal to the product of the mass and acceleration, or mv_o^2/L. This statement is false.]

d.) The net force on the mass when at the bottom of the arc is $(mv_o^2/L)(j)$. [There are no forces in the direction of motion, so the mass will free wheel through the bottom. There must be a force perpendicular to the motion and pointing toward the center of the arc, though, or the body would not *move* in a curved path. The magnitude of the associated center-seeking acceleration has to be v_o^2/L, where L in this case is both the length of the string and the radius of the arc, and the magnitude of that center-seeking <u>force</u> is mv_o^2/L. This statement is true.]

21.) A flatbed truck coasts forward frictionlessly along on a flat surface. The surface between the truck bed and a large, massive crate sitting on the bed is *frictional*. At *t = 0*, you begin to push the crate toward the rear of the bed. At *t = 1 second*, the crate leaves the truck.

a.) Between *t = 0* and *t = 1 second*, the momentum of the truck is conserved. [If you take *the system* to be the truck alone, there will be two external forces acting on it. Specifically, *you* will be applying force to the bed as you traction off of it to push the crate, and there will be *friction* acting between the crate and the truck. As these external forces exist, the truck's momentum alone cannot be conserved and this response is false.]

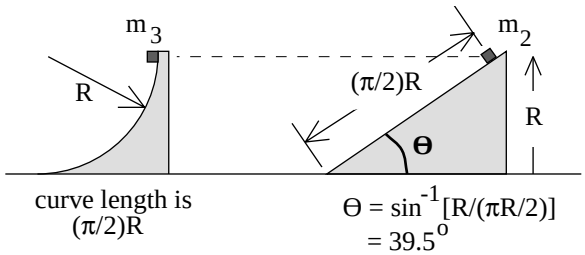
b.) Between *t = 0* and *t = 1 second*, the momentum of the truck/crate system is conserved, and the truck will have slowed by the time the crate leaves it. [Friction between the truck and crate is an internal force (Newton's Third Law--the frictional force the crate applies to the truck is equal and opposite the frictional force the truck applies to the crate), and the force you apply to the crate is equal and opposite the force your feet apply to the truck (without friction between your feet and the truck bed, you wouldn't be able to push at all). That means you are effectively a part of the system, whether the problem states this or not. As there are no external forces acting on the truck/crate system (at least not in the horizontal), momentum must be conserved in the horizontal and the first part of this response is true. What about the second part? Relative to the ground, the crate has to be moving more slowly than the truck or the two will continue moving with the same speed and the crate won't traverse over the bed's surface. What is slowing the crate, relative to the ground? You are pushing! But as you push the crate toward the rear of the truck, you must also be pushing the truck forward (Newton's Third Law and the presence of friction). As such, the speed of the truck will increase as you push, and this response is false.]

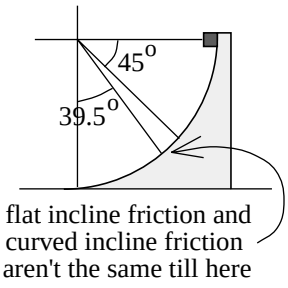
c.) Between *t = 0* and *t = 1 second*, the momentum of the truck/crate system is conserved, and the truck will have sped up by the time the crate leaves it. [It looks like this is the one.]

d.) Between *t = 0* and *t = 1 second*, the momentum of the truck/crate system is not conserved as *you* are applying an external force to the crate. [Nope.]

e.) None of the above. [Nope.]

-----x-----





Additional NOTE about Problem 5: If you include friction, the energy loss down a flat incline is *more than* the energy loss down a curved incline, assuming the total distance traveled in both cases is the same. This may seem counterintuitive, so I'm taking a moment to explain why. As you know, the amount of work that friction does on a body moving over a frictional surface depends solely upon the size of the frictional force and the distance over which that force acts. The frictional force, on the other hand, depends upon the coefficient of friction (a constant in this case) and the magnitude of the normal force. For the flat incline, the normal force is constant so the frictional force f_{flat} is constant. For our curved incline the *coefficient of friction* is the same as in the flat incline case, but the frictional force at the top of the curved incline is small because the normal force at

the top is small. By the same token, the frictional force at the bottom of the curve is large because the normal force at the bottom is large. If, for half of the motion, f_{curve} was less than f_{flat} and, for the other half of the motion, f_{curve} was more than f_{flat}, then the net effect would be the same as that observed with the flat incline. The problem is that for a flat incline of height R to have the same length as a quarter circle of radius R, the angle of the incline has to be 39.5^{o}. The normal force on the curved incline does not become less than the normal force on the flat incline until the body has reached the 39.5^{o} point. The lesser amount of work done before that point is not countered by an appropriately greater amount of work below the 39.5^{o} mark, and the net effect does not match the flat incline. And yes, this is very obscure.

MULTIPLE CHOICE SOLUTIONS--MECHANICS

TEST II

Note:
--The following information pertains to Problems 1 through 5: Projectiles A, B, C, and D are fired at the same time from a height h meters above the ground. With the exception of Projectile D, which is dropped from rest, all the projectiles (i.e., Projectiles A, B, and C) have the same muzzle velocity v_o, (though each is fired at a different angle--see the sketches below and note that the angle defined as θ is the same in all cases). It takes t_1 seconds for Projectile A to get to the top of its flight. It takes t_2 seconds for Projectile D to reach the ground.

Projectile A Projectile B Projectile C Projectile D

1.) At time t_1, Projectile A's:

a.) Velocity will be perpendicular to its acceleration. [At t_1, the projectile is at the top of its arc. At that point, its velocity vector is comprised of its x-component (that component stays the same throughout the motion as there are no x-direction forces and, hence, no x-direction acceleration acting to change it) but no y-direction velocity (it's at the top of its arc and, hence, will go no farther upward). That means the velocity direction at the top is in the horizontal. The acceleration vector throughout the motion is in the y-direction (i.e., gravity pointing down), so the velocity vector and the acceleration vectors are perpendicular to one another. This response is true, but are there others?]

b.) Velocity will be $v_o \cos \theta$. [At the top, the only velocity component that is non-zero is the horizontal component. As the x-component of the velocity will be constant throughout the motion, and as that component does, indeed, equal $v_o \cos \theta$, this response is true. Are there others?]

c.) X-component of acceleration will be twice what it was at t = 0. [If this were an A.P. test and you were thinking clearly, you would never have gotten to this question as you would have already determined that both a and b were true and, hence, d must be the answer. Nevertheless, this is an interesting and tricky question. The x-component of the acceleration is zero (see Response a for more discussion). Twice zero is still zero, so this statement is true.]

d.) All of the above responses are true. [Given all that has been said above, this is the one.]

2.) Projectile A's:

a.) Acceleration is greater on the way up than on the way down. [The only acceleration acting is gravity in the y-direction. It is a constant, so this statement is false.]

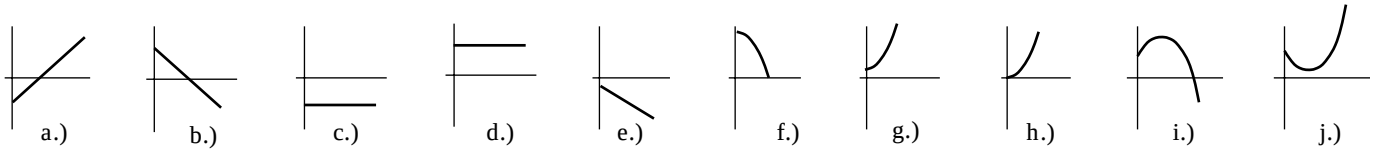
b.) Velocity changes at the same rate going up as going down. [This is the same as saying that the acceleration of the body is a constant, which it is. This statement is true.]

c.) Y-component acceleration sign is the same as its y-component velocity sign while going up. [The y-component of acceleration is that of gravity. Its sign is negative. The y-component of the velocity going upward is in the direction of motion, or positive. Clearly the two are not the same.]

d.) Velocity, when at h going upward, will be the same as its velocity when at h coming down. [This is tricky. Assuming there is no friction, the velocity magnitude when the body is at h going up will be the same as when going down, but the directions will be different. As we are dealing with vectors, this difference in direction makes the velocities different.]

e.) Both b and d. [Nope, but this was included to be tricky in case you hadn't realized that d was not a legitimate statement.]

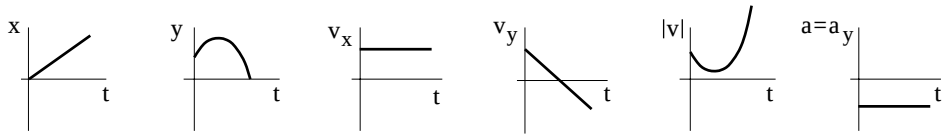
3.) Consider the graphs shown below.



Projectile A's:

a.) Y-component of Position vs. Time graph looks like graph a.
b.) X-component of Position vs. Time graph looks like graph j.
c.) Y-component of Velocity vs. Time graph looks like graph b.
d.) X-component of Velocity vs. Time graph looks like graph c.
e.) Y-component of Acceleration vs. Time graph looks like graph d.

[Commentary: A graph for each of the major parameters for this situation is shown below. This is something you should have been able to both visualize and sketch on your own. If you think you wouldn't have been able to do that, use the graphs provided as a stimulus to do the visualization part.]



4.) The time t_2:

a.) Depends only on h and constant(s). [Using $x_2 = x_1 + v_1t + .5at^2$ with $a = -g$, $v_1 = 0$, and $x_2 - x_1 = 0 - (h) = -h$, we get the relationship $-h = .5(-g)t^2$. This selection is evidently true, but are there other true statements?]

b.) Is the same time it takes Projectile B to hit the ground. [The time it takes to hit the ground is a y-motion related question. As the initial velocity in the y-direction for both cases is the same (it's zero), and as the gravitational acceleration is the same in both cases, the two projectiles should hit the ground in the same amount of time. This statement is true. Other truths?]

c.) Is more than the time it takes Projectile C to hit the ground, but less than the time it takes Projectile A to hit. [Because Projectile C had a downward initial velocity in the y-direction, it will take less time to hit the ground than does Projectile D which had no initial velocity in the y-

direction. And as Projectile A had an upward y-component of its velocity, it will take more time to reach the ground. This statement is true.]

d.) Both a and b. [Nope.]

e.) All of the above except d. [This is the one.]

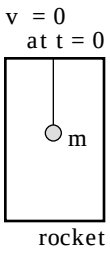
5.) If h were doubled, Projectile D's:

a.) Time to touch down will double. [Using $x_2 = x_1 + v_1t + .5at^2$ with $a = -g$, $v_1 = 0$, and $x_2 - x_1 = 0 - (h) = -h$, we get the relationship $-h = .5(-g)t^2$. This means that $t = (2h/g)^{1/2}$. If h is doubled, t goes up by a factor of $(2)^{1/2}$, not by a factor of 2. This response is false.]

b.) Velocity just before touch down will double. [Using $v_2 = v_1 + at$ with v_2 being the velocity just before hitting the ground, $v_1 = 0$, and $a = -g$, we get $v_2 = -gt$. We have already determined that doubling h does not mean t doubles, so this statement is false.]

c.) Acceleration just before touch down will double. [Acceleration in these cases is always constant in both the x and y-direction. False.]

d.) None of the above. [This must be the one.]



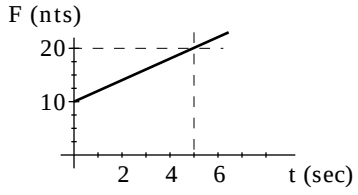
6.) A mass m is attached to a string hung from the ceiling of a rocket. The rocket accelerates from rest upward with an acceleration equal to a_1. Just an instant after t = 0:

a.) The tension in the line is mg. [If the rope were stationary, or if the rocket were moving with constant velocity (i.e., no acceleration) either up or down, this statement would be true. But in this case, summing the force and using N.S.L. on m yields $T - mg = ma$. T does not equal mg. Response false.]

b.) The tension in the line is $m(g + a_1)$. [From above, this is true.]

c.) The tension in the line is $m(g - a_1)$. [Nope.]

d.) None of the above. [Nope.]



7.) A 2.5 kg mass moves one-dimensionally over a frictionless surface. A single, external force $F = 2t + 10$ is applied (see graph to the right). The body's velocity changes from 0 m/s to 30 m/s during the first 5 seconds:

Important Note: When taking the multiple choice part of an AP test, it is important to do the easy problems immediately, leaving the doable but more difficult problems for later. This problem is neither. If I were taking this as a test, I'd skip it!

a.) The net work done by the force is 75 joules (the area under the line). [The area under a force vs time graph has the units of newton·seconds. The units of work are newton·meters, or joules. Just from a consideration of the units, this statement must be false.]

b.) The net work done by the force is 1125 joules. [To begin with, as both the force and direction of motion are positive (a body going from 0 to 30 m/s is moving in the positive direction), the work must be positive. That excludes Responses c and e below. As far as the numerical answer to the work question goes, there is the hard (but educational) way and the easy way. First, the hard way.

We know that $W = \int F \bullet dr$, that F = 2t + 10, and that as F = ma, a = F/m = (2t+10)/m. Knowing a as a function of time, we can also use Calculus to derive an expression for the velocity as a function of time (in this case, noting that the initial velocity is zero, that expression is $v = (t^2 + 10t)/m$).]

Note: Why might we need that last bit of information? F is a function of time, which means we must express dr as a function of time before we can deal with the integral. Using the Chain Rule, that relationship is dr = (dr/dt)dt (notice how the dt's could cancel leaving dr if one were interested in doing such a thing), or dr = v dt. This is why we need v(t).

With all of this, we can write:

$$
\begin{aligned}
W &= \int F \bullet dr \\
&= \int (F)(dr)\cos 0^o \\
&= \int (F)\left(\frac{dr}{dt}dt\right)(1) \\
&= \int (F)((v)dt) \\
&= \int (2t+10)\left(\frac{t^2+10t}{m}\right)dt \\
&= \frac{1}{m}\int_{t=0}^{5}(2t^3+20t^2+10t^2+100t)dt \\
&= \frac{1}{m}\int_{t=0}^{5}(2t^3+30t^2+100t)dt \\
&= \frac{1}{2.5}\left(\frac{1}{2}t^4+10t^3+50t^2\right)_{t=0}^{t=5} \\
&= 1125 \text{ joules.}
\end{aligned}
$$

Now, the easy way. We know from the Work/Energy theorem that $W_{net} = KE_2 - KE_1$. That means that $W_{net} = (.5)(2.5)(30)^2 - (.5)(2.5)(0)^2 = 1125$ joules. In either case, this response is true.]

c.) The net work done by the force is -1125 joules. [Wrong sign.]

d.) The net work done by the force is 2813 joules. [Got this by forgetting to divide by the mass m. This response is false.]

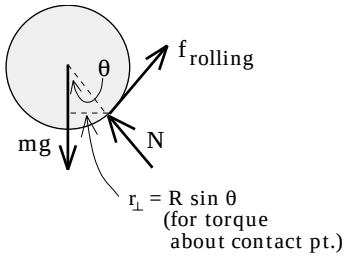
e.) The net work done by the force is -2813 joules. [Nope.]

8.) A 20 centimeter diameter ball whose moment of inertia is $(2/5)mR^2$ accelerates down a 30^o incline plane. After traveling .2 meters down the incline, its angular velocity is 14 rad/sec. Assume the magnitude of g is 10 m/s^2.

a.) The velocity of the ball's center of mass at the .2 meter point is 140 m/s. [The relationship between the velocity of a ball's center of mass and its angular velocity at some point in its motion is $v_{cm} = R\omega$, where R is the number of meters per radian there are associated with the body's motion. Putting in the numbers yields: v_{cm} = (.1 m/rad)(14 rad/sec) = 1.4 m/s. If you correctly halved the diameter to get the radius but didn't convert from

centimeters to meters, you got a numerical value for the velocity of 140. Unfortunately, its units are cm/s instead of m/s, so this response is false.]

b.) After traveling an additional .2 meters, the ball's angular velocity will be 28 rad/sec. [A constant angular acceleration situation, which this is, means that the angular velocity will change linearly in time. That is, the body will pick up twice the angular velocity in twice the time (not twice the distance). As the second .2 meter stretch of the trip will take considerably less time than did the first .2 meter leg, the angular velocity will not double during the second leg, and this response is false.]



c.) The ball's angular acceleration is 35.7 rad/sec^2. [Before summing up the torques acting about the contact point between the ball and the incline, we need the moment of inertia about the contact point. Using the parallel axis theorem, we can write: $I_{contact} = I_{cm} + md^2$, where d is the distance between the two parallel axes (in this case, that distance is the radius R of the ball). Doing the math yields $I_{contact} = (7/5)mR^2$. Summing the torques about the contact point yields:

$mg(R\sin\theta) = I_{contact}\alpha = [(7/5)mR^2]\alpha$, or $\alpha = \dfrac{5g\sin\theta}{7R} = \dfrac{5(10m/s^2)(\sin 30^o)}{7(.1m)}$. This equals 35.7 rad/sec^2, which means that this response is true.]

d.) Both a and c are true statements. [As Response a is incorrect, this is false.]

9.) A .5 kg ball moving at v = (20 m/s)i strikes a massive wall over a .02 second period and, reversing itself, leaves with a velocity magnitude equal to 16 m/s.

a.) The impulse applied to the ball during the collision is (2 nt·sec)i, and the magnitude of the average applied force is 100 nts. [The body's change of momentum vector and the impulse vector are equal to one another. If the incoming velocity is positive, the outgoing velocity will be negative. If you were NOT careful about the sign of each velocity, you might have taken the difference between .5(16) - (.5)(20) and ended up with 2 nt·sec. This is wrong. With the signs taken into account, we can write: impulse = $mv_{after} - mv_{before}$ = (.5 kg)(-16 m/s) - (.5 kg)(20 m/s) = -18 nt·sec. Although the i unit vector should technically be included, the fact that the motion is one dimensional means that only the sign of the impulse is important. This response is false.]

b.) The impulse applied to the ball during the collision is (18 nt·sec)i, and the magnitude of the average applied force is 900 nts. [Nope. Wrong sign for the impulse.]

c.) The impulse applied to the ball during the collision is (-2 nt·sec)i, and the magnitude of the average applied force is 100 nts. [Nope.]

d.) The impulse applied to the ball during the collision is (-18 nt·sec)i, and the magnitude of the average applied force is 900 nts. [From the calculation in Response a, Δp = -18 nt·sec. With the collision occurring over .02 seconds and the expression $F_{avg}\Delta t = \Delta p$, we can write that F_{avg} (.02 sec) = (-18 nt.·sec), or F_{avg} = 900 nts. In short, this is the one.]

e.) None of the above. [Nope.]

10.) A satellite 3R units from the surface of a planet of radius R has potential energy U_1. Its gravitational potential energy 6R units from the surface (call this potential energy U_2) will be:

a.) Larger and equal to $(1/2)U_1$. [This is tricky if you over think the problem. In general, a body's gravitational potential energy gets smaller the closer you get to a field producing planet. This makes sense even though the gravitational potential energy function associated with $F_{grav} = -Gm_1m_2/r^2$ is inherently negative (at infinity, U = 0; in closer, it is <u>negative</u> Gm_1m/r--the biggest the potential energy ever gets is zero!). That means that the farther you are out from a planet, the larger the potential energy value will be (this eliminates Response d). As for algebraic values: The distance between the center of the two bodies is 4R (not 3R--you want the center to center distance, not the center to surface distance). Using our gravitational potential energy expression, we get $U_1 = -Gm_sm_p/(4R)$. At the second position, $U_2 = -Gm_sm_p/(7R)$. Writing U_2 in terms of U_1, we get $U_2 = U_1(4/7)$, which means that this option/response is not true.]

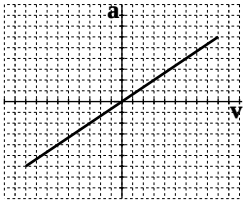
b.) Larger and equal to $2U_1$. [From above, this option is false.]

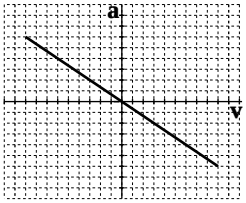
c.) Larger and equal to $(4/7)U_1$. [This is the one.]

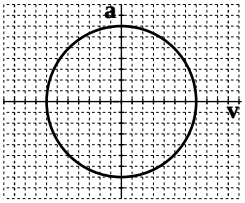
d.) Smaller and equal to $(4/7)U_1$. [From above, this option is false.]

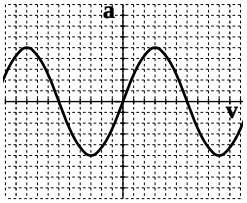
e.) None of the above. [Nope.]

11.) A graph of the acceleration vs. velocity of a body oscillating in simple harmonic motion looks like:

a.) 

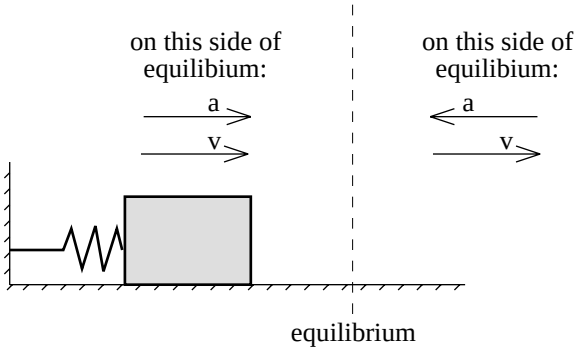
b.) 

c.) 

d.) 

e.) None of these.

[Commentary: If you will remember, the position function for this kind of motion is characterized by a sine wave whereas the velocity function is a cosine and the acceleration function is a negative sine. Put a little differently, the velocity and acceleration are out of phase with one another by a quarter of a cycle. What does this mean? Consider the sketch to the right. If the body's velocity at the point shown is, say, +2 m/s (note that the body is picking up speed because the acceleration is in the direction of motion), there will be a point on the other side of x = 0 where the body's velocity will again be +2 m/s (in that case, the body will be slowing down because the acceleration will be opposite the direction of motion). That means there are two accelerations to be assigned to +2 m/s. How does one graph such a function? With a circle! In short, graph c does the trick.]



12.) A 30 newton force acting at 30^o below the horizontal acts over a 12 meter distance. The work done by F:

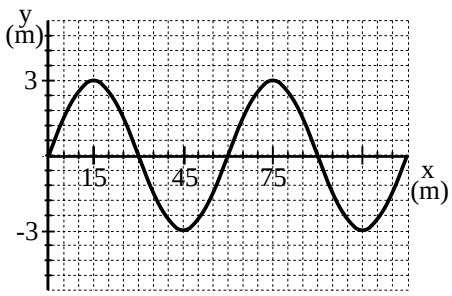
a.) Is -360 joules. [In the first place, you don't know if the direction of motion is the same as the direction of the force, which means you don't know whether the angle is 30^o or 150^o. If you assume the displacement and force are in the same direction (something you haven't the right to do), the work done by the force would be $W = Fd \cos 30^o = (30 \text{ nt})(12 \text{ m}) \cos 30^o = +312$ joules. The answer in this response was obviously obtained by making the displacement/force assumption coupled with forgetting to multiply by the cosine term in the dot product. In short, this answer is false for all sorts of reasons.]

b.) Is +360 joules. [For the same reasons outlined in Response a, this statement is false.]

c.) Would be the same if the angle had been above the horizontal. [This response makes no assumptions about the displacement and force direction. It simply observes that if the angle is the same, it doesn't matter whether the force vector is above or below the line of the displacement. The amount of work will be the same. This statement is true.]

d.) Both a and c. [Nope.]

e.) Both b and c. [Nope.]



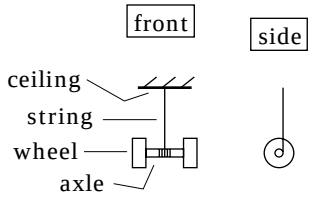
13.) At a particular point in time during the motion of a wave, the displacement vs. position function is as graphed to the right.

a.) The amplitude of the wave is 3 meters, its wavelength is 60 meters, and its frequency is .5 Hz. [From inspection, the amplitude of this wave is 3 meters. The wavelength can be determined by measuring the distance between two crests or two troughs. In this case, that value is 60 meters. The frequency is a time related quantity, and as this graph tells us the displacement of the wave at various points in space AS THEY EXIST AT ONE TIME AND ONE TIME ONLY, we have no way of determining either the period or the frequency. As such, this response is false.]

b.) The amplitude of the wave is 6 meters, its wavelength is 30 meters, and its frequency is impossible to tell with the information given. [The amplitude is not 6, it's 3. This response is false.]

c.) The amplitude of the wave is 3 meters, its wavelength is impossible to tell with the information given, and its frequency is impossible to tell with the information provided. [From above, everything is OK here except the determination of the wavelength, which should be 60 meters. This statement is false.]

d.) None of the above. [This has to be the one.]



14.) A string attached to the ceiling at one end has itself wrapped around an axle whose radius is R at the other end (see sketch). Attached to the axle are two wheels of mass m each, one at each end. The moment of inertia of the entire assembly along the axle's length is $1.4mR^2$. All numerical values are measured in the MKS system. The system is released to freefall. During the fall:

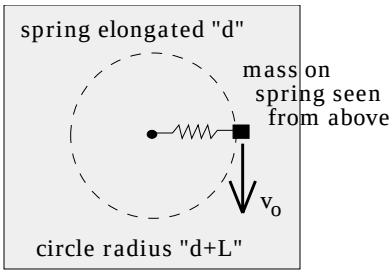
a.) Energy is conserved but angular momentum is not conserved. [In this case, tension does work that is not taken into account via a potential energy function, so energy is not conserved (though the modified conservation of energy equation could be used to determine the velocity of the body after it has dropped some known distance). This statement is false.]

b.) The body's angular acceleration α is numerically greater than the acceleration a of the body's center of mass. [The temptation may be to actually solve this problem to see if α>a. That would be a waste of time. Translational acceleration and rotational acceleration are related by the expression $a = R\alpha$. If R is greater than 1, then the numerical value of α will be less than a. If R is less than 1, the opposite will be the case. In any case, this statement will not always be true, so it must be false.]

c.) The body's angular acceleration α will be numerically less than the acceleration a of the body's center of mass only if the radius of the axle is greater than 1 meter. [According to what was said in Response b, this statement is true.]

d.) Both a and b. [Nope.]

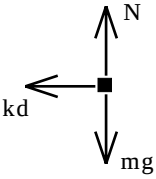
e.) Both a and c. [Nope.]



15.) A spring produces a force equal to kx, where x is the elongation of the spring and k is a constant. Our spring's unstretched spring length is L. One end of the spring is attached to a mass m and the other end of the spring is hooked frictionlessly over a pin at the center of a table. The spring is elongated a distance d beyond L, and the mass is given a constant velocity magnitude v_o so that the whole system moves in a circular path as shown to the right (the view looks down from above).

a.) With the velocity magnitude a constant, the acceleration for this motion must be zero. [For an object to move along a curved path, there must exist within the system a center-seeking force. That, in turn, means that there must be a center-seeking (centripetal) acceleration. This statement is false.]

b.) The acceleration is non-zero, and k = mg. [The acceleration is non-zero, but the rest is nonsense. To get k, we must use N.S.L. The f.b.d. for the mass looking head-on (i.e., as though the mass were coming straight at us) is shown to the right. Summing the forces in the center-seeking direction (kd in this case) and putting that acceleration $a_c = v_o^2/R = v_o^2/(d+L)$, we get:



f.b.d. of mass looking head on--radius of curve "d+L" is directed to the left

$$kd = m[v_o^2/(d+L)]$$

$$\Rightarrow \quad k = mv_o^2/[d(d+L)].$$

This statement is false.]

c.) The acceleration is non-zero, and $k = (mv_o^2)/[x(d + L)]$. [According to the expression derived in Response b, this statement is true.]

d.) The acceleration is non-zero, and $k = (mv_o^2)/(x)$. [This is false.]

16.) Two 2 kilogram masses collide. The first, Body A, has 25 joules of kinetic energy before the collision. The second mass is initially stationary. The two masses experience a perfectly inelastic collision. After the collision, mass B's kinetic energy is:

a.) 12.5 joules. [This is a typically sneaky, A.P. type question. Energy information is given, an energy related question is asked, but energy isn't conserved in the situation, so using conservation of energy will take you nowhere. Cleverly, though, the answer you would have gotten if you had treated energy as conserved is provided. That is, the after-collision kinetic

energy of mass B would simply be half the before-collision kinetic energy in the system (remember, the masses are the same), and that value is 12.5 joules. Unfortunately, this response is false.]

b.) 2.5 joules. [To actually do the problem, we know that before the collision, mass A's kinetic energy is $KE_o = .5mv_o^2 = .5(2\text{ kg})v_o^2 = 25$ joules, or mass A's before-collision velocity is $v_o = 5$ m/s. Using the conservation of momentum through the collision, we can write: $mv_o = 2mv_1$, or the after-collision velocity of the two stuck-together bodies is $v_1 = 2.5$m/s. Plugging that into mass B's after-collision kinetic energy equation, we get: $KE_1 = .5mv_1^2 = .5(2\text{ kg})(2.5\text{ m/s})^2 = 6.25$ joules. This response is false.]

c.) 6.25 joules. [From above, this is the one.]

d.) None of the above. [Nope.]

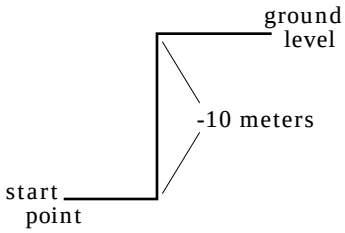
17.) An orbiting 1200 kg satellite has a period of 43 minutes (2580 seconds) in an orbit that is 15,000 meters above a planet of radius 10^6 meters. The satellite's speed is:

a.) 2472 m/s. [This is a little tricky as you don't really need all of the information given. The heart of the problem is in the fact that a satellite's period T and its velocity v are related by the expression: $v = 2\pi r/T$, where r is the radius of the orbit. For this situation, we get: $v = 2\pi(1{,}015{,}000\text{ meters})/(2580\text{ sec.}) = 2472$ m/s. This response is true.]

b.) 2435 m/s. [If you had taken the radius of motion to be 10^6 meters instead of $(10^6 + 1.5\text{x}10^4)$ meters, you would have gotten this incorrect answer.]

c.) 32.5 m/s. [If you had taken the radius of motion to be $1.5\text{x}10^4$ meters, you would have gotten this incorrect answer.]

d.) None of the above. [Nope.]



18.) A 10 kg projectile moving vertically has 1500 joules of gravitational potential energy at the same time that it has 1500 joules of kinetic energy. (Assume g's magnitude is 10 m/s^2). The potential energy is defined as zero at a point 10 meters below ground level where the body begins its motion (see sketch):

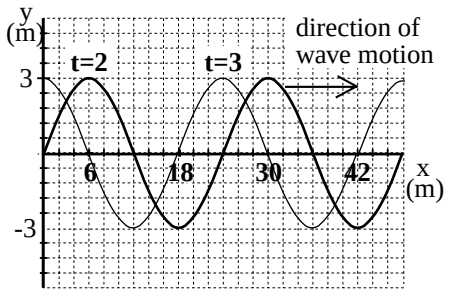
a.) The work gravity does to the top of the body's flight is -3000 joules, and the body's maximum height is 30 meters above ground level. [Knowing where ground level is relative to the body's initial position is inconsequential here. To use the gravitational potential energy function mgy, all you need is knowledge of the level at which the potential energy is zero. As that level is the so-called start point of the motion, the total energy in the system at that point (3000 joules) is all kinetic. Gravity does negative work in bringing the body to rest at the top of the flight, which means the total work gravity does over that portion of flight is, indeed, -3000 joules. As for the maximum height, $mgy_{max} = (10\text{ kg})(10\text{ m/s}^2)y_{max} = 3000$ joules, or $y_{max} = 30$ meters ABOVE THE ZERO POTENTIAL ENERGY LEVEL. As this is 10 meters below ground level, the body will fly 20 meters above ground level and this statement is false.]

b.) The work gravity does to the top of the body's flight is -3000 joules, and the body's maximum height is 20 meters above ground level. [From the reasoning above, this statement is true.]

c.) The work gravity does to the top of the body's flight is -4000 joules, and the body's maximum height is 30 meters above ground level. [If ground had been the zero potential energy level, then at ground level all of the energy in the system (3000 joules) would be kinetic and the velocity of the body at that point would have been $1/2mv_{gr}^2 = 3000$, or $v_{gr} = 24.5$ m/s. In that case, gravity would have to do -3000 joules of work to bring the body to rest, NOT INCLUDING the work it did while that body went from y = -10 meters to ground level (that amount of work was (10 kg)(10 m/s^2)(-10 meters) = -1000 joules). The total work done in this scenario would, indeed, have been -4000 joules. The height in that case would have been such that $(1/2)mv_{gr}^2 + 0 = mgy_{top}$, or .5(10 kg)(24.5 m/s)^2 = (10 kg)(10 m/s^2)y_{top}, or $y_{top} = 30$ meters. This statement would have been true under those circumstances, but isn't under the circumstances presented in the original problem.]

d.) The work gravity does to the top of the body's flight is -4000 joules, and the body's maximum height is 20 meters above ground level. [Nope.]

19.) A graph of a traveling wave as seen at t = 2 seconds and t = 3 seconds is shown to the right. At the rate it is traveling, how far will it travel between t = 2 seconds and t = 5 seconds?



a.) 12 meters. [According to the graph, the wave is moving 18 meters every second (the distance the wave traveled between the t = 2 seconds and t = 3 seconds is 18 meters, so its velocity is (18 meters)/(3 sec - 2 sec) = 18m/s). Over a 3 second period (i.e., the time between t = 2 seconds and t = 5 seconds) moving at 18 m/s, the wave will travel 54 meters, and this response is false.]

b.) 18 meters. [Nope, although this is the answer you would have gotten if you mistakenly took the distance traveled in one second to be 6 meters.]

c.) 54 meters. [This is the one.]

d.) None of the above. {Nope.]

20.) Newton's Second Law is applied to a system. After a free body diagram is drawn and the forces summed, the equation - $32x^2 = 2a$ emerges, where x is the position and a is the acceleration of the body at an arbitrary point in time.

a.) This equation does not characterize an oscillatory system. [If this equation had been -32x = 2a, we would have had the characteristic equation for an ideal spring (i.e., -kx = ma). If that had been the case, a positive displacement x would yield a negative force and a negative displacement x would have yielded a positive force. That is, the force would always have been oriented back toward the equilibrium position. With the expression $-32x^2 = 2a$, the $-x^2$ term guarantees that, no matter what the sign of x,, the net force will always be negative. In short, this force function is not a restoring force and, as a consequence, will not produce oscillatory motion of any type. As such, this response is true.]

b.) This equation does characterize an oscillatory system and the motion is simple harmonic in nature. [Nope.]

c.) This equation does characterize an oscillatory system and the motion's frequency is 4 radians per second. [Nope.]

d.) Both b and c. [Nope.]

e.) None of the above. [Nope.]

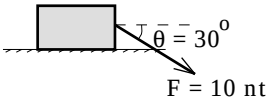
21.) A spring oscillates with frequency 1 cycle/second. What approximate length must a simple pendulum have to oscillate with that same frequency?

a.) 25 cm. [A frequency of 1 cycles/second corresponds to an angular frequency of 2π radians per second. The angular frequency function for a pendulum is $(g/L)^{1/2}$, where L is the pendulum length. Putting it all together, we can write $2\pi = (g/L)^{1/2}$, or $L = g/4\pi^2$, or approximately .25 meters. This response is true.]

b.) 50 cm. [Nope.]

c.) 67 cm. [Nope.]

d.) 90 cm. [Nope.]



22.) A body moves 25 meters. For the entire motion, a 10 newton force acts on the body as shown to the right.

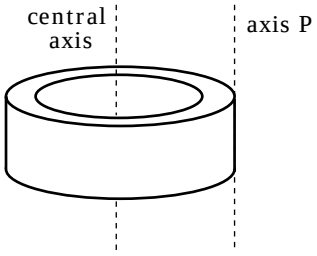
a.) The work done by the force is 250 joules. [This is a tricky question on a number of counts. First, the direction of the motion was not defined. As such, you have no idea whether the work quantity is going to be positive or negative. In short, you can't calculate a number response; Solution e is the answer. If the direction of motion was to the right, $Fd\cos\theta = (10)(25)\cos 30^o = 217$ joules. 250 joules (determined by multiplying the force times displacement without the cosine factor) wouldn't have been the correct response even if the motion had been to the right.]

b.) The work done by the force is -250 joules. [If the motion had been to the left, the calculated work value would have been -217 joules. This answer is obviously wrong.]

c.) The work done by the force is 217 joules. [From above, this would have been correct if the motion had been to the right. As we don't know if that was the case, this response is false.]

d.) The work done by the force is -217 joules. [Same problem. This response if false.]

e.) None of the above. [Given the trickiness of the problem, this is it.]



23.) The moment of inertia about the central axis of a ring is $(1/2)m(R_1^2+R_2^2)$, where R_1 is the inside radius and R_2 is the outside radius of the ring (see sketch). The moment of inertia about the axis labeled P in the sketch is:

a.) $(1/2)mR_1^2 + (1/2)mR_2^2 + (1/2)m[(R_1+ R_2)/2]^2$. [The Parallel Axis Theorem states: $I_p = I_{cm} + mh^2$, where h is the distance between the two axes. Using that expression yields: $I_p = [(1/2)m(R_1^2 + R_2^2) + mR_2^2]$. Regrouping yields: $I_p = (1/2)mR_1^2 + (3/2)mR_2^2$. This response is false.]

b.) $(3/2)mR_1^2 + (1/2)mR_2^2$. [According to the analysis done above, this response is false.]

c.) $(1/2)mR_1^2 + (3/2)mR_2^2$. [According to the analysis done above, this response is true.]

d.) None of the above. [Nope.]

24.) A .3 kg body moves down a 30^o incline. At a point .7 meters from the bottom the body is found to be moving with a velocity magnitude of .2 m/s. At that point a frictional force of .5 newtons begins to act. Assume the magnitude of the gravitational acceleration is 10 m/s^2.

a.) The body will not make it to the bottom of the ramp. [To make it to the bottom of the ramp, the body needs more total energy at the .7 meter point than friction will remove from the system. On the body's way to the bottom, friction will extract energy in the amount -fd = -(.5 nt)(.7 m) = .35 nt·m. At the .7 meter point, the body has $.5mv^2 + mgh = .5(.3\text{ kg})(.2\text{ m/s})^2 + (.3\text{ kg})(10\text{ m/s}^2)(.7 \sin 30^o \text{ meters}) = 1.06$newtons. The body will have enough energy to make it to the bottom of the ramp, and this statement is false.]

b.) The body will just barely make it to the bottom of the ramp. [As shown in Response a, the body will make it to the bottom of the ramp with energy and, hence, velocity to spare. This statement is false.]

c.) The body will make it to the bottom of the ramp, and its velocity at that point will be 4.73 m/s. [We know it will make it to the bottom. We don't know its velocity there. To get it: The potential energy was taken to be zero at ground level in Response a. Continuing in that vein, the energy at the bottom will be the total energy associated with the .7 meter point minus the energy extracted by friction as the body traveled to the bottom. Mathematically, then, E_{bot} = (1.06 joules) - (.35 joules) = .71 joules. As all of that energy will be kinetic, we can write $.5mv^2 = .5(.3\text{ kg})v^2 = .71$, or $v^2 = 4.73\ m^2/s^2$, or v = 2.18 m/s. The individual doing the calculation here did not take the square root in determining the velocity. This statement is false.]

d.) None of the above. [This is true by default.]

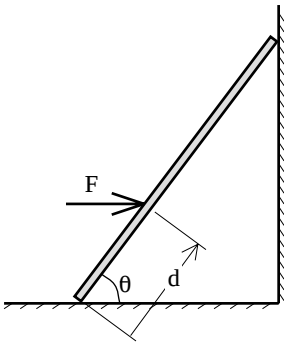
25.) The position function for an oscillating body is $x = 20 \sin (.6t - \pi/2)$. The approximate frequency of the motion is:

a.) 20 Hz. [As the angular frequency is .6 radians per second, the frequency is (.6 rad/sec)/(2π), or approximately .1 cycles/second. This response is not true.]

b.) 1.57 Hz. [Nope.]

c.) .6 Hz. [Nope.]

d.) .1 Hz. [This is the one.]



26.) A ladder of mass m and length L sits on a frictionless floor perched against a frictionless wall. A force F acting at a distance d units up the ladder (see sketch) keeps the ladder from angularly accelerating.

a.) The torque applied by mg about the point of contact with the floor is negative because mg is directed in the -j direction. [A torque's direction is negative if the force providing the torque motivates the body to angularly accelerate clockwise about the point about which the torque has been taken (this generality was first concluded after using the right hand rule as it applies to rxF). According to that criterion, mg does do this, so the torque mg applies about the contact point at the floor will, indeed, be negative . . . but not for the reason given. This response is false.]

b.) The torque applied by the normal force at the wall about the contact point with the floor is negative because N is directed in the -i direction. [In fact, N produces a positive torque, as defined in the discussion given in Response a. This statement is false.]

c.) About the floor contact, the torque due to F is negative as it tends to motivate the ladder to angularly accelerate clockwise. [This statement is true.]

d.) Both a and b. [Nope.]

27.) A mass m blows into two pieces whose mass ratio is 5 to 9 (call these pieces m_a to m_b):

a.) The ratio of the kinetic energies (E_a to E_b) will be 5 to 9 respectively, and the ratio of the momenta (p_a to p_b) will be 5 to 9. [If the mass is split into a 5 to 9 ratio, that is the same as saying the 5/14 of the mass went one way and 9/14 of the mass went the other. Letting m = 1 kg for simplicity, that observation and the conservation of momentum allows us to write: $0 = (5/14)v_1 - (9/14)v_2$. From this, we get the ratio $v_1/v_2 = 9/5$ (note that the ratio of $v_1^2/v_2^2 = 81/25$--we'll need this information when we look at the kinetic energy of both). Energy isn't conserved, but we can write out an expression for the kinetic energy of both bodies, then take a ratio. Doing so yields: $KE_1 = .5(5/14)v_1^2$ and $KE_2 = .5(9/14)v_2^2$. Taking a ratio of the two yields $KE_1/KE_2 = [.5(5/14)v_1^2]/[.5(9/14)v_2^2] = (5/9)(v_1^2/v_2^2)$. Substituting in for the velocity squared terms, we get $KE_1/KE_2 = (5/9)(81/25) = 9/5$. In short, this has the right numbers, but they are in the wrong order. This statement is false.]

b.) The ratio of the kinetic energies (E_a to E_b) will be 9 to 5 respectively, and the ratio of the momenta (p_a to p_b) will be 9 to 5. [From above, the energy part of this is true. What about the momenta? It might be tempting to write out the conservation of momentum equation for this situation, or $0 = (5/14)v_1 - (9/14)v_2$. From this, we get the ratio $v_1/v_2 = 9/5$. Unfortunately, this is the ratio of velocities, not the ratio of momenta. In fact, if we had written out the conservation of momentum equation in a more general fashion, we would have found that: $0 = p_a - p_b$, or $p_a = p_b$. In other words, the momentum ratio is 1:1 and this response is false.]

c.) The ratio of the kinetic energies (E_a to E_b) will be 9 to 5 and the ratio of the momenta (p_a to p_b) will be 1 to 1. [From above, this response is true.]

d.) None of the above is true. [Nope.]

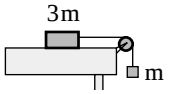
28.) Two stars circle in elliptical orbits about their collective center of mass. If one star is three times the size of the second star, which of the following is true?

	momentum of center of mass	angular momentum	energy	for the smaller star, T^2 is proportional to r^3
a.)	conserved	conserved	not conserved	yes
b.)	not conserved	not conserved	not conserved	no
c.)	conserved	conserved	conserved	yes
d.)	conserved	conserved	conserved	no

e.) None of the above.

[Commentary: What is conserved and what is not conserved in planetary motion problems? Well, there are no external torques acting on the system, so angular momentum must be conserved (this eliminates Response b). Assuming there is no friction, which is always a good approximation in these kinds of problems, there are no non-conservative forces acting, so

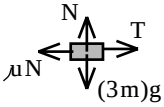
energy is conserved (this eliminates Responses a and b). And as there are no external forces acting, the momentum of the system's center of mass will be conserved, so Responses c and d are still alive. The question comes down to whether the square of the smaller star's period is proportional to the cube of its semi-major axis (i.e., whether Kepler's Third Law applies here). What you will hopefully remember is that Kepler called his third Law a law only because Brahe's data made the T^2 proportional to r^3 phenomenon look like a universal truth. What Kepler didn't evidently take into consideration was the fact that Brahe's data had been taken by observing very small objects (i.e., planets) orbiting an immense object (i.e., the sun). In such cases, the system's center of mass and the center of the star are essentially the same, and T^2 is, to a very good approximation, proportional to r^3 (theory predicts this only after approximations are made--see your book if this isn't clear). As the two stars in this problem are almost the same size, the approximation that allows Kepler's Third Law to hold cannot be made, and T^2 is NOT proportional to r^3. In short, Response d is the one.]



29.) The frictional force acting on the 3m mass shown to the right is .6mg. If the hanging mass is m, the system's acceleration is:

a.) .1g. [This is a tricky question. If you ASSUME the mass is moving to the right, the f.b.d. on the 3m mass will look like the f.b.d. shown to the right. The N.S.L. equation you get from that set-up coupled with the N.S.L. evaluation of the hanging mass will yield an acceleration equal to .1g. Unfortunately, although the direction of acceleration is obvious, we haven't been told the direction of motion of the body (it could initially be moving in either direction). As such, this statement is not necessarily true.]

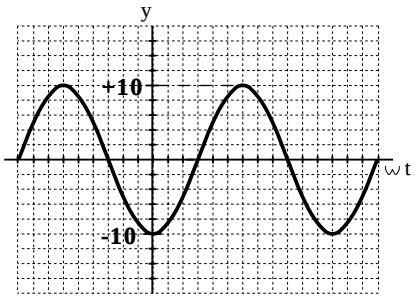
assume 3m mass is moving to right



b.) .4g. [If you assumed the direction of the acceleration was to the left, the direction of the frictional force in Part a's f.b.d. would have been reversed and the acceleration would have calculated to .4g. Again, unfortunately, we haven't been given the motion's direction. This statement is also false.]

c.) .133g. [If you assumed the motion's direction was to the right AND made the mistake of believing that the tension was equal to the weight of the hanging mass (i.e., mg), the acceleration would have calculated to .133g. As these assumptions are not necessarily true, this is a false statement.]

d.) Indeterminable as there isn't enough information. [By default, this is the one.]



30.) A graph of the position function for a body oscillating with an angular frequency of π radians per second in simple harmonic motion is shown to the right.

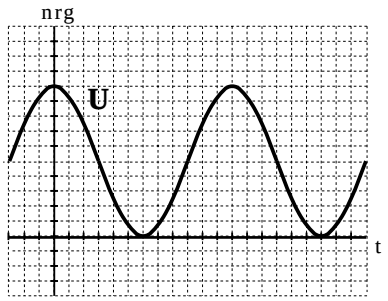
a.) The amplitude is 10, the period is 2 seconds per cycle, and the phase shift is π/2 radians. [The amplitude for this wave is 10 meters, which eliminates Response b. The period is the inverse of the frequency, where the frequency in this case is the angular frequency (i.e., π radians) divided by 2π, or .5 cycles/second. This yields a period of 2 seconds per cycle. So far, so good for this selection. As for the phase shift, if you were to shift a sine wave π/2 radians to the right (i.e., a quarter of a cycle), you would end up with a cosine wave. That is not what the graphs shows (in fact, the phase shift is -π/2). This response is false.]

b.) The amplitude is 20, the period is 5 seconds per cycle, and the phase shift is $-\pi/2$ radians. [The amplitude is off and this is false.]

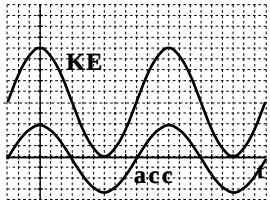
c.) The amplitude is 10, the period is 2 seconds per cycle, and the phase shift is $-\boldsymbol{\pi}/2$ radians. [From what has been said above, this is the one.]

d.) None of the above. [Nope.]

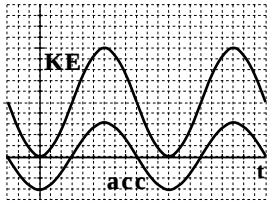
31.) The potential energy vs. time graph for an oscillating system is shown to the right. The kinetic energy vs. time and acceleration vs. time graphs look like:



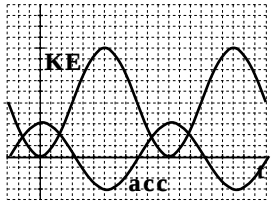
a.)



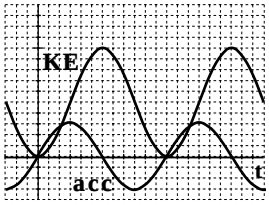
b.)



c.)



d.)



e.) None of these.

[Commentary: Looking at the potential energy graph, the maximum potential energy does not appear to be diminishing with time. That means that we can assume that energy is, to a good approximation, conserved. Due to this, the kinetic and potential energy functions should be mirror images of each other (this eliminates Graph a).

The first time I did this problem, my commentary continued on to say that because velocity and acceleration functions are out of phase with one another by a quarter of a cycle (see next sentence for justification of this should it not be obvious), the kinetic energy function (a velocity dependent quantity) should be out of phase with the acceleration by a quarter of a cycle, and graphs b and c could not be "it."

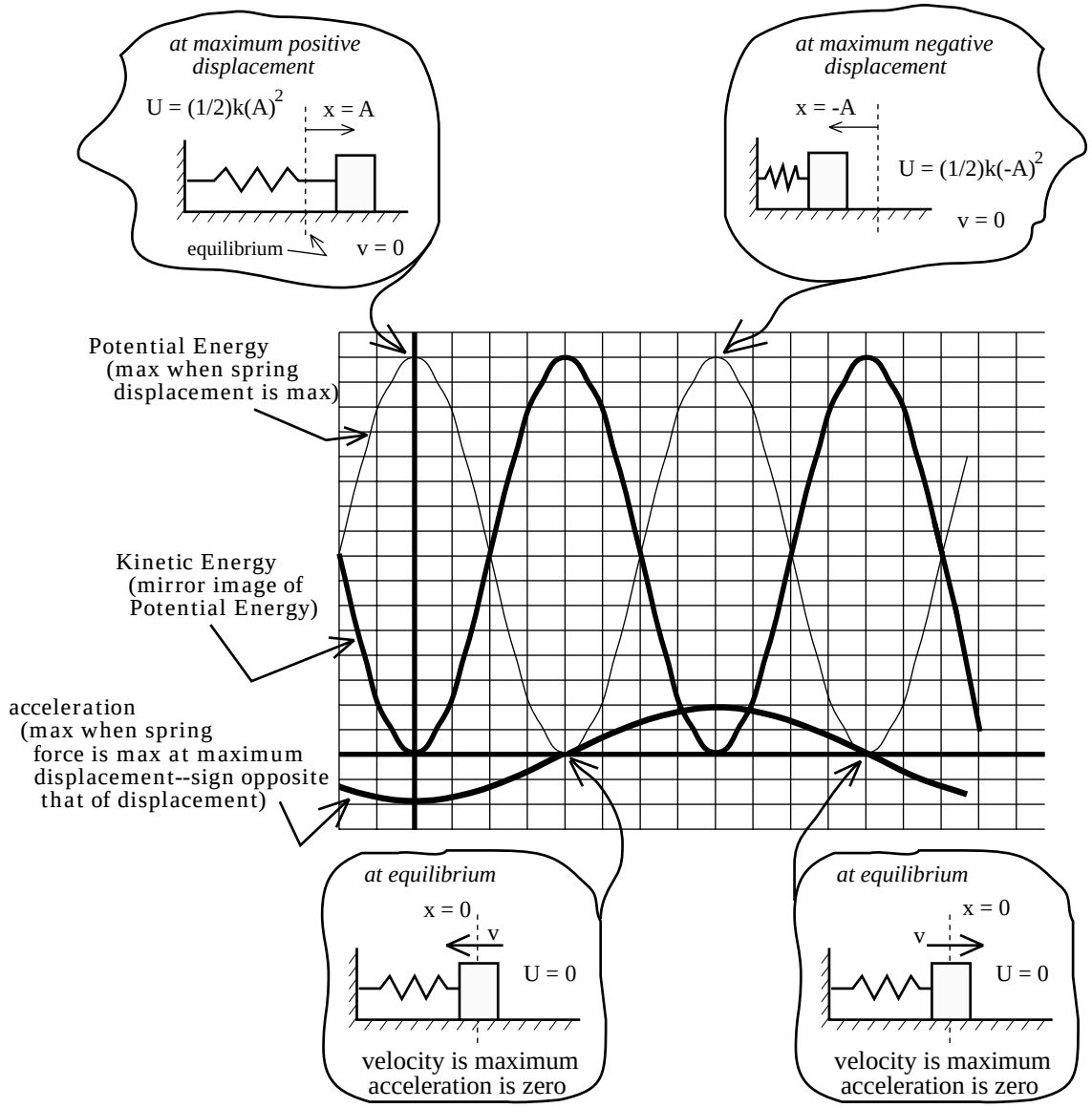
(Note that the justification for the velocity/acceleration phase relationship goes as follows: if we assume the velocity function is a sine wave--the acceleration, or derivative of the velocity function, will be a cosine wave. As sine curves are out of phase with cosine curves by a quarter of a cycle, the velocity and acceleration functions should be out of phase with one another by a quarter of a cycle.)

So what about graph d? When I generated that graph, I used the fact that the maximum kinetic energy occurs when a body is at equilibrium (visualize this by thinking about a mass attached to a spring), and that at equilibrium the force and acceleration are zero. Look at graph d's first KE peak (that's where I started when I constructed the graphic). The acceleration at that point is graphed as zero. To finish off the graphic, I constructed a sine wave around that point and came up with what you see. In fact, I presented this as the right answer, thereby committing the sin I yell and scream at you folks NOT to commit. Specifically, don't get so involved in the math that you completely lose track of the reality of the situation.

What am I talking about? A student noticed that although the graph may match at the first kinetic energy peak, it doesn't match at the origin of the axes. At that point, the kinetic energy is zero and the acceleration is shown to be zero--a clear no-no. When brought to my attention, I did the more complete evaluation shown on the next page.

The key to the problem? Kinetic energy is NOT a function of velocity, it's a function of velocity SQUARED. Squaring the velocity essentially halves the amount of time it takes for the kinetic energy to go from being maximum to being maximum again. That is, in the time it takes to go from a positive maximum acceleration to a negative maximum acceleration (this is half the acceleration cycle), the kinetic energy goes from positive maximum to positive maximum. Squaring the velocity halves the period of the kinetic energy waveform.

The answer is e. When I mess up, I do it good.



32.) At a particular point in its motion, a 10 kg projectile moving vertically has 1500 joules of gravitational potential energy at the same time that it has 1500 joules of kinetic energy. The body starts its motion at ground level, and it takes t_1 seconds to reach the point alluded to above.

(Assume the gravitational potential energy function is defined to be zero at ground level, and assume g's magnitude is 10 m/s^2). At the point:

a.) The maximum height the body reaches will be 15 meters. [The sum of the potential and kinetic energies is 3000 joules. This is the total mechanical energy in the system. At the top of the flight, all of that is wrapped up in potential energy. That is: $mgy_{max} = 3000$ joules. Putting in the numbers yields a maximum height $y_{max} = 30$ meters. This statement is false.]

b.) It will take $2t_1$ seconds for the body to reach the top of its flight. [With half the energy being potential, and with gravitational potential energy equaling mgy near the earth's surface, the assumption that most people would make here is that the body is halfway to the top. It is, indeed, halfway from the top, but it is not at all clear whether it is moving upward toward the top or downward away from the top. Both situations have the same potential and kinetic energy associated with them (remember, energy is not a vector). As such, we don't really know if it took t_1 seconds to get from ground level to the halfway point, or t_1 seconds to get from ground level to the top and then back down to the halfway point. Not knowing this makes it impossible to evaluate the situation, which means this statement must be false. For those who care, if the body was halfway going toward the top, the governing equation relating the body's y position and time would be the kinematic equation $(y_2 - y_1) = v_o t + .5at^2$. With the body at the halfway point, it would have to travel 2d to get to the top. From the equation, it should be clear that that would take $(2)^{1/2}t_1$ seconds to accomplish (try some numbers if you don't believe me). On all counts, this statement is false.]

c.) The body is exactly halfway to the top of its flight. [From Response b, this is true.]

d.) None of the above. [Nope.]

33.) A skater with arms outstretched goes into a spin during which her arms are brought in near her body. During the movement:

a.) Energy is conserved and angular momentum is conserved. [Although it may not be obvious, energy is not conserved here. How so? Chemical energy, which is neither kinetic nor potential in nature--is required to motivate the muscles of the arms to pull them in (they would naturally fly outward if given the freedom). That means energy is being drawn into the system from a source that is not part of the mechanical energy family, and this statement is false.]

b.) Energy is conserved and angular momentum is not conserved. [This statement is false on the grounds that energy is not conserved.]

c.) Energy is not conserved and angular momentum is conserved. [We have already established that energy is not conserved, so that part is true. Because the forces that pull the arms in produce no torque about the axis of rotation (a force aimed at the axis of rotation will produce no torque about the axis of rotation), angular momentum will be conserved. As for momentum, skaters are not usually moving when they perform spins, but even if they were, standard momentum would be conserved as long as there were no external forces acting on the body to change the momentum. In short, momentum is also conserved and this response is true.]

d.) Energy is not conserved and angular momentum is not conserved. [Nope.]

34.) A sound wave moving at 330 m/s has a frequency of 220 Hz. What is the phase difference between two points .3 meters apart?

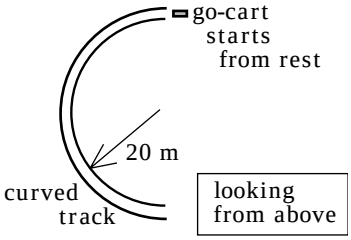
a.) (1/5) of a cycle, or .4π radians. [The velocity of a wave will numerically equal the product of the wave's frequency and wavelength. Knowing that the frequency is 220 Hz and the wave velocity is 330 m/s, the wavelength must be 1.5 meters. A distance of .3 meters is one-fifth of 1.5 meters, so the points are one-fifth of a wavelength away from each other. As there are 2π radians in a wavelength, one fifth of that will be $2\pi/5 = .4\pi$, which is to say that this response is correct.]

b.) (2/3) of a cycle, or $4\pi/3$ radians. [Not so.]

c.) (3/2) of a cycle, or 3π radians. [Not so.]

d.) None of the above. [Not so.]

35.) A go-cart sitting on a flat, circular section of track of radius 20 meters accelerates at a rate of 1 m/s^2. If the car's mass is 100 kg and the coefficient of static friction between the road and its tires is .5, approximately how long will it take before friction can no longer hold the cart in circular motion and the cart spins out? Approximate g to be 10m/s^2.



a.) 2.4 seconds. [There is no way to do this problem shorthand--you have to work it out. To start, we need to determine the maximum velocity the body can handle, given what we know about the coefficient of static friction. Noting that in this situation, the normal force N equals the weight of the car mg, we can sum the forces in the center-seeking direction and write:

$$\mu_s N = ma_c$$
$$\Rightarrow \mu_s(mg) = m(v_{max}^2/R)$$
$$\Rightarrow v_{max} = (\mu_s gR)^{1/2}$$
$$= [(.5)(10\ m/s^2)(20\ m)]^{1/2}$$
$$= 10\ m/s.$$

An acceleration of 1 m/s^2 means the body changes its velocity by 1 m/s every second. At that rate, it will get to 10 m/s in 10 seconds. This statement is false.]

b.) 8.2 seconds. [From above, this statement is false.]

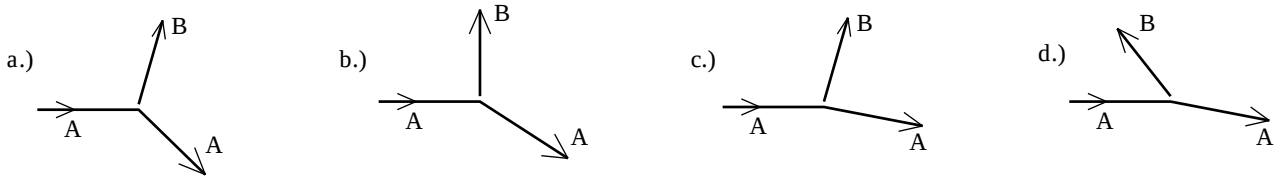
c.) 10.0 seconds. [From above, this statement is true.]

d.) 13.6 seconds. [From above, this statement is false.]

MULTIPLE CHOICE SOLUTIONS--MECHANICS

TEST III

1.) An atomic particle whose mass is 210 atomic mass units collides with a stationary particle whose mass is 40 atomic mass units. Which of the sketches below most closely approximates a possible collision scenario?



[Commentary: Due to its size, A isn't going to deviate much upon colliding with B (this effectively eliminates graphs a and b). Due to momentum considerations, neither of the masses would be expected to move backwards after the collision (this eliminates graph d). Graph c comes closest.]

2.) A satellite orbits with a constant velocity v in a circular path around a planet. At one point in the flight, a short thruster burst is used to do work on the satellite, motivating it into an elliptical path. After the burst, the average velocity per orbit of the satellite is less than v. In what direction were the thrusters fired?

a.) In the direction of the satellite's motion. [Firing the thrusters forward in the direction of the satellite's motion will momentarily slow the satellite, causing it to move into an oblique elliptical path. As a consequence, the satellite will be pulled closer to the planet as it passed by and, as a consequence, will travel faster than otherwise expected at that point. In fact, the velocity throughout the motion will, on average, be faster than would have otherwise been the case. In short, this option is not true.]

b.) In the direction opposite the satellite's motion. [Firing the thrusters backward in the direction opposite the satellite's motion will momentarily speed the satellite up, causing it to move out farther away from the planet than would otherwise be expected. As a consequence, the satellite will pass farther from the planet at its closest point, moving more slowly in the process. In fact, the average orbital velocity will be slower than would have otherwise been the case. In short, this option will do the job.]

c.) Perpendicular to and inward relative to the satellite's motion. [Interesting observation: a force applied perpendicular to the motion will do no work. According to the problem's statement, work was done. This option will not do.]

d.) There is not enough information given to tell. [Nope.]

3.) A jackhammer pounds the ground with a frequency of 5 hertz. If each cycle is driven by a hydraulic spring whose effective spring constant is 800 nt/m, and if the throw of the device (i.e., the distance over which the bit is accelerated before contact) is .32 meters:

a.) The energy provided by each stroke will be approximately 40 joules. [The potential energy wrapped up in what is effectively a spring system here is $.5kA^2$, where k is the spring

constant and A is the amplitude of the motion. In this case, that quantity is: .5(800 nts/m)(.32 m)^2 = 40.96 joules. As this is approximately 40 joules, this response is true.]

b.) The work done per stroke will be approximately 40 joules. [This is tricky in its simplicity. The work done, to a good approximation, will equal the energy available in the system (true, some of the work might be done overcoming friction, but it will still be done). That means that the answer to Response a and the answer to this response should be the same, and this response is true.]

c.) The power provided per stroke will be approximately 200 watts. [Power is defined as the work per unit time. If one stroke does 40 joules of work, and if one stroke takes 1/5 of a second (remember, the frequency is 5 cycles/second, so the period is 1/5 second per cycle), then the number of joules per second being provided to the system will be equal to the product (40 joules/cycle)/(1/5 second/cycle) = 200 watts. This response is true.]

d.) All of the above. [Yup.]

e.) None of the above. [Nope.]

4.) A 12 kg projectile is fired straight up. Halfway to the top of its flight, its kinetic energy is 300 joules. Taking g's magnitude to be 10 m/s^2:

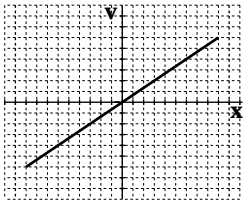
a.) The projectile's velocity at the halfway point is 50 m/s. [The kinetic energy is (1/2)mv^2 = .5(12 kg)v^2 = 300 joules. The math yields a velocity v = (50)$^{1/2}$ m/s. This statement is false.]

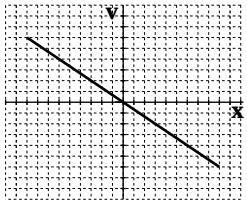
b.) The projectile's maximum height is 5 meters. [At the halfway point, the total mechanical energy will be U + KE = mg(h/2) + 300. At the top, the total mechanical energy will be U + 0 = mgh. As the total energy will not change (no extraneous work being done during the motion), these two can be equated. Solving for h yields 5 meters. This response is true.]

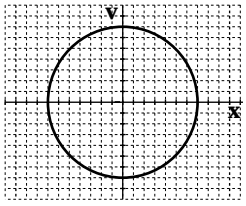
c.) The projectile's initial velocity (i.e., at ground level) is 8 m/s. [The total mechanical energy is equal to the potential energy at the top of the flight. That quantity is mgh = (12 kg)(10 m/s^2)(5 m) = 600 joules. The total energy is also equal to the kinetic energy at the beginning of the flight. That quantity is (1/2)mv^2 = .5(12)v^2. Equating the two total energy expressions yields an initial velocity v = 10 m/s. This response is false.]

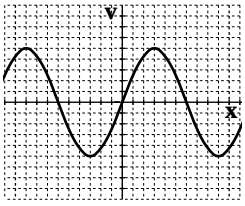
d.) None of the above. [Evidently not.]

5.) A graph of the velocity vs. position of a body oscillating in simple harmonic motion looks like:

a.) 

b.) 

c.) 

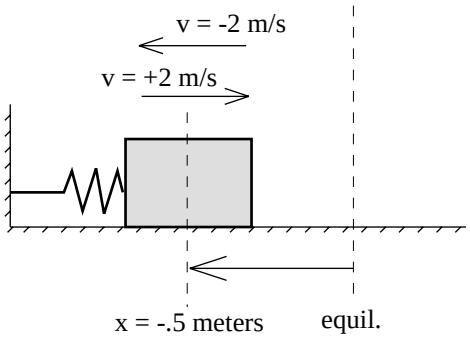
d.) 

e.) None of these.

[Commentary: If you will remember, the position function for this kind of motion is characterized by a sine wave whereas the velocity function is a cosine. Put a little differently, the position and velocity are out of phase with one another by a quarter of a cycle. What does this mean? Consider the sketch on the next page. Assume that when the body is at x = -.5 meters, its velocity is+2 m/s. There will be a point in time when the bodyis at x = -.5 meters

two velocities possible at one point

going the other direction at -2 m/s. In other words, there will be two velocities associated with x = -.5 meters. How does one graph such a function? With a circle! In short, graph c does the trick.]



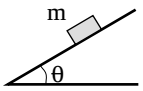
6.) During a 10 second period, a wheel has a constant torque applied to it that diminishes its angular speed from 1 rad/sec to .8 rad/sec. In 20 seconds, the angular speed will be:

a.) .64 rad/sec. [A constant torque will provide a constant angular acceleration. As such, we can use kinematics or, better yet, remember that a constant acceleration means that velocity is changing linearly with time. That being the case, doubling the time simply doubles the slowdown. The correct response should be .6 rad/sec, and this response is false.]

b.) .60 rad/sec. [This is the one.]

c.) .40 rad/sec. [Nope.]

d.) .10 rad/sec. [Nope.]



7.) At a given instant, a mass m is observed to be moving with velocity v_o up a frictional incline plane of angle θ.

a.) The direction of the mass's acceleration is opposite the direction of its velocity. [The two accelerating forces, a gravitational component and friction, are oriented down the incline. As the velocity is up the incline, this statement is true.]

b.) The acceleration would be greater if the block were traveling down the incline. [If the body were traveling down the incline, the gravitational component would still be down the incline but the frictional force would be up the incline. As the accelerating forces would subtract from one another, the acceleration would be less than if the body was moving up the incline. This statement is false.]

c.) If θ were made smaller, the coefficient of kinetic friction would change, and the acceleration would be lessened. [The coefficient of friction is a constant. This statement is false.]

d.) Both a and b. [Nope.]

e.) Both a and c. [Nope.]

--The following information pertains to Problems 8 through 10: Projectiles A and B are fired at the same time from a height h meters above the ground. Both projectiles have the same muzzle velocity v_o. It takes t_1 seconds for Projectile A to get to the top of its flight. A sketch of the two situations is shown to the right.

projectile A projectile B

8.) If Projectile A travels a net horizontal distance equal to x_1 meters:

a.) Projectile B will travel a net horizontal distance less than x_1 meters. [If Projectile A is angled steeply--at almost 90^o--it obviously won't go very far, and this statement will be false.]

b.) Projectile B will travel a net horizontal distance equal to x_1 meters. [From the information given in Response a, this is clearly not true.]

c.) Projectile B will travel a net horizontal distance greater than x_1 meters. [If Projectile A is angled close to 45^o, the projectile will have both an x and y component of velocity and, in fact, will land close to its maximum range--definitely farther than x_1. This statement is clearly false.]

d.) The relative distances between Projectiles A and B will depend upon the angle θ. [Everything said above makes it clear that the angle is crucial in determining the range.]

9.) The initial velocity of Projectile B will:

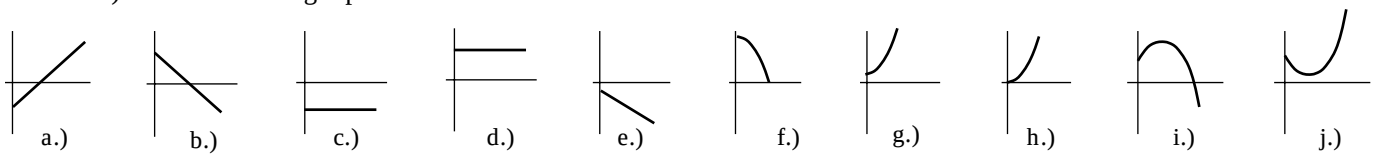
a.) Be equal to the initial velocity of Projectile A. [Velocity is a vector. It has direction. The magnitudes of the two velocities are the same, but the directions are not. This statement is false.]

b.) Be equal to the velocity of Projectile A at time t_1. [At time t_1, Projectile A will be at the top of its arc moving only in the horizontal. For that case, the direction of the two velocity vectors will be the same (both will be in the horizontal), but the magnitudes will be different (one is v_o, the other $v_o \cos\theta$). This statement is false.]

c.) Be equal to the velocity of Projectile A just an instant before touch down. [This is obviously false.]

d.) None of the above. [This is it.]

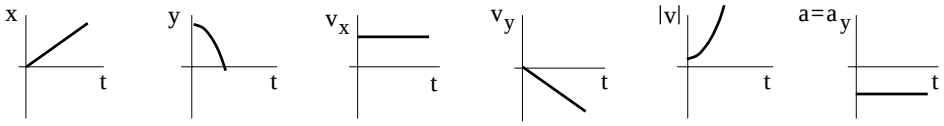
10.) Consider the graphs shown below:



Projectile B's:

a.) Y-component of its Position vs. Time graph looks like graph a.
b.) X-component of its Position vs. Time graph looks like graph j.
c.) Magnitude of its Velocity vs. Time graph looks like graph g.
d.) X-component of its Velocity vs. Time graph looks like graph c.
e.) Y-component of its Acceleration vs. Time graph looks like graph b.

[Commentary: A graph for each of the major parameters for this situation is shown below. This is something you should have been able to both visualize and sketch on your own. If you think you wouldn't have been able to do that, use the graphs provided as a stimulus to do the visualization part.]



----------end of section----------

11.) A car moving with velocity v_o rounds a flat curve of radius R. If the car had been moving a bit faster, it would have lost traction and skidded.

a.) If the car had broken traction and skidded, it would have moved away from the curved path it was supposed to take. The force that moved it out of its curved path is called a centrifugal force. [This is twisted. As any driver who has every hit an ice patch knows, a car that loses traction while rounding a curve will simply move in a force free straight line (the apparent outward movement of the car is an illusion brought on by the fact that the road is curved). The idea of a centrifugal force is misleading. Centrifugal force, in fact, does not exist (though centrifugal effects do exist) except as a fictitious force that must be assumed to exist if one is to use N.S.L. while examining a problem from the perspective of an accelerated frame of reference (like the frame that is attached to you as you sit behind the wheel of a car moving in a curved path). Summarily, this statement is false.]

b.) Kinetic friction in this situation will be less than mv_o^2/R. [This is a tricky question. Because the set-up has nothing to do with kinetic friction (it is static friction that keeps a car from skidding sideways when rounding a turn), the temptation is to assume that we are looking at apples and oranges and that the statement is false. What is important to notice is the fact that there still is a relative relationship between static and kinetic friction in the sense that the kinetic friction between two surfaces is always less than the static friction between those same two surfaces. As the static frictional force must equal mv_o^2/R in this case, the kinetic frictional force must be less than that amount, and this statement is true.]

c.) The coefficient of static friction is equal to mv_o^2/R. [The coefficient of static friction is a unitless constant. If for no other reason, this statement can't be true because mv_o^2/R is not a unitless quantity. In fact, this is supposed to be a trick question. If you had read static friction instead of coefficient of static friction, the statement would have been true. That is, Newton's Second Law states that the net force acting on a body must equal the product of two variables--the body's mass and the body's acceleration. In the case of a car rounding an unbanked curve, the center-seeking force that must exist naturally within the system to pull the car out of straight-line motion and into curved motion is provided solely by the static friction (or a component of the static friction) between the car and the road. We don't know anything about static friction in this problem, but because it is the only force acting it must equal $ma = m(v^2/R)$. If the static frictional force is equal to this value, the car will be right at its limit and will not slide. This response is false.]

d.) None of the above. [Nope.]

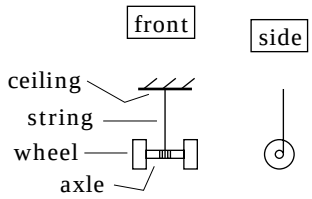
12.) A 10 kg projectile moving vertically has 1500 joules of gravitational potential energy at the same time that it has 1500 joules of kinetic energy. (Assume the gravitational potential energy function is defined as zero at ground level, and assume g's magnitude is 10 m/s^2). At this point:

a.) The body's velocity is exactly half that of its maximum value. [With the total energy equal to 3000 joules, we can write $3000 = .5mv_{max}^2 = .5(10)v_{max}^2$. Solving yields a maximum velocity of $(600)^{1/2} = 24.5$ m/s. The velocity at the point we are dealing with can be determined using $1500 = .5mv_{max}^2 = .5(10)v_1^2$. That velocity is $(300)^{1/2} = 17.3$ m/s. This statement is false.]

b.) The body's velocity is more than half that of its maximum value. [From Response a, this is true. This might be obvious, depending upon whether you are good at looking at situations in a qualitative way. The body is at its kinetic energy halfway point. If kinetic energy were a linear function of velocity, the body's velocity at the halfway point would be $v_{max}/2$. Because kinetic energy is related to the velocity squared, the body in this case would have a velocity of $v_{max}/2^{1/2}$.]

c.) The body's velocity is less than half of its maximum value. [From above, this is false.]

d.) This cannot be answered as the maximum velocity is not a quantity we can calculate. [Nope.]



13.) A string is attached to the ceiling at one end and has its other end wrapped around an axle of mass m and radius R (see sketch). Attached to the axle are two wheels of mass m and radius 3R each, one at each end. The moment of inertia of the entire assembly along the axle's length is $1.5mR^2$. All numerical values are measured in the MKS system. The system is released to freefall. During the fall, the acceleration of the body's center of mass is:

a.) .40g. [Summing the torques about the center of mass yields: $-TR = -I_{cm}\alpha$. Using $a = R\alpha$ and $I_{cm} = 1.5mR^2$, and multiplying by -1, we get $T = I_{cm}a/R = (1.5mR^2)(a/R)/R = 1.5ma$. Summing the forces in the vertical yields: $T - (3m)g = -(3m)a$. Substituting in for T yields: $1.5ma - 3mg = -3ma$, or $a = .67g$. If you got the negative sign right on the translational acceleration part, but inadvertently used m instead of 3m for your mass, you undoubtedly got .4g for your solution. Unfortunately, that is wrong and this response is false.]

b.) .67g. [This is the one.]

c.) 2.0g. [Nope.]

d.) None of the above. [Nope.]

14.) A 1 kg mass is found to be moving 18 m/s up a 30^o incline. How fast is the mass moving 3 seconds later? Take g to be 10 m/s^2.

a.) 2 m/s. [The modified conservation of momentum equation can be written as $\Sigma p_1 + \Sigma F_{ext}\Delta t = \Sigma p_2$. What it says is that in a particular direction, the momentum of a system will stay the same over time Δt unless there are external impulses (the $F_{ext}\Delta t$ quantities) acting on the system during that time. If external impulses are present, the total momentum at time t_2 will equal the total momentum at time t_1 plus or minus the extra impulses that are applied. Taking the line of the incline to the x direction, we can use the modified conservation of momentum equation to write: $\Sigma p_1 + \Sigma F_{ext}\Delta t = \Sigma p_2$, or $(1kg)(18\ m/s) + [-(1kg)(10\ m/s^2)(\sin 30^o)](3\ sec) = (1\ kg)v_2$, or $v_2 = 3$ m/s. If this be the case, this response is false.]

b.) 3 m/s. [From the above analysis, this is a true statement.]

c.) 6 m/s. [Nope.]

d.) None of the above. [Nope.]

15.) A satellite of mass m orbits a planet of mass M and radius R in a circular orbit. If the orbit is 3R units above the planet's surface, the velocity of the satellite must be:

a.) $[GM/(3R)]^{1/2}$. [Looking for a velocity in an ORBIT problem usually means that you will have to use N.S.L. coupled with the fact that the orbiting body's acceleration is centripetal. As the distance between the center of masses of the two objects is 4R, N.S.L. yields: $GmM/(4R)^2 = m[v^2/(4R)]$, or $v = [GM/(4R)]^{1/2}$. If you inadvertently took the distance part of the force expression to be 3R (i.e., the distance between the satellite and the surface of the planet), you got the incorrect response . . . which is to say the one offered in Response a.]

b.) $[GM/(4R)]^{1/2}$. [From the discussion above, this is the correct response.]

c.) $[GM/(16R^2)]^{1/2}$. [Nope.]

d.) GM/(4R). [Nope.]

16.) During time t_1 a single, constant force is applied to body A changing its velocity from 0 m/s to 30 m/s. For an unknown period of time t_2 the same force is applied to a second, identical body (call it B). During t_2, B's velocity changes from zero to 15 m/s.

a.) The net work done on A will be twice as much as done on B. Also, the distance over which the force is applied to A is twice that applied to B. [For the first part, we know that $W_{net} = KE_2 - KE_1$. As both bodies start from rest, the question is whether $(1/2)mv_{2,A}^2 = 2(1/2)mv_{2,B}^2$? Clearly, $v_{1,A}^2 = (30\ m/s)^2 = 900\ m^2/s^2$ is not the same as twice $v_{2,B}^2 = (15\ m/s)^2 = 225\ m^2/s^2$. This answer is false without even having to look at the second part.]

b.) The net work done on A will be twice as much as done on B, but the distance over which the force is applied will not be twice as long. [This is false for the same reason Response a was wrong.]

c.) The net work done on A will be less than twice as much as done on B. Also, the distance over which the force is applied to A is less than twice that for B. [The first part of this is clearly wrong. This statement if false.]

d.) The net work done on A will be more than twice as much as done on B. Also, the distance over which the force is applied to A is more than twice that for B. [As shown in Response a, the first half of this statement is true. As for the second part: We know that $W = F \cdot d$. Put another way, the amount of work a constant force does is proportional to the distance over which it is applied. If the work done to A is more than twice as much as is done to B, then the distance over which the force is applied must also be more than twice for A than for B. This statement is true.]

e.) None of the above. [Nope.]

17.) A pendulum bob's mass is decreased by a factor of 4 while its length is increased by a factor of 4.

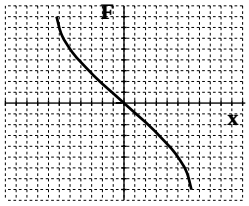
a.) Its frequency will stay the same as will its period. [The angular frequency of a pendulum is equal to $(g/L)^{1/2}$. This means that changing the mass of the bob does nothing to the angular frequency, the frequency, or the period, but changing the length L does change things. In fact, increasing L decreases the angular frequency and, by extension, the frequency. As the period is the inverse of the frequency, decreasing the frequency increases the period. On every level, this response is false.]

b.) Its frequency will increase and its period will increase. [From what has been said above, this response is false.]

c.) Its frequency will decrease and its period will increase. [From what has been said above, this is the one.]

d.) Its frequency will decrease and its period stays the same. [Nope.]

e.) Its frequency will increase and its period will decrease. [Nope.]



18.) For small amplitude oscillations, the force on an unusually designed spring is very close to being proportional to the displacement of the spring. For large amplitude oscillations, the force becomes non-linear. A graph of the force vs. displacement is shown to the right. When this spring executes large amplitude oscillations:

a.) Its period will be the same as in the small oscillations case, and its frequency will also be the same. [Period and frequency are both qualitatively related to the vigorousness of the restoring force. As the restoring force is not linear in this case for larger amplitude oscillations, this response must be false.]

b.) Its period will be smaller than in the small oscillations case, and the frequency will also be smaller. [The fact that the period and frequency are both getting smaller tells you that this response is false. Frequency and period are inversely related.]

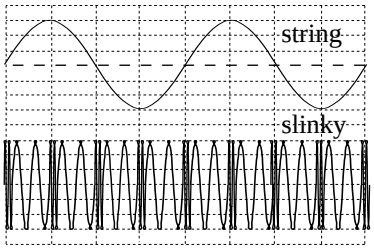
c.) Its period will be smaller than in the small oscillations case, but the frequency will be larger. [If the force had been linear with position, it would have taken a certain amount of time for one cycle. If at any time during the motion the force is larger than would have been the case for the linear situation, the body will accelerate more quickly back toward equilibrium and the amount of time for one cycle (i.e., the period) will be less than for the linear situation. With a smaller period, we get a larger frequency, and this response is true.]

d.) Its period will be larger than in the small oscillations case, but the frequency will be smaller. [Nope.]

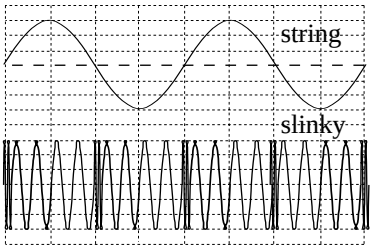
e.) None of the above. [Nope.]

19.) A string carries a transverse wave down its axis. A slinky carries a longitudinal wave down its axis. At a given instance, the string looks like the top sketches on each grid below. Assuming the slinky's frequency is twice that of the wave on the string, which of the slinky waveform matches this situation?

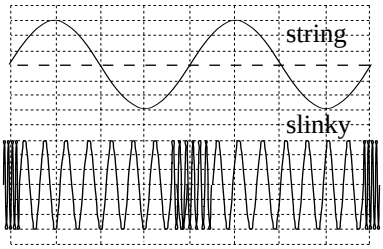
a.)



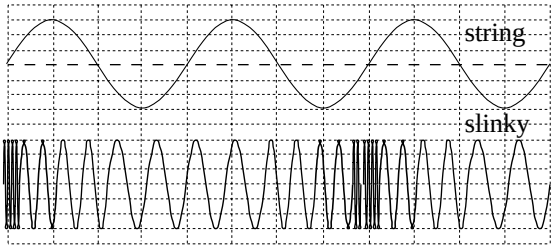
b.)



c.)



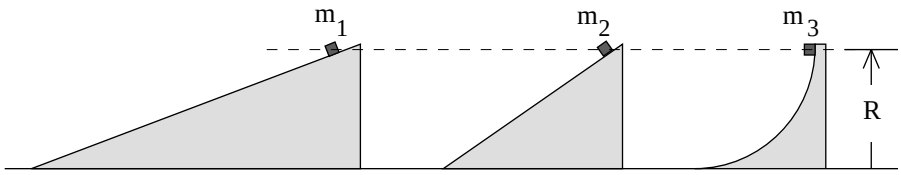
d.)



e.) There is not enough information to tell.

[Commentary: The frequency tells you how many cycles pass you by per unit time. That means that if the string's frequency is 1 cycle/second, the slinky's frequency will be 2 cycles/second. If the wave velocity for the two waveforms was the same, that would mean that the string's wavelength would be twice that of the slinky (wavelength and frequency are inversely related) and graph d would have been the correct solution. Unfortunately, we don't know that the wave velocities are the same. In fact, they most probably will not be the same. As such, the only response we can give is Response e.]

20.) Three frictionless inclines are shown below. The sketches are drawn approximately to scale with the curved incline being a quarter circle. The masses are different sizes. All three masses are released at once. Mass m_2 travels the same distance as does mass m_3.



The first mass to the bottom will be:

a.) Mass m_1. [The mass that gets to the bottom first will be the mass with the largest acceleration and the shortest distance to travel. This isn't the one.]

b.) Mass m_2. [Although masses m_2 and m_3 travel the same distance, m_3's acceleration, on average, is greater than m_2's acceleration. This isn't the one.]

c.) Mass m_3. [From what has been said above, this is the true statement.]

d.) Mass m_2 and m_3. [Nope.]

e.) They will all reach the bottom at the same time. [Nope.]

21.) A metal hoop and a hollow rubber ball have the same radius and mass. The two are placed side by side at the top of an incline.

a.) The hoop will arrive at the bottom first, but its rotational kinetic energy will be less than the rotational kinetic energy of the ball when it arrives at the bottom. [The body with the greatest moment of inertia (i.e., the body that most resists changes of its angular motion) will take the most time to get to the bottom. As the hoop's mass is distributed as far from the axis of rotation as possible, the hoop has the greatest rotational inertia. As such, it will not reach the bottom first. This response is false.]

b.) The hoop will arrive at the bottom first, and its rotational kinetic energy will be greater than the rotational kinetic energy of the ball when it arrives at the bottom. [Again, this is false because the hoop doesn't arrive first at the bottom.]

c.) The ball will arrive at the bottom first, but its rotational kinetic energy will be less than the rotational kinetic energy of the hoop when it arrives at the bottom. [From what has been stated above, the first part of this is correct--the ball does reach the bottom first. As for the rotational kinetic energy part, let's think through the problem: Both objects begin with the same amount of potential energy (same masses, same height above the incline's bottom). As each object begins to roll, energy is redistributed between translational kinetic energy ($.5mv^2$) and rotational kinetic energy ($.5I\omega^2$). The body that makes it to the bottom first will have the

larger translational velocity, so it will have the larger translational kinetic energy. As the translational and rotational kinetic energies must always add to the same amount (energy is conserved), the ball with the greatest translational kinetic energy must have a smaller rotational kinetic energy. In short, this response is true.]

d.) The ball will arrive at the bottom first, and its angular velocity will be greater than the angular velocity of the hoop when it arrives at the bottom. [From the analysis above, this response is false.]

22.) A block of mass 2 kg is found sliding with velocity 5 m/s against a frictional, circular wall of radius 3 meters. If the frictional force is a constant 4 newtons:

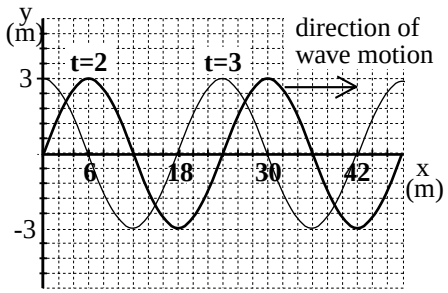
a.) The net force acting on the block at the instant alluded to is 4 newtons, and the distance the block travels before coming to rest is 1.25 meters. [The net force acting on the block must include the center seeking force that is motivating the block into circular motion. As 4 newtons is only the frictional component of the net force, this statement is false.]

b.) The net force acting on the block at the instant alluded to is 16.7 newtons, and the distance the block travels before coming to rest is 1.25 meters. [One approach in determining the net centripetal force is to sum the forces acting in the center seeking direction. Unfortunately, that sole force is the normal force between the block and the wall. If we knew the coefficient of friction between the block and wall, we could use our knowledge of the frictional force to determine N. Unfortunately, we haven't that information. The only other alternative is to determine a number that is equal to the net center seeking force. We can do that because we know from N.S.L. that the sum of the forces will equal ma, where $a = v^2/R$ in this case. Putting in the numbers, we get $mv^2/R = (2\text{ kg})(5\text{ m/s})^2/(3\text{ meters}) = 16.7$ newtons. Unfortunately, that is only the centripetal component to the net force (we have forgotten to include friction). This statement is false.]

c.) The net force acting on the block at the instant alluded to is 17.1 newtons, and the distance the block travels before coming to rest is 6.25 meters. [From above, we know that $F_c = 16.7$ nt and $F_{tangential} = 4$ newtons. As these are at right angles to one another, the magnitude of the two added vectorially together yields $F = (F_c^2 + F_{tangential}^2)^{1/2} = [(16.7\text{ nt})^2 + (4\text{ nt})^2]^{1/2} = 17.1$ newtons. Great jumping huzzahs, this is the answer to the first part. As far as the second part goes, conservation of energy maintains that $.5mv^2 - fd = 0$, or $d = .5mv^2/f = .5(2\text{ kg})(5\text{ m/s})^2/(4\text{ nts}) = 6.25$ meters. This statement is true.]

d.) None of the above. [Nope.]

23.) A graph of a traveling wave as seen at t = 2 seconds and t = 3 seconds is shown to the right. The wave velocity of the wave is:

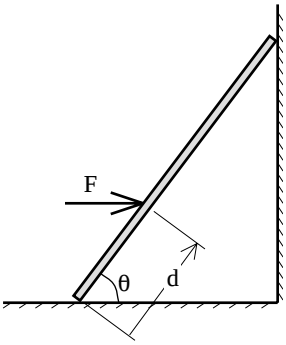


a.) 18 m/s. [Between t = 2 seconds and t = 3 seconds, the wave traveled 18 meters. As such, the wave velocity is 18m/s, and this response is true.]

b.) 6 m/s. [Not this one.]

c.) 4 m/s. [Not this one.]

d.) None of the above. [Not this one.]



24.) A ladder of mass m and length L sits on a frictionless floor perched against a frictionless wall. A force F acting at a distance d units up the ladder keeps the ladder from angularly accelerating. If F is removed:

a.) There will be only two forces acting on the ladder as it begins to accelerate. [Nope, there are three--the vertical force at the floor, gravity, and a normal force at the wall. This response is false.]

b.) The vertical force V acting at the floor will equal mg. [N.S.L. yields $V - mg = ma_y$. If the system was not accelerating, the vertical force acting at the floor (i.e., the only force acting at the floor) will, indeed, equal mg. As a_y is not zero, this statement is false.]

c.) The normal force at the wall must go to zero. [Again, if the body was not accelerating, a_x would be zero, and N, the only force acting in the horizontal, would have to be zero. Unfortunately, the body is accelerating, so N.S.L. yields $N = ma_x$, and N is not equal to zero. This response is false.]

d.) None of the above. [From what has been said above, this is it.]

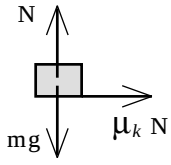
25.) A spinning skater has rotational kinetic energy equal to 32 joules when spinning with an angular velocity of 4 rad/sec. If the skater had kept the same position but had spun with an angular velocity of only 2 rad/sec, his kinetic energy would have been:

a.) Half the original amount, or 16 joules. [The question is, how does halving the angular velocity affect kinetic energy, assuming that all else is kept the same? As kinetic energy is a function of angular velocity squared, halving the angular velocity should quarter the kinetic energy. As such, the answer is 8 joules, and this response is false.]

b.) A quarter the original amount, or 8 joules. [This is the one.]

c.) Double the original amount, or 64 joules. [Nope.]

d.) Quadruple the original amount, or 128 joules. [Nope.]



26.) A box of mass m sits on the bed of a flatbed pick-up truck. The maximum acceleration the truck can experience without the box breaking loose is a_1. Assume the truck accelerates to the right.

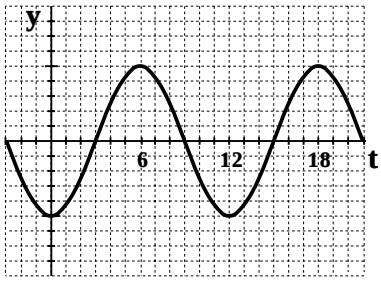
a.) The coefficient of static friction must be equal to a_1/g. [To determine this, we must do an f.b.d. on the box (shown to the right), then sum the forces. To begin with, note that the STATIC FRICTIONAL FORCE is to the right. (There are two ways to see this. First, from a fixed frame of reference (i.e., the ground) the box must move to the right. It might not move to the right as fast as the truck does (hence appearing to move to the left relative to the truck's bed), but it will move to the right somewhat. The only force available to effect that acceleration is friction. The other way to look at it is to consider the direction the block would slide relative to the truck bed if the box broke loose and began to slide. Relative to the bed, it would slide leftward. The static frictional force must be opposite that direction, or to the right.) N.S.L. yields $\mu_k mg = ma_1$, or $\mu_k = a_1/g$. This statement is true, but are there others?]

b.) The direction of the static frictional force acting on the box is to the left. [According to the arguments used above, this is false.]

c.) The direction of the static frictional force acting on the box is to the right. [According to the arguments presented above, this statement is true.]

d.) Both a and b. [Not so.]

e.) Both a and c. [This is the one.]



27.) A position versus time graph for a body oscillating in simple harmonic motion is shown to the right. The motion's angular frequency is:

a.) $\pi/12$ radians/second. [The time span between any two consecutive peaks or troughs is 12 seconds, so the period of the oscillation is 12 seconds per cycle. The inverse of that is the frequency at (1/12) cycle/second, and the angular frequency is that value multiplied by 2π, or $\pi/6$ radians per second. As such, this response is false.]

b.) $\pi/6$ radians/second. [From above, this is the one.]

c.) 1/12 radians/second. [Nope.]

d.) None of the above. [Nope.]

28.) When a 10 kg body is .5 meters up a 30^o incline plane, its velocity up the incline is observed to be 20 m/s. A constant 50 newton frictional force acts on the body.

a.) The net work done on the body will be negative and the body will slow down. {Note: This is an example of a problem that starts out looking like a beast only to be found to be a pussy cat. It is also an example of a typical A.P. maneuver--putting an incredibly EASY question in the middle of the test (versus putting the easy stuff at the beginning and having things get harder as you go). The moral of the story: When taking the A.P. test, don't assume that the last few questions will be hard--they may be gimmees.} [With the body moving up the incline, it is obviously slowing down. That means that energy is being removed from the system by negative work. This answer is true.]

b.) The net work done on the body will be positive and the body will slow down. [Just because the body is moving in the positive direction doesn't mean positive work is being done on it. This statement is false.]

c.) The body will have come to rest at the end of the 4 meters of motion. [Hopefully you have deduced that Response a is true. If Response d had been "Both a and c," then you'd have to determine whether this statement was true along with Response a. As Response d includes a choice you know is not true, and as there can only be one true response per question, this has to be false. I will show that this is true by doing the math ONLY because it might be educational for you, but if this was a real A.P. test you were taking, you'd be dumb to do the same. Time is of the essence on A.P. tests. Be clever--don't do any more than you absolutely have to do. The solution: The modified conservation energy equations is $KE_1 + U_1 + F_{extraneous} \cdot d = KE_2 + U_2$. Assuming that d is the distance up the incline (that means that d sin θ is the change of vertical position) to the stop point and the zero potential energy level is defined as the position of the body when its velocity is 20 m/s (i.e., .5 meters up the incline), and remembering that (1.) the mass is 10 kg, (2.) the initial velocity is 20 m/s, and (3.) friction does -fd's worth of work as the body moves to its resting point, we can write:

$$.5mv_1^2 + 0 + (-fd) = 0 + mg(d\sin\theta)$$

$$.5(10\text{ kg})(20\text{ m/s})^2 - (50\text{ nt})d = (10\text{ kg})(10\text{ m/s}^2)(d\sin 30^o)$$

$$\Rightarrow \quad d = 20 \text{ meters.}$$

This statement is false.]

d.) Both b and c. [Nope.]

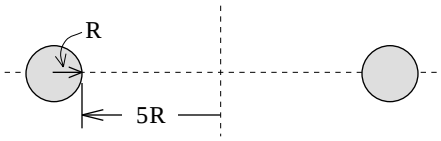
29.) A 2 kg mass with 100 joules of kinetic energy strikes a second body that is half the mass of the first. The collision is perfectly inelastic. What is the approximate velocity of the two bodies after the collision?

a.) 10.0 m/s. [Noting that $.5mv^2 = .5(2\text{ kg})v^2 = 100$ joules implies that v = 10 m/s, this response is actually the initial velocity of the 2 kg mass. As this would not be the after-collision velocity as well, this is false.]

b.) 8.16 m/s. [If energy were conserved through the collision, you could write 100 joules $= .5(2\text{ kg} + 1\text{ kg})v_{after}^2$, or $v_{after} = 8.16$ m/s. Unfortunately, energy is not conserved, and this response is false.]

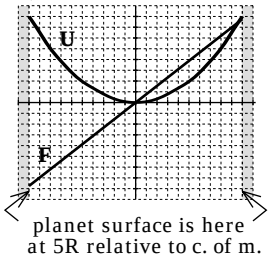
c.) 6.7 m/s. [Conservation of momentum coupled with the initial velocity found in Response a (i.e., 10 m/s) yields $(2\text{ kg})(10\text{ m/s}) = (3\text{ kg})v_{after}$, or $v_{after} = 6.7$ m/s. This response is true.]

d.) 3.3 m/s. [Nope.]

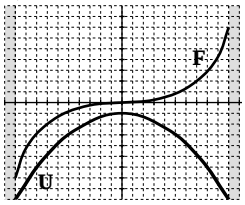


30.) Two asteroids, both of mass M and radius R, orbit one another in circular paths (the origin of the coordinate axis is at the center of mass of the two asteroids). A mass m is moved from the left asteroid to the right asteroid. The distance between the two asteroids is 10 R (see sketch). Which of the graphs shown describe the force on the mass and the mass's potential energy at various points as it makes its trip?

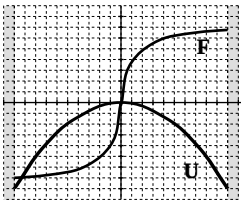
a.)



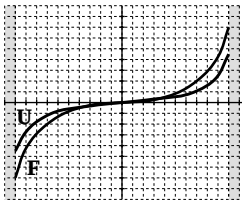
b.)



c.)



d.)



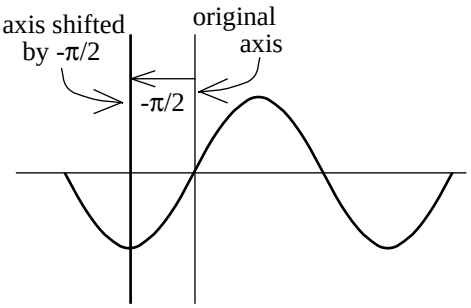
e.) None of these.

[Commentary: Think about the gravitational force acting on the mass when it is at -5R (i.e., when it is on the surface of the left-hand asteroid). The predominate pull at that point will be from the left-hand asteroid in the negative direction, relative to the coordinate axis centered at the system's center of mass (taking into account the considerably smaller force in the positive direction exerted by the right-hand asteroid, the net force on the mass is: $-GMm/(5R)^2 + GMm/(10R)^2$). As the mass proceeds toward the center of the system, the force will diminish until it reaches zero right at the center of mass of the system. How will the force diminish? It will drop by $1/r^2$ (think about the force expression). This eliminates Response a and c (Response c is eliminated because that force function does not approach the origin as $1/r^2$). Once the mass has passed the center of the system, the direction of the net force changes to positive and gets larger as the body gets closer to the right-hand asteroid. That part of the graph will be an inverted mirror image of the force function as it exists on the left-hand side of the system's center (that means Response b and d are still viable). Looking at the potential energy function, Response d can be immediately eliminated if we remember that the gravitational potential energy function associated with far-distant objects is always negative (i.e., it's - GmM/r). The graph shown in

Response b is a -1/r function, but one might question whether it is acceptable. Why might there be a problem? Because the object clearly has no net gravitational force acting on it at the center (the sum of the asteroid forces at that point will equal zero), but the graph suggests that the mass nevertheless has potential energy there. This is where a simplistic understanding of potential energy functions can get you into trouble. If you believe that when a body has potential energy it will always accelerate when released (common example: hold a pencil above the floor and let it go--it has potential energy and, when released, it accelerates downward), you must conclude that Response b cannot be true. Why? Because there is no net gravitational force at that point, yet the graph maintains that there is gravitational potential energy. The problem here can be resolved if we remember that, by definition, the gravitational potential energy associated with each object in the celestial system is defined as zero at infinity. With that being the case, the net potential energy due to the presence of an assemblage of large bodies turns out to be $- Gm_1m/r_1 - Gm_2m/r_2 - Gm_3m/r_3$ . . . etc. In short, the net potential energy held by the mass when at system's center will not be zero, it will be $- Gm_p m_o/(6R) - Gm_p m_o/(6R) = -2Gm_p m_o/(6R)$. Is this kosher? Sure it is! As long as the difference in the evaluation of the potential energy between any two points tells you how much work the net gravitational field does as a body moves between the two points, that's all that is important. The fact that the body has potential energy at some point but will not accelerate if released at that point is inconsequential. In short, Response b is OK.]

31.) The position function for an oscillating body is $x = 20 \sin (.6t - \pi/2)$. At $t = 0$, the body is at:

a.) Equilibrium. [You can do this in one of two ways. You can either plug $t = 0$ into the position expression and see what you come up with, or you can note that at $t = 0$, the position of the body is governed by the phase shift, which for this problem is $-\pi/2$. This phase shift pushes the axis to the left a quarter of a cycle (see sketch), which means the x position will be negative and as big as it can be. In short, it will not be at equilibrium. This is false.]



b.) Its maximum positive displacement. [Nope.]

c.) Its maximum negative displacement. [Yup.]

d.) None of the above. [Nope.]

32.) The moment of inertia about the central axis of a disk (call it disk A) of radius R and mass m is known to equal I_A. A similar disk (call this disk B)has half the mass of A and half its radius.

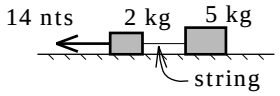
a.) The moment of inertia about disk B's central axis will be the same as that of disk A. [Even if you don't remember the actual expression for the moment of inertia of a disk (it happens to be $(1/2)mR^2$), all moment of inertia expressions have mass and a distance squared term multiplied together (i.e., an mR^2-type quantity). Completely ignoring the constant that goes out in front of that expression, we can deduce that halving the radius and mass will take on a mathematical form that is something like $(m/2)(R/2)^2 = (1/8)mR^2$. In short, the moment of inertia will drop to 1/8 of disk A's moment of inertia. This response is false.]

b.) The moment of inertia about disk B's central axis will be $(1/2)I_A$. [From the discussion above, this response is false.]

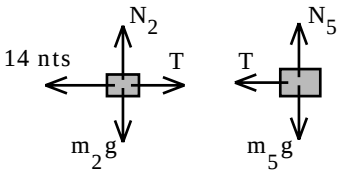
c.) The moment of inertia about disk B's central axis will be $(1/4)I_A$. [From the discussion above, this response is false.]

d.) None of the above. [This is the one.]

33.) Two blocks on a frictionless surface are attached by a string (see sketch). If a 14 newton force is applied as shown, the tension in the string will be:



a.) 2.0 nts. [An f.b.d. on the two masses is shown to the right. Using it, the acceleration of the system is found to be 2 m/s^2. From m_5's f.b.d., T = 5a = 10 nts, and this statement is false.]



b.) 2.8 nts. [The statement is false.]

c.) 4.67 nts. [The statement is false.]

d.) 10.0 nts. [The statement is true.]

34.) A stationary mass A is struck head on elastically by a second body B of the same mass and whose velocity before the collision is v_o. After the collision, mass A's velocity is:

a.) Zero. [When mass B strikes an equal sized stationary mass A, B will stop dead and A will leave with B's initial velocity (this is all a consequence of the conservation of both energy and momentum for the situation). That means mass B's final velocity must be zero, while mass A's final velocity is v_o. This response is false.]

b.) $v_o/4$. [From above, this won't do.]

c.) $v_o/2$. [From above, this won't do.]

d.) v_o. [From above, this is the one.]

e.) None of the above. [From above, this won't do.]

35.) A 1 kg body sits a distance h = 2 meters above a tabletop. The tabletop is d = 1.2 meters high. Three students calculate the body's potential energy. Assuming the magnitude of the gravitational acceleration is 10 m/s^2, which student got it right?

a.) Student A with a calculated potential energy of 20 joules. [If you take the tabletop to be the zero potential energy level, mgh = (1 kg)(10 m/s^2)(2 m) = 20 meters. Although this statement is true, there is a problem. There may be other acceptable responses.]

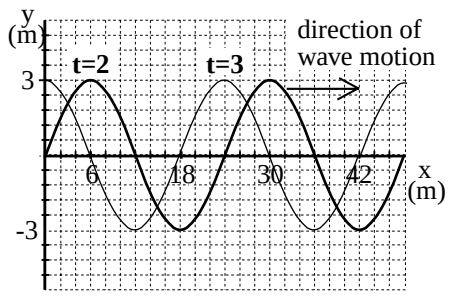
b.) Student B with a calculated potential energy of 32 joules. [If you take the ground to be the zero potential energy level, mgh = (1 kg)(10 m/s^2)(3.2 m) = 32 meters. Although this statement is true, there is a problem. There may be other acceptable responses.]

c.) Student C with a calculated potential energy of 0 joules. [If you take the two meters above the tabletop to be the zero potential energy level, the potential energy of the body at that point will be zero. There is nothing wrong with this. Potential energy functions are usually defined as zero where the force is zero, but there is no place near the surface of the earth where gravity is zero. As such, this constant force's potential energy function can be defined to be zero wherever it's convenient. This statement, along with all of the others, is true.]

d.) Any of the three could be correct depending upon where the zero level for the potential energy function was defined. [For the reasons outlined above, this is it.]

MULTIPLE CHOICE SOLUTIONS--MECHANICS

TEST IV



1.) A graph of a traveling wave as seen at t = 2 seconds and t = 3 seconds is shown to the right. The wavelength of the wave is:

a.) 18 meters. [We can take the wavelength directly off the graph. One cycle spans 24 meters, which means the wavelength is 24 meters and this response is false.]

b.) 24 meters. [According to the analysis above, this is the one.]

c.) 30 meters. [Nope.]

d.) Can't tell with the information given. [Nope.]

e.) None of the above. [Nope.]

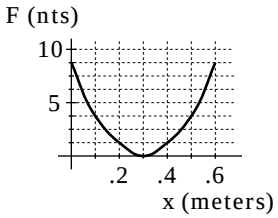
2.) The position function for an oscillating body is $x = 20 \sin (.6t - \pi/2)$. At t = 0, the body's velocity is:

a.) 20 m/s. [There are two ways to do this: you can grunting through the math or be clever. The math yields a velocity expression equal to $v = (.6)(20)\cos(.6(0)-\pi/2) = 0$. Being clever requires you to notice that at t = 0, the displacement is at the negative maximum (i.e., $A\sin(-\pi/2) = -A$). The velocity at the extremes is zero (it is a turn around point), so the velocity here at t = 0 must be zero. In any case, this response is false.]

b.) 12 m/s. [Nope.]

c.) 2.0 m/s. [Nope.]

d.) None of the above. [This is the one.]



3.) A 4 kg body moves from x = 0 to x = .6 meters one-dimensionally over a frictionless surface. It takes 12 seconds to execute the trip. The single force acting on the body during the motion is graphed to the right. The force function in this case is $F = (x - 3)^2$.

a.) The net work done by the force over the first .3 meters is negative, whereas the work done over the final .3 meters is positive. [As the force is always in the same direction (it gets larger and smaller, but it always does so in the positive region), the work will always be positive. This statement is false.]

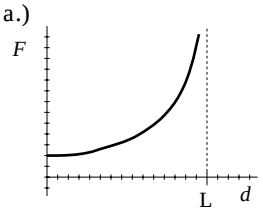
b.) The net work over the entire interval is $(x^3/3) - 3x^2 + 9x$. [This is the integral of Fdx. This is a work quantity, but it is not evaluated between x = 0 and x = .6 meters. As picky as this is going to seem, this statement is false.]

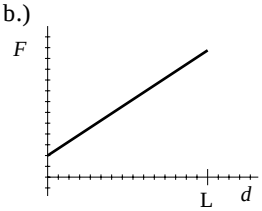
c.) The work done between x = .1 meters and x = .6 meters is 3.5 joules. [Evaluating the integral of Fdx between the limits does, indeed, give you 3.5 joules.]

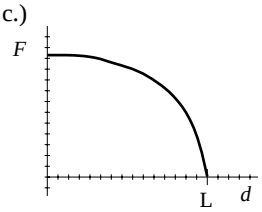
d.) None of the above. [Nope.]

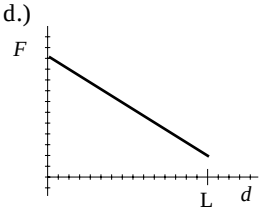
4.) A ladder of mass m and length L sitting on a frictionless floor at an angle θ is perched against a frictionless wall. A horizontal force F acting at a distance d units up the ladder

keeps the ladder from angularly accelerating. Which graph characterizes the relationship between F and d?

a.) 

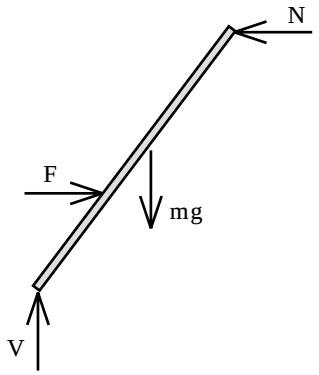
b.) 

c.) 

d.) 

e.) None of these.

[Commentary: The temptation is to assume that the <u>torque</u> required of F to keep the ladder stationary must be a constant. If that is so, the larger we make d (i.e., the farther F is from the point of contact with the floor), the smaller F has to be. There is a problem with this thinking which is noticed if you actually do the problem: Using the f.b.d. shown to the right, the sum of the forces in the horizontal yields $F - N = ma_y = 0$, or $F = N$. Summing the torques about the floor yields: $-F(x \sin\theta) - mg[(L/2)\cos\theta] + NL\sin\theta = I\alpha = 0$. Substituting F for N and solving yields:



$$F = \frac{mg_{[(L/2)\cos\theta]}}{-d\sin\theta + L\sin\theta} = \frac{mgL\cos\theta}{2(L-d)\sin\theta} = \frac{mgL\cot\theta}{2}\frac{1}{(L-d)}.$$ What does this tell us? When the force is at the floor and d = 0, the force is $\frac{mg}{2\cot\theta}$--a finite quantity. But as d approaches L (i.e., as the force gets closer and closer to the wall), the denominator gets smaller and smaller and F goes toward infinity. The graph that characterizes this kind of function is Graph a.]

5.) An ideal spring is hung free in the vertical. A 5 kg mass is attached without elongating the spring, and the mass is released from that point to oscillate up and down. If the mass had been gently lowered (it wasn't, but if it had been), it would have moved 2.5 meters downward before coming to equilibrium. Approximate g to equal 10 m/s^2.

a.) The spring constant is 2 nts/meter, the angular frequency of the motion will be 2 radians/second, and the maximum velocity will be 5 m/s. [The spring constant is defined as the force per unit displacement required to stretch or compress a spring. In this case, the weight of a 5 kg mass (that is, (5 kg)(10 m/s^2) = 50 nts) elongates the spring 2.5 meters. The ratio of (50 nt)/(2.5 meters) = 20 nt/m is the force constant. As such, this response is false. The fact that the spring and mass, when gently lowered, would have reached 2.5 meters, means that the amplitude of the motion will be 2.5 meters (the amplitude is the maximum displacement from equilibrium, which is what 2.5 meters is in this problem).]

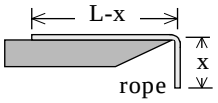
b.) The spring constant is 20 nts/meter, the angular frequency of the motion will be 4 radians/second, and the maximum velocity will be 2 m/s. [This has the correct spring constant. The angular frequency will equal $(k/m)^{1/2} = [(20\text{ nt/m})/(5\text{ kg})]^{1/2} = 2$ radians per second. The angular frequency here is wrong, so this response is false.]

c.) The spring constant is 2 nts/meter, and neither the angular frequency nor the maximum velocity can be calculated as there is not enough information to do so. [Wrong spring constant.]

d.) The spring constant is 20 nts/meter, the angular frequency of the motion will be 2 radians/second, and the maximum velocity will be 5 m/s. [The spring constant and angular

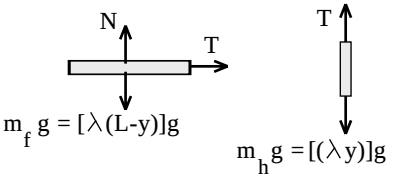
frequency are both correct and the maximum velocity will be $v_{max} = \omega A = (2 rad/sec)(2.5 m) = 5 m/s$, so this response is true.]

e.) None of the above. [Nope.]

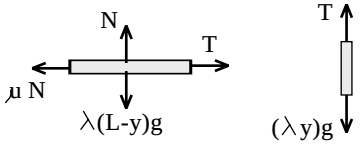


6.) At a given instant, a fat rope of length L and mass density λ (note: mass density is the mass per unit length of rope) is as shown hanging over the edge of a table.

a.) Assuming the table is frictionless, the rope's acceleration will be dependent only upon λ. [The only thing accelerating the rope is gravity pulling down on the part of the rope that is hanging over the table's edge (though there is a tension force due to the inertia of the flat-lying rope). The amount of hanging mass is equal to the mass per unit length λ times the length y of rope hanging, or $m_h = \lambda y$. By similar logic, the amount of flat-lying mass is $m_f = \lambda(L-y)$. The f.b.d.'s for the situation are shown above. N.S.L. on the hanging mass yields: $T - m_h g = m_h a$, or $T - (\lambda y)g = -(\lambda y)a$. N.S.L. on the flat-lying mass yields: $T = m_f a$, or $T - [\lambda(L-y)]a = 0$. Eliminating T yields $a = [y/(L-y)]g$. As the rope's acceleration isn't dependent upon the mass density of the rope at all, this statement is false.]



b.) Assuming the table is frictional and the rope is stationary and not accelerating, the coefficient of static friction between the rope and the table will be y/(L-y). [F.b.d.'s on both sections are shown to the right. N.S.L. applied to the tabled piece yields (after substituting $N = \lambda(L-y)g$ and putting $a = 0$):



$$T - \mu_k \lambda(L-y)g = 0$$

N.S.L. on the hanging piece yields:

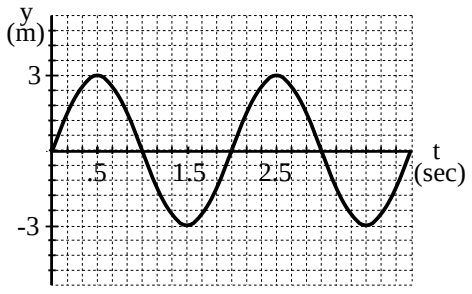
$$T - \lambda y g = 0.$$

Substituting in for T and dividing out λ, we get $\mu_k = y/(L-y)$. This statement is true.]

c.) Back to the frictionless situation, the acceleration will be a constant. [From Part a, the acceleration will be $a = [y/(L-y)]g$. Clearly, a changes as y increases. This statement is false.]

d.) Both b and c. [Nope.]

7.) At a particular position along the path of a wave, the displacement vs. time is graph to the right.



a.) The amplitude of the wave is 3 meters, its wavelength is 2 meters, and its frequency is .5 Hz. [Looking at the graph, it is obvious that the amplitude is 3 meters. The wavelength is a quantity that is distance related, not time related. If we knew the wave velocity, which we don't, we could use the time related information on the graph to determine the frequency of the motion, then use $v = \lambda\nu$ to

determine the wavelength. But because we don't have the velocity information, we don't have enough information to determine the wavelength for this situation, and this response is false.]

b.) The amplitude of the wave is 6 meters, it is impossible to tell what the wavelength is from the information given, and its frequency is 2 Hz. [Wrong amplitude--this response is false.]

c.) The amplitude of the wave is 3 meters, it is impossible to tell what the wavelength is from the information given, and its frequency is .5 Hz. [The amplitude is correct as is the observation that the wavelength can't be determined for this situation. As for the frequency, it would appear from the graph that one cycle takes 2 seconds to sweep out. That gives us a frequency of (1/2) cycles/second, which means this response is true.]

d.) None of the above. [Nope.]

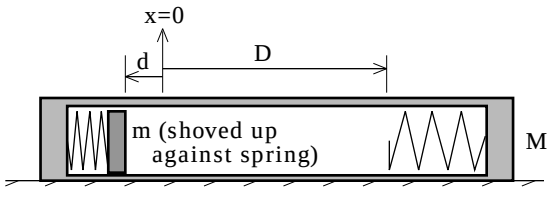
8.) A mass m is released 6R meters from the surface of a moon whose radius is R and whose mass is M. If released from rest, how fast will the mass be traveling just before it hits the surface of the planet?

a.) $[2GM/(6R)]^{1/2}$. [This is a conservation of energy problem. There are two things that can potentially be tricky about it. The first is that you have been given the distance between the object and the surface of the planet, not the object and the planet's center (the latter is what is needed for the potential energy function). The second is that the body will have gravitational potential energy when it is on the planet's surface. If you neglect to notice either of these potential pitfalls, you will write the conservation of energy equation as: KE_1 (this is zero) + U_1 + W_{ext} (this is zero) = KE_2 + U_2 (incorrectly assumed to be zero). That is, $-GmM/(6r) = .5mv^2$, concluding that $v = [-2GM/(6R)]^{1/2}$. Hopefully, you would notice the negative sign under the square root and not peremptorily change it to make the equation work. This response is not true.]

b.) $[-2GM/(7R)]^{1/2}$. [If you didn't notice that there is potential energy associated with the planet's surface, but you did get the right distance value for the potential energy function, you wrote $-GmM/(7r) = .5mv^2$, concluding that $v = [-2GM/(7R)]^{1/2}$. If you ignored the negative sign under the radical (ARGH), you got the incorrect response quoted above.]

c.) $[12GM/(7R)]^{1/2}$. [If you noticed all of the twists, you wrote: $-GmM/(7R) = .5mv^2 - GmM/R$, concluding that $v = [12GM/(7R)]^{1/2}$. This response is true.]

d.) None of the above. [Nope.]



9.) An ideal spring (spring constant k) is mounted inside a hollowed out block whose mass is M. With M held stationary, a second mass m is pushed up against the spring until the spring's displacement is d. A second spring with the same spring constant is positioned at the end of the hollow. The net distance between the ends of the two unsprung springs is D (see sketch). The shaft applies a constant frictional force f on m only over the distance marked. Assume M is fixed to the ground so that it cannot move. The spring is released accelerating m.

a.) Just after leaving the spring, m's velocity will be $(k/m)^{1/2}d$. [The spring is ideal, so energy is conserved through the acceleration. Using conservation of energy for that period, we get:

$.5kd^2 = .5mv_1^2$, or $v_1 = k(/m)^{1/2}d$. This response is true. Are there any other responses that are true?]

b.) The impulse provided to m by the spring will equal $(km)^{1/2}d$. [The impulse is either $F\Delta t$ or $\Delta(mv)$. We can easily determine the latter. That yields: $\Delta p = (mv_1 - mv_o) = (m\ (k/m)^{1/2}d - 0) = (km)^{1/2}d$. This statement is true. Are there others?]

c.) If m = .1 kg, D = 1 meter, f = 1 nt, and m's velocity just as it leaves the spring is 7 m/s, the time it takes m to reach the second spring will be approximately .16 seconds. [We need to use the modified conservation of energy theorem to determine the velocity of m as it reaches the second spring, then use the modified conservation of momentum equation to determine the time interval. Noting that the frictional force will do negative work in the amount -fD, conservation of energy yields: $.5m(v_{clear})^2 - fD = .5mv_2^2$. With numbers, this becomes $.5(.1\ kg)(7m/s)^2 - (1\ nt)(1\ m) = .5(.1\ kg)v_2^2$. From this, $v_2 = 5.4$ m/s. Friction is acting as an external force here (M is restrained from moving, hence the force M applies to m in the way of friction is external), so the modified conservation of momentum equation yields: $mv_{clear} - f\Delta t = mv_2$. Putting in the numbers yields $(.1\ kg)(7\ m/s) - (1\ nt)\Delta t = (.1kg)(5.4\ m/s)$, or $\Delta t = .16$ seconds. This response is true.]

d.) Both a and b but not c. [Response c is true also. This response is false.]

e.) Response a, b, and c. [This is the one.]

10.) Moving under the influence of gravity, a body's kinetic energy is KE_o. If, due to its natural motion in the gravitational field, the body's velocity changes to one-third of its original value, the change of the body's potential energy will equal:

a.) $(1/3)KE_o$. [Kinetic energy is a function of v^2. If the velocity decreases to one-third its original value, the kinetic energy decreases to $(1/3)^2KE_o$, or (1/9) of its original value. That means that 8/9 of the kinetic energy is converted to potential energy. This statement is false.]

b.) $(1/9)KE_o$. [From above, this statement is false.]

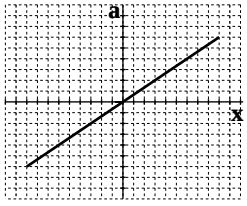
c.) $(8/9)KE_o$. [From the commentary in Response a, this statement is true.]

d.) $(3)KE_o$. [From above, this statement is false.]

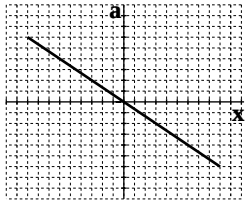
e.) $(9)KE_o$. [From above, this statement is false.]

11.) A graph of the acceleration vs. position of a body oscillating in simple harmonic motion looks like:

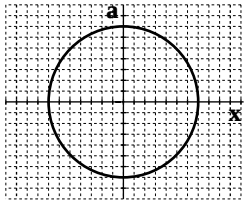
a.)



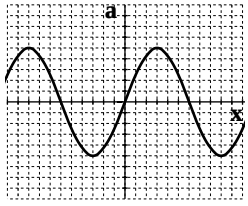
b.)



c.)



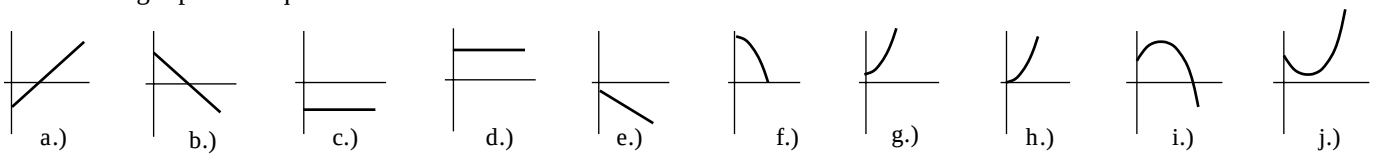
d.)



e.) None of these.

[Commentary: The position and acceleration are π radians out of phase with one another, which means that when one is big and positive, the other is big and negative. The graph that characterizes that kind of relationship is Graph b, which is the correct response for this question.]

--The graphs for questions #12 and #13:



12.) If all of the graphs were that of Position vs. Time, the only graphs listed below that would fit a kinematic situation would be:

a.) Graphs a, b, and e. [The only criteria that must be met to use kinematics is that the acceleration of the motion be a constant. Position vs. Time graphs that depict constant changes of position (graphs a, b, and e) correspond to constant velocity situations in which the accelerations are also constant and equal to zero. So, is this the right response? The answer to that hinges on whether there are other graphs in addition to these that will work. If there are, this response is not COMPLETE and won't do.]

b.) Graphs a, b, c, d, and e. [Along with graphs a, b, and e, graphs c and d also correspond to constant velocity situations (these are no-motion, constant position situations in which the velocities are a constant zero). Is this the right response? The answer to that hinges on whether there are other graphs in addition to these that will work. If there are, this response is not complete and won't do.]

c.) Graph f, g, h, i, and j. [Parabolic Position vs. Time graphs correspond to situations in which the velocity is changing in a constant manner--exactly what is needed for kinematics. So, is this the right response? Given that we have already determined that Responses a and b fit the bill, this response is incomplete and, hence, lacking.]

d.) All of the graphs. [This should now be obvious from what has been said above.]

e.) None of the graphs. [Nope.]

13.) If all of the graphs were that of Velocity vs. Time, the only graphs listed below that would fit a kinematic situation would be:

a.) Graphs a, b ,and e. [Velocities that change at a constant rate correspond to constant acceleration (i.e., kinematic) situations. Graphs a, b, and e do this, but are there others?]

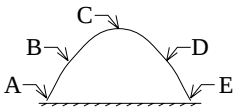
b.) Graphs a, b, c, d, and e. [In addition to graphs a, b, and e, graphs c and d also depict constantly changing velocities (their changes may be zero, but they are still constant zeros). Are there other possibilities? The answer is NO (see the next possibility). This response is true.]

c.) Graphs f, g, h, i, and j. [The parabolic curves associated with these graphs do not represent velocity changes that are constant, which means that they do not correspond to constant acceleration situations. This response is false.]

d.) All of the graphs. [Nope.]

e.) None of the graphs. [Nope.]

----------end of section----------



14.) A projectile is fired in a frictionless environment as shown to the right.

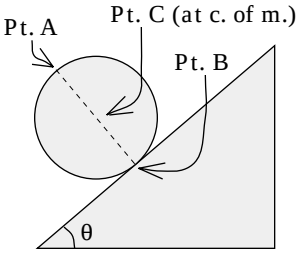
a.) The work gravity does while the body moves from Point B to Point D is zero. [Qualitatively, gravity will do equal but opposite work going up as coming down between these two points, so the net work done by gravity should be zero. From a mathematical standpoint, gravity is a conservative force. As such, the work it does as a body moves from one point to another is endpoint dependent. In the case of the gravitational potential energy function, what is important is the y coordinate of the initial and final positions (remember, U_{grav} = mgy near the earth's surface). As Points B and D are at the same y coordinate, the body's potential energy when at those two positions will be the same. As $W_{grav} = -\Delta U_{grav}$, the net work done in this case will be zero and this statement is true.]

b.) The work gravity does while the body moves in its downward arc between Point C and Point D is negative. [The force is in the negative direction and the y component of the displacement is in the negative direction, but that does not mean that the work done by the force is negative. It isn't the direction of the force alone, or the displacement alone, that determines the sign of a work quantity. What determines the sign is the direction of the two vectors relative to one another. If both are down, the angle between the two is less than 90^o and the cosine of the angle will be positive. Viewed in a different light, the force is motivating the body to pick up speed, which means the energy of the system is increasing and the work done must be positive. This statement is false.]

c.) The amount of work gravity does as the body moves from Point A to Point E is twice the amount of work done between Point A and Point C. [In fact, because the two points have the same y coordinate, the change of potential energy will be zero and the work done by gravity between those two points will be zero. Put another way, gravity does negative work on the body on its way up, and an equal amount of positive work on the way down. The net work round trip is zero. This statement is false.]

d.) Both a and b. [Nope.]

e.) Both a and c. [Nope.]



15.) A ball rolls down an incline. Its angular acceleration is:

a.) $mg(R\cos\theta)/I_{cm}$, where I_{cm} is the moment of inertia about the ball's center of mass. [If torques had been summed about the point of contact between the ball and incline (Point B), and if $r_\perp$ for mg had been miscalculated to be R(cosθ) instead of R(sin θ), and if that torque sum was set equal to $I_{cm}\alpha$ instead of $I_B\alpha$, we would have gotten the answer given. In short, this statement is false.]

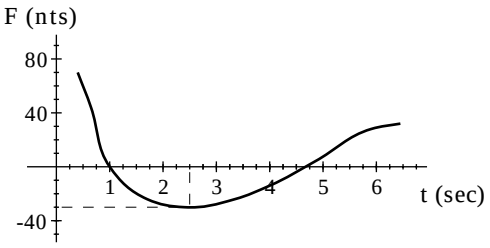
b.) fR/I_B, where f is the frictional force (it's equal to mg sinθ - ma_{cm}) and I_B is the moment of inertia about Point B. [The frictional force produces no torque about Point B as the frictional force passes <u>through</u> Point B. This response is false.]

c.) $a_A/(2R)$, where a_A is the acceleration of Point A. [The relationship between the ball's angular acceleration α and the translational acceleration of the ball's center of mass is a_{cm} = $(1R)\alpha$. Why? Because the acceleration we are looking at is that of a point located a distance R units from the instantaneous axis of rotation of the ball at its point of contact with the incline (remember, we can evaluate the motion either by looking at the rotation about and translation of the center of mass, or from the perspective of a pure (instantaneous) rotation about the point of contact (Point B)--in this case, we are using the latter approach). Likewise, a point 2R units from the instantaneous axis of rotation (i.e., the acceleration at Point A) will have a translational

acceleration $a_A = (2R)\alpha$. Rearranging this expression yields $\alpha = a_A/(2R)$. This statement is true.]

d.) Both a and c. [Nope.]

e.) Both b and c. [Nope.]



16.) A non-linear frictional force in the x direction is graphed to the right. If it is the only force acting on a body, and if the body's mass is 3 kg:

a.) At t = 2.5 seconds, the mass is moving in the -x direction. [There is no way of telling the direction of the motion given the direction of force. All the force will do is change the motion, speeding the body up or slowing it down. This response is false.]

b.) At t = 2.5 seconds, the mass's velocity will be approximately zero. [The slope of the force function is zero at t = 2.5 seconds, but all that means is that the force is not changing at that point in time. It tells us nothing about the velocity. This statement is false.]

c.) The acceleration of the mass at t = 2.5 seconds is approximately -10 m/s^2. [Using N.S.L. to evaluate the situation (approximating F to equal 30 newtons at t = 2.5 seconds) yields $F = ma \Rightarrow (-30\text{ nt}) = (3\text{ kg})a$, or $a = -10\ m/s^2$. This statement is true.]

d.) The velocity of the body would either have been positive until t = 1 second, whereupon it would have changed, or vice versa. [Just because the force changes direction at t = 1 second doesn't mean the body changes directions at that point in time. This statement is false.]

e.) None of the above. [Nope.]

--The following information pertains to Problems 17 through 20: Projectiles A, B, C, and D are fired at the same time from a height h meters above the ground. With the exception of Projectile D, which is dropped from rest, all the projectiles (i.e., Projectiles A, B, and C) have the same muzzle velocity v_o, (though each is fired at a different angle--see the sketches below and note that the angle defined as θ is the same in all cases). It takes t_1 seconds for Projectile A to get to the top of its flight. It takes t_2 seconds forProjectile D to reach the ground.

Projectile A Projectile B Projectile C Projectile D

17.) The initial velocity of Projectile C will be:

a.) The same as the initial velocity of Projectile B. The two will also have the same acceleration. [They have the same acceleration and initial velocity magnitudes, but their initial velocity directions will be different.]

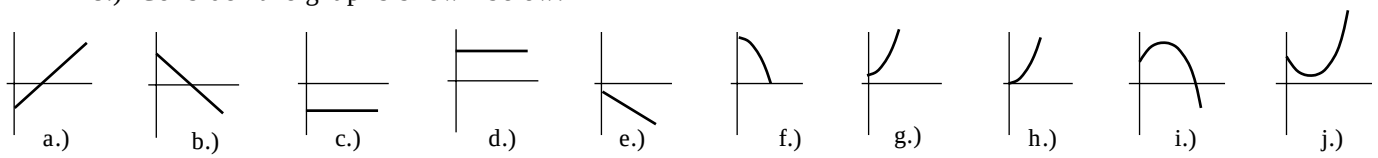
b.) The same as Projectile B after time t_1. The two will also have different accelerations. [At time t_1, Projectile B will have the same velocity as Projectile A has after time

$2t_1$. At this time, the velocity magnitude of both will be v_o at an angle θ <u>below</u> the horizontal. In other words, this part of the statement is true. Unfortunately, the "different accelerations" part of the statement makes it a false response.]

c.) The same as Projectile A after time $2t_1$. The two will also have the same acceleration. [From the discussion in Response b, this statement is true.]

d.) None of the above. [Nope.]

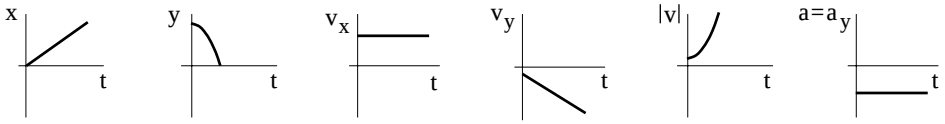
18.) Consider the graphs shown below:



Projectile C's:

a.) Y-component of Position vs. Time graph looks like graph e.
b.) X-component of Position vs. Time graph looks like graph a.
c.) Y-component of Velocity vs. Time graph looks like graph e.
d.) X-component of Velocity vs. Time graph looks like graph c.
e.) Y-component of Acceleration vs. Time graph looks like graph b.

[Commentary: A graph for each of the major parameters for this situation is shown below. This is something you should have been able to both visualize and sketch on your own. If you think you wouldn't have been able to do that, use the graphs provided as a stimulus to do the visualization part. Response c above is the one.]



--For questions 19 and 20:At the same time Projectiles A, B, and C are fired, Projectile D is dropped from rest a distance h meters above the ground. It takes this projectile t_2 seconds to reach the ground.

19.) The time t_2:

a.) Depends only on h and constant(s). [Using $x_2 = x_1 + v_1t + .5at^2$ with a = -g , $v_1 = 0$, and $x_2 - x_1 = 0 - (h) = -h$, we get the relationship $-h = .5(-g)t^2$. This selection is evidently true, but are there other true statements?]

b.) Is the same time it takes Projectile B to hit the ground. [The time it takes to hit the ground is a y-motion related question. As the initial velocity in the y-direction for both cases is the same (it's zero), and as the gravitational acceleration is the same in both cases, the two projectiles should hit the ground in the same amount of time. This statement is true. Other truths?]

c.) Is more than the time it takes Projectile C to hit the ground, but less than the time it takes Projectile A to hit. [Because Projectile C had a downward initial velocity in the y-direction, it will take less time to hit the ground than does Projectile D which had no initial velocity in the y-direction. And as Projectile A had an upward y-component of its velocity, it will take more time to reach the ground. This statement is true.]

d.) Both a and b, but not c.

e.) All of the above except d. [This is the one.]

20.) If h were doubled, Projectile D's:

a.) Time to touch down would double. [Using $x_2 = x_1 + v_1t + .5at^2$ with $a = -g$, $v_1 = 0$, and $x_2 - x_1 = 0 - (h) = -h$, we get the relationship $-h = .5(-g)t^2$. This means that $t = (2h/g)^{1/2}$. If h is doubled, t goes up by a factor of $(2)^{1/2}$, not by a factor of 2. This response is false.]

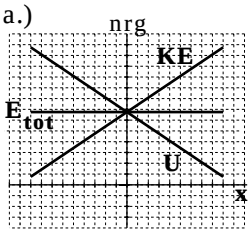
b.) Velocity just before touch down would double. [Using $v_2 = v_1 + at$ with v_2 being the velocity just before hitting the ground, $v_1 = 0$, and $a = -g$, we get $v_2 = -gt$. We have already determined that doubling h does not mean t doubles, so this statement if false.]

c.) Acceleration just before touch down would double. [Acceleration in these cases is always constant in both the x and y-direction. False.]

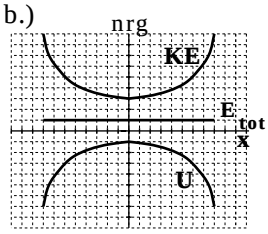
d.) None of the above. [This must be the one.]

21.) Which graph combination depicts the potential energy, the kinetic energy, and the total energy of an ideal vibrating spring?

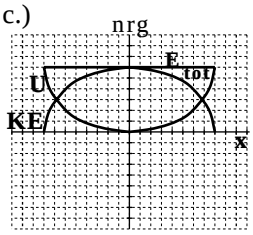
a.)



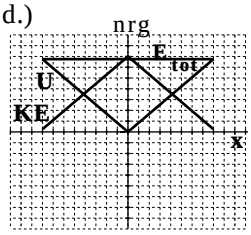
b.)



c.)



d.)



e.) None of these.

[Commentary: The kinetic energy and potential energy associated with a vibrating spring have two characteristics that will be helpful here. Both are positive ($.5mv^2$ and $.5kx^2$), which eliminates Graph b, and both must add to the constant total energy, which eliminates Graph a. We know that the potential energy function is related to the square of the displacement, so Graph d isn't going to do the job, and we can see that Graph c has all of the characteristics we are looking for. In fact, that is the correct response.]

22.) <u>In theory</u>, a mass must travel how fast to escape the earth's gravitational field?

a.) It is not possible to tell for this situation because the mass of the object was not given. [Just as the acceleration of a freely falling body is not a function of mass (due to the inertial mass/gravitational mass interplay), an object's escape velocity is also not a function of mass. This response is false.]

b.) A mass can never escape the earth's gravitational field. [In theory, it can when it reaches infinity . . . IN THEORY! This response is false.]

c.) $(2GM/R_e^2)^{1/2}$. [Oh, so close. Problems like this require the use of conservation of energy. As the body leaves earth, its kinetic energy is $.5m(v_{esc})^2$. Its gravitational potential energy at the earth's surface is $-GmM/R_e$. As the energy boost that initially accelerated the satellite has already happened (that's how the body got its kinetic energy), W_{ext} is zero. Once

the mass has just barely escaped the earth's field, all of its energy will have been spent leaving it with no kinetic energy (i.e., $KE_2 = 0$), and as the body is theoretically at infinity, its gravitational potential energy is zero (i.e., $-GmM/\infty = 0$). In other words, the conservation of energy equation can be written as: $.5mv_{esc}^2 + (-GmM/R_e) = 0$, or $v_{esc} = (2GM/R_e)^{1/2}$, and this response just misses the mark.]

d.) None of the above. [This is the one.]

23.) When a 10 kg body is .5 meters up a 30^o incline plane, its velocity is observed to be 20 m/s. A constant 50 newton frictional force acts on the body. Over the next 4 meters (note that a 4 meter run at 30^o will produce a rise of 2 meters):

a.) The work gravity does on the body will be greater than the work friction does. Also, when the body finally gets to the top of its run, it will stop momentarily, then begin to accelerate back down the incline. [The first part of this is straightforward. Taking the zero potential energy level to be where the body has velocity 20 m/s, the work gravity does is $W_{grav} = -\Delta U = -(mgy - 0) = -(10\text{ kg})(10\text{ m/s}^2)(2) = -200$ joules. The work friction does is $-fd = -(50\text{ nt})(4\text{ m}) = -200$ joules. This statement is false.]

b.) The work gravity does on the body will equal the work friction does. Also, when the body finally gets to the top of its run, it will stop and remain there (i.e., it won't be accelerated down the incline). [From the analysis in Response a, the first part of this statement is true. As for the second part, the component of friction down the incline will be $mg\sin\theta = (10\text{ kg})(10\text{ m/s}^2)\sin 30^o = 50$ newtons. At the top, the body will stop and kinetic friction will yield to static friction. Static friction is always greater than kinetic friction, so even if the component of gravity had been a tad larger than the kinetic frictional force, static friction would have stopped the body from accelerating down the incline. This statement is true.]

c.) The work gravity does on the body will be the same as the work friction does. Also, when the body finally gets to the top of its run, it will stop momentarily, then begin to accelerate back down the incline. [The second part of this statement is analyzed in Response b. From that, this statement is false.]

d.) The work gravity does on the body will be less than the work friction does. Also, when the body finally gets to the top of its run, it will stop and remain there (i.e., it won't be accelerated down the incline). [From the analysis in Response a, the first part of this statement is false.]

24.) Newton's Second Law is applied to a system. After a free body diagram is drawn and the forces summed, the equation $-32x^3 = 2a$ emerges, where x is the position and a is the acceleration of the body.

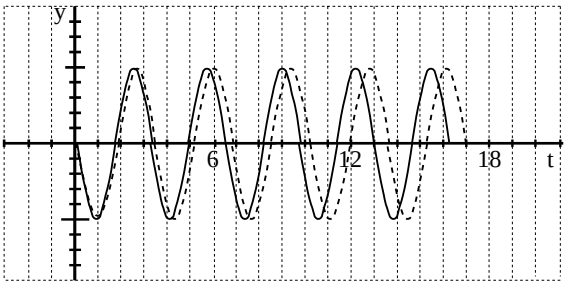
a.) This equation does not characterize an oscillatory system. [If this equation had been $-32x = 2a$, we would have had the characteristic equation for an ideal spring (i.e., $-kx = ma$). If that had been the case, a positive displacement x would yield a negative force and a negative displacement x would have yielded a positive force. That is, the force would always have been oriented back toward the equilibrium position. With the expression $-32x^3 = 2a$, we certainly don't have simple harmonic motion, but the $-x^3$ term does generate a restoring force that will motivate this system to oscillate. In short, this response is false.]

b.) This equation does characterize an oscillatory system and the motion is simple harmonic in nature. [This is an oscillatory system, but the equation is not that of simple harmonic motion. This statement is false.]

c.) This equation does characterize an oscillatory system and the motion's frequency is 4 radians per second. [As it is not simple harmonic motion, we have no way to tell what the frequency is, so this response must be false.]

d.) Both Response b and c. [Nope.]

e.) None of the above. [This is the one.]



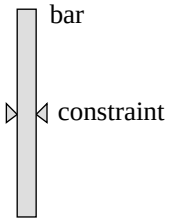
25.) Two waves exist simultaneously in the same medium (they are graphed on the same axis to the right). The beat frequency for the situation is approximately:

a.) (2/13) - (2/17). [Beats are produced when two very close frequencies superimpose. In doing so, the net waveform will sometimes be in phase and, hence, constructively interfere, and sometimes be out of phase, destructively interfering. The frequency of the warble that ensues is called the beat frequency, and the beat frequency will always equal the difference between the two superimposing waves. In this case, the frequency of the shorter wavelength is 4 cycles per 13 seconds while the frequency of the longer wavelength is 4 cycles per 17 second. The difference between these two frequencies is (4/13) - (4/17), and this response is false.]

b.) (4/13) - (4/17). [From what was said above, this is the one.]

c.) (8/13) - (8/17). [Nope.]

d.) None of the above. [Nope.]



26.) A 1.2 meter long bar is clamped at its center and tapped gently once at one end. The bar rings with a frequency equal to 200 hertz. The wave velocity of the disturbance as it moves through the bar will be:

a.) 960 m/s. [We need to determine the wavelength that will resonate on this length bar. Noting that a waveform with antinodes at both ends and a node in the middle is required, we conclude that the longest wavelength whose frequency does the job (this will be the lowest frequency to do the job--there are others) will be equal to 2L, or 2.4 meters. Using the relationship $v = \lambda\nu = (2.4m)(200hz) = 480m/s$, this response is false.]

b.) 480 m/s. [Looks like this is the one.]

c.) 240 m/s. [Nope.]

d.) None of the above. [Nope.]

27) During a 10 second time interval, the power provided to a body by a constant force is 375 watts. If the body was initially at rest:

a.) The amount of work done during the interval is 3750 joules. [Power is defined as the amount of work done per unit time, or P = W/t. Using that equation, we get (375 watts) = W/(10 sec), or W = 3750 joules. This statement is true, but it might not be the only true statement in the bunch (note that Response d says, "All of the above.")]

b.) Doubling the time over which the power-producing force is applied will quadruple the distance traveled. [How is distance related to time? The force is constant, which means the acceleration is constant. Using kinematics, the net distance d traveled (i.e., $x_2 - x_1$) is $d = v_o t$

$+ .5at^2$. With the initial velocity equal to zero, $d = .5at^2$, or the distance is proportional to the square of the time. Doubling the time should quadruple the distance, which means this statement is true. Note: This has nothing to do with power--it's essentially a kinematics problem. This is not atypical of A.P. test problems. They set you up with information given in one context (in this case, in the context of work and power), then require you to use some other approach entirely to analyze one or more of the responses. BEWARE!]

c.) Doubling the time over which the power-producing force is applied will double the power provided to the body. [From above, doubling the time quadruples the distance (i.e., the distance goes to 4d). Using the power expression, $P_{\text{time doubled}} = F(4d)/2t. = 2(Fd/t)$. This statement is true.]

d.) All of the above. [This statement is true.]

28.) A rotating disk has moment of inertia I and kinetic energy K. The body's angular momentum will be:

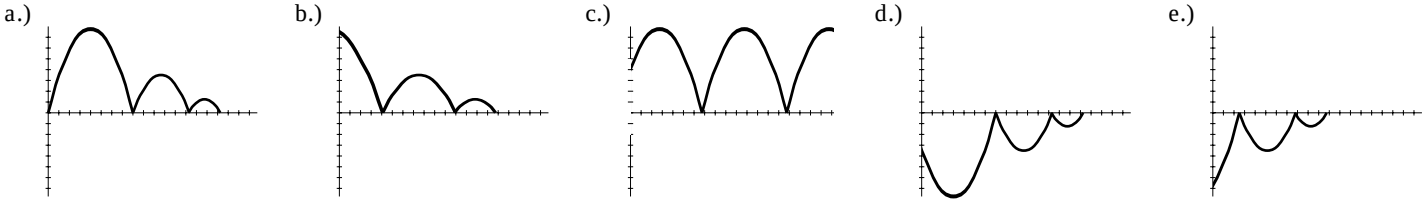
a.) K^2I. [The way to do problems like this is to write out the units for the quantity you are interested in, write out the units for the quantities you are given, then match the two. Angular momentum has the units of $I\omega$. Kinetic energy has the units of $(1/2)I\omega^2$. Multiplying the kinetic energy expression by 2I yields $I^2\omega^2$. Taking the square root yields $I\omega$. Combining the units of kinetic energy and moment of inertia to get the units of angular momentum suggests that $L = (2KI)^{1/2}$, and this response is false.]

b.) $2K^2/I$. [Nope.]

c.) $(K^2I)^{1/2}$. [Nope.]

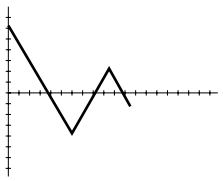
d.) $(2KI)^{1/2}$. [Yup.]

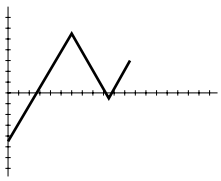
29.) A mass m is fired downward (i.e., vertically) with velocity v_0 off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's position as a function of time?

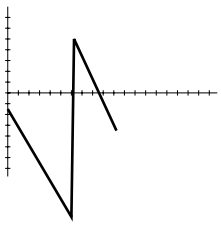


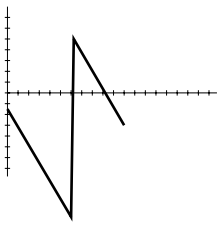
[Commentary: Visualize a ball being thrown downward. The position of the ball will always be in the positive region (i.e., above the ground), so graphs d and e cannot be right. The first motion of the ball is downward, so it can't be graphs a and c (it can't be c because energy is lost with each bounce. That leaves graph b.]

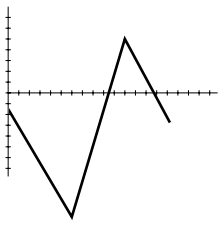
30.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's velocity as a function of time?

a.) 

b.) 

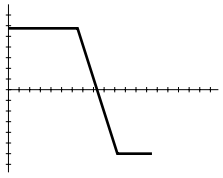
c.) 

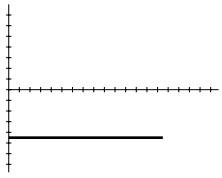
d.) 

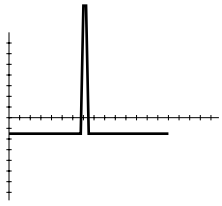
e.) 

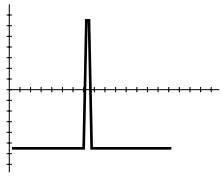
[Commentary: Again, visualize a ball being thrown downward. Its velocity will start out downward with a sign that is negative (this eliminates graph a), and its freefall velocity gets larger and larger in the negative direction until the ball hits the ground (graph b starts negative, but its magnitude gets less negative). At the ground, the velocity changes from large and negative to large and positive almost instantly. Graph e's transition from negative to positive is slow, so that's not the one. Graphs c and d appear to be the same with one small difference. The slope of the velocity on graph c <u>after the transition</u> is not the same as its slope before the transition. As the change of velocity (the slope) has to be the same at all times during freefall, graph d gets the nod.]

31.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's acceleration as a function of time?

a.) 

b.) 

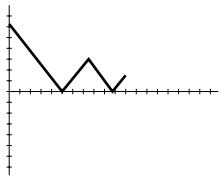
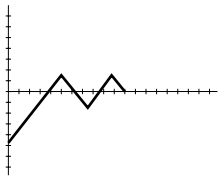
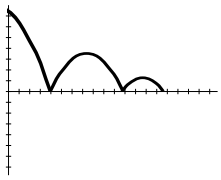
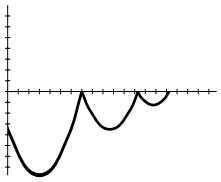
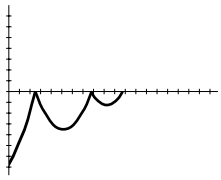
c.) 

d.) 

e.) None of these graphs.

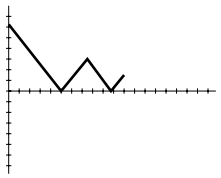
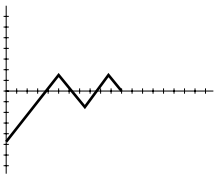
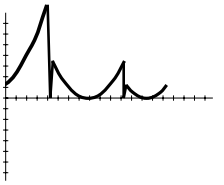
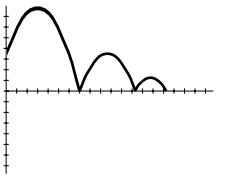
[Commentary: The mass's freefall acceleration will be negative (this eliminates graph a) and constant until the ball hits the ground. During contact with the ground, the mass's acceleration becomes positive (this eliminates graph b), very large in comparison to g (this eliminates graph d), and lasts for only an instant. Graph c fits the bill.]

32.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's potential energy as a function of time, assuming $U_{grav} = 0$ at ground level?

a.)  b.)  c.)  d.)  e.) 

[Commentary: The gravitational potential energy function is linear with distance (mgy) near the earth's surface. Distance, on the other hand, is related to the square of the time (remember $(x_2 - x_1) = v_1t + .5at^2$?). As such, the gravitational potential energy should vary non-linearly (this eliminates graphs a and b). As gravitational potential energy will always be positive in this case (eliminating graphs d and e), graph c is the best fit. (Notice also that after the bounce, the potential energy increases before decreasing again. Notice also that this function mimics the position vs. time function previously found).]

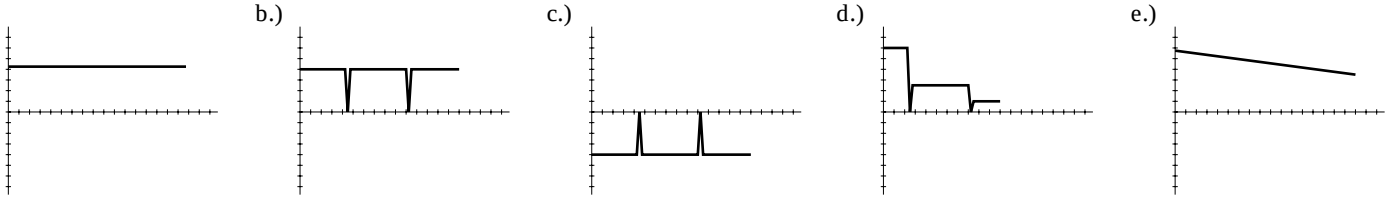
33.) A mass m is fired downward (i.e., vertically) with velocity v_0 off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's kinetic energy as a function of time?

a.)  b.)  c.)  d.)  e.) None of the graphs.

[Commentary: As potential energy is proportional to t^2, kinetic energy must also be non-linear (this eliminates graphs a and b). In fact, KE and U should be mirror images of one another (as the total energy of the system is conserved, an increase of KE must be met with an equal decrease of U, and vice versa). The mirror image to Problem 7.6c (this situation's potential energy vs time graph) is graph c. This may be difficult to see due to the long vertical lines that take KE to zero at the bounce, but if you ignore those lines you will see that the general contour of the two graphs are, indeed, inverted images of one another. If you didn't see the mirror image effect, the thing to notice is that the kinetic energy should get larger until the bounce, should go to zero at the bounce, should be very large just after the bounce, should decrease to zero as the mass rises into the air and finally stops, then should increase as it heads back toward the ground. Graph c does that. Graph d doesn't.]

34.) A mass m is fired downward (i.e., vertically) with velocity v_0 off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's total energy as a function of time?

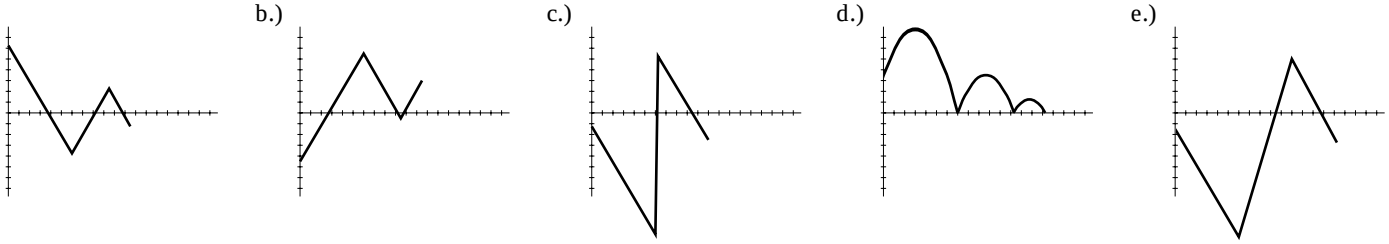
a.) b.) c.) d.) e.)



[Commentary: The total energy will be positive (this eliminates graph c) and conserved (i.e., constant--this eliminates graph e) as long as the mass is in the air. When it hits, the total energy (potential plus kinetic) momentarily goes to zero, then goes to a non-zero value that is less than before the bounce (this eliminates graph b). Graph d is the one.]

35.) A mass m is fired downward (i.e., vertically) with velocity v_o off a building that is h meters high. The mass hits the ground and bounces inelastically. Which graph shown below best depicts the body's momentum as a function of time?

a.) b.) c.) d.) e.)



[Commentary: Think again about the mass and its initial motion. Its velocity is negative, which means its momentum will be negative (this eliminates graphs a and d). The velocity magnitude gets bigger as the mass falls, hence momentum magnitude will also get bigger in a negative sense (this eliminates graph b). The velocity changes from a big negative value to a relatively big positive value almost instantaneously (i.e., through the bounce), hence the momentum does the same (this quick change eliminates graph e). Graph c looks like the one.]

Semester Equations in a Nutshell

When studying for a semester final, most people spend at least some time memorizing formulas. In fact, what makes finals difficult is not knowing relationships. It is being able to look at a problem with absolutely no insight as to which chapter the problem is from, identify the approach that will most probably allow you to analyze the situation, then be able to follow through with the execution of that approach. Obviously, to execute *any* of the approaches, you will need to know some equations. What you need to be aware of is the fact that KNOWING THE EQUATIONS WILL NOT GUARANTEE PASSING THE TEST. If you don't understand what the variables within a formula mean, or don't know when a formula is applicable, or don't know what the formula can do for you (and what it *can't* do), formulas will not help a bit.

Having said all of that, you will nevertheless have to review to be sure you do know the equations you are most likely going to need. To aid in this, the following is designed to allow you to determine just how much you actually know off the top of your head at this point in time.

Take a few minutes and see how many of the following relationships you can write down cold (i.e., without studying first).

-- the kinematic equations for translational motion:
-- the kinematic equations for rotational motion:
-- N.S.L. (the translational version):
-- N.S.L. (the rotational version):
-- centripetal acceleration relationship:
-- static frictional force:
-- kinetic frictional force:
-- the steps to N.S.L.
-- kinetic energy for translational motion:
-- kinetic energy for rotational motion:
-- definition of work for a constant force/angle/displacement situation:
-- definition of work for a variable force/angle/displacement situation:
-- relationship between work and a potential energy function:
-- Work/energy theorem:
-- gravitational potential energy function near the surface of the earth:
-- gravitational potential energy function far from the surface of the earth:
-- gravitational force function near the surface of the earth:
-- gravitational force function far from the surface of the earth:
-- force function for an ideal spring:
-- potential energy function for an ideal spring:
-- derivation of potential energy function from force function:

-- derivation of force function from potential energy function:
-- modified conservation of energy expression:
-- momentum expression:
-- impulse on a single body:
-- impulse equation:
-- deriving center of mass expression for a discrete distribution of mass:
-- deriving center of mass expression for a continuous distribution of mass:
-- modified conservation of momentum expression:
-- moment of inertia for a point mass:
-- integral form of the definition of moment of inertia:
-- parallel axis theorem:
-- center of mass acceleration of an angularly accelerating, rotating body:
-- center of mass velocity of a body rotating with known angular velocity:
-- *r perpendicular*--how to get it:
-- torque determine using the *r perpendicular* approach
-- torque determine using the *F perpendicular* approach
-- torque determine using the *definition* approach
-- angular momentum determine using rotational parameters:
-- angular momentum determine using translational parameters:
-- expression for the position of a vibrating object as a function of time, or x(t) :
-- characteristic equation associated with *simple harmonic motion*:
-- period as function of the frequency of a vibrating body:
-- angular frequency of a vibrating spring whose spring constant is k:
-- angular frequency of a simple pendulum whose length is L:
-- angular frequency as function of the frequency:
-- a spring system's maximum velocity (and where does it happen?):
-- a spring system's maximum acceleration (and where does it happen?):
-- a spring system's maximum energy:
-- relationship between wave velocity, frequency, and wavelength

INDEX

acceleration
  angular, 240
  average, 56
  centripetal, 119-121
  definition of, 56
  instantaneous, 57
  sign significance, 59-62
  velocity vs. time graph, 58
angular acceleration
  average, 240
  instantaneous, 241
  of massive pulley, 271
  torque and Newton's Second Law, 269
angular displacement, 236
angular momentum
  in general, 291-297
  in orbital systems, 314
angular velocity
  as function of translational velocity, 241-242
  average, 236
  instantaneous, 239
  sign significance, 237-239
anti-derivative, 46
Atwood machine
  energy considerations, 186
  N.S.L. with massive pulley, 272

beats, 376-377
Brahe, Tycho, 307

center of mass
  area mass-density fct., 202
  for continuous masses, 199
  for discrete masses, 198
  frame of reference, 204, 225-230
  in general, 197-203
  volume mass-density fct., 200
centrifugal force, 124
centripetal acceleration
  definition of, 116
  derivation of, 119-122
centripetal force
  deadman's curve, 114
  hammer throw, 113
  in general, 110-122
  inclined curve, 116
  M.O.B. maneuver, 110
  planetary motion, 113
  tires on a road, 112
chain rule, 31-33
coefficient of friction
  kinetic, 90
  static, 92
collisions
  elastic, inelastic, and perfectly inelastic, 218
  in general, 209-220
  with energy conserved, 217-219
conservation of energy
  derivation of modified version, 175-185
  in celestial systems, 323
  in multi-body systems, 185-187
  in rotational systems, 284-290
  symmetry in celestial systems, 328-329
conservation of momentum
  derivation of modified version, 205-209
  in general, 205-220
conservative force
  commentary, 164
  example--gravity, 157-160
  in general, 160
cross product
  direction of, 16, 22
  in polar notation, 15
  in unit vector notation, 17
  matrix evaluation, 18
  physical significance, 20-21
  right hand rule, 17
curl, 38-40

del operator
definition of, 34
used on potential energy fct's., 174-175
derivatives
a geometric approach, 30
families of fcts., 29
formal definition, 27
Gauss's Law, 37
general rules, 30
notation, 26
partial derivatives
in theory, 33
slope of a tangent, 26
the chain rule, 31, 33
the del operator
the curl, 38
the divergence, 36
the gradient, 35
the del operator, 34
vector fields, 36
differential displacement dx, 30
differential displacement vector
in polar notation, 146, 148
in unit vector notation, 144, 150
differential time interval dt, 30
displacement from time-dependent velocity function, 56
displacement from velocity vs. time graph, 55
divergence, 36-37
Doppler shift, 377-381
dot product
in polar notation, 12
in unit vector notation, 13
physical significance, 14

energy
a definition of, 153
gravitational potential energy (general fct.), 169, 321
gravitational potential energy (near-Earth fct.), 162
gravitational potential energy and rotating systems, 283
in center of mass frame of ref., 227
in collisions, 218
energy (con't.)
in orbital systems, 324-329
in rotational systems, 281-290
in the rotation of a pinned beam, 284
in vibrational system, 349
of a rolling ball
from center-of-mass perspective, 287
from pure-rotation perspective, 288
rotation and translation combined, 285-290
rotational kinetic energy, 249
spring potential energy, 172
translational kinetic energy, 153
equilibrium and rigid bodies, 266

fictitious forces
centrifugal force, 124-127
linear, 127-128
forces
centripetal, 110
derived from potential energy fct., 174-175
frictional
kinetic friction, 90
static friction, 92
gravitational, 87
normal, 88
push-me pull-you, 92
tension, 89
freefall with friction, 106-110

Gauss's Law
differential form, 37
gradient, 35
gravitation
energy considerations, 323-328
energy symmetry, 328-329
inside a massive object, 317
Newton's Law of, 309-312
potential energy
multibody systems, 322
potential energy fct. for, 321

Hooke's Law, 172, 336

ideal spring, 171
impulse
    definition of, 195
    the integral form, 224
impulse, the integral form
    when work/energy won't do, 224-225
inertia, 122
integral
    area under curve, 43
    differential interval, 44
    notation, 45
    theoretical basis, 46
integrals
    Examples, 47

Kepler's Laws
    in general, 312-317
    Law of Areas, 313
    Law of Orbits, 312
    Law of Periods, 315
Keppler, Johannes, 307
kinematics
    equations derived, 64-69
    graphical relationships, 67
    projectile motion
        horizontal displacement, 75
        in general, 72-79
        maximum height, 77
        preliminaries, 73
        time of flight, 75
        touchdown velocity, 77
        velocity at max. height, 76
    simple examples, 69-72
    statement of, 62-64
    two-dimensional motion, 72-79
kinetic energy
    derivation of, 151-153
    rotational, 289
    translational, 153

mass
    area density fct., 202, 252
    gravitational, 123
    inertial, 122
    linear density fct., 254
    volume density fct., 200, 253
moment of inertia
    area mass-density fct., 252
    chart showing various forms, 257
moment of inertia (con't.)
    for a continuous mass, 251-256
    for discrete masses, 249-251
    in general, 248-256
    linear mass-density fct., 254
    the parallel axis theorem, 251
    volume mass-density fct., 253
momentum
    conservation of, 205-219
    definition of, 195
    vector addition of, 196
momentum and energy
    what when, 222-224
momentum and impulse, the integral form
    when work/energy won't do, 224-225

Newton's Laws
    a statement of, 85-87
    types of forces, 87-92
Newton's Second Law
    angular acceleration, 269
    approach to use,
    approaches, 93
    examples
        Atwood Machine with massive pulley, 272-274
        ball rolling down incline, 274
        centripetal motion, 110-119
        gravitational force inside massive object, 319
        in a pendulum system, 351
        incline plane with massless pulley, 99-106
        orbital motion, 312
        rigid bodies, 266
        translational motion only, 94-110
        with Kepler's Third Law, 316
    free body diagrams, 94
    non-linear acceleration (freefall with friction), 106-110
    rolling objects
        summary of approaches, 281
    rotation and translation combined, 270-281
    rotation viewed from center of mass perspective, 274-276
    rotation viewed from pure rotation perspective, 276-278
    rotational analogue, 269-281
    the formal approach, 94-98, 99-104

Newton's Second Law (con't.)
the seat-of-pants approach, 98, 104-106
time dependent velocity, 128-131
torque and pure rotation, 269-270
Newton, Sir Isaac, 85, 307

parallel axis theorem, 251
potential energy
derivation for known conservative force, 166-173
for an ideal spring, 172
gravitational
preamble, 160-163
in general, 160-174
Newton's general gravitational fct., 321
with work/energy theorem, 164-166
power
definition of, 187-188
units, 188
projectile motion, 72-79

radian measure, 235
resonance
in a string system, 362
in an air column, 373
in general wave motion, 361
sound in a steel bar, 369
standing waves
constraints test, 372
rigid bodies, 266-269
rolling
angular and translational motion related, 241-242
rolling with sliding (skidding)
the bowling ball problem, 297-300
rotational kinematics, 242-246
rotational parameters
a plug for, 246-248

scalar
definition, 1
simple harmonic motion
mathematics of, 336-340, 352
slope of a secant, 25
speed
average, 51
instantaneous, 52
standing waves
in a column of air, 373-376
in a steel bar, 369-371
on a string system, 362-368
systems of units, 88

torque
direction of, 263
in general, 261-266
magnitude of, 262
rotational version of Newton's Second Law, 269
to calculate
by definition, 263
by F perpendicular, 265
by r perpendicular, 263-265

vectors
direction of, 2
example of, 1
graphical manipulation, 3
graphical representation, 2
in polar notation, 5
in unit vector notation, 7
magnitude of, 2
multiplied by scalar, 4
polar to unit vector, 11
resultant, 3
unit vector to polar
in general, 9
quadrants problem, 10
velocity
angular, 236
definition of, 52
direction of, 53
instantaneous, 53
magnitude of, 53
position vs. time graph, 54
sign significance, 59
time dependent
frictional freefall, 106-110
general, 128-131
vibrational motion
amplitude of, 335
angular frequency of, 341
criteria for, 333
displacement, 335
force functions, 335
frequency of, 335

vibrational motion (con't.)
in an ideal spring, 333
period of, 335
periodic motion, 335
simple harmonic motion
angular frequency, 340-342
derivation of, 336-340
energy within, 349
phase shift, 342-349
summary, 349-350
simple harmonic motion, 335
simple pendulum, 350-352

waves
definition of, 355
Doppler shift
in astronomy, 380
frequency of, 359
longitudinal waves, 357
mathematical characterization of, 360
nodes and anti-nodes, 359
period of, 359
reflection of, 358
resonance
explanation of, 361-362
sonic boom, 379
standing waves
constraints test, 372
in a column of air, 373-376
in a steel bar, 369-371
on a string system, 362-368
superposition of, 359
transverse waves, 357
velocity of, 359
wave train
longitudinal, 356
transverse, 355
wave-length of, 359
work
definition, 139
differential displacement
in polar notation, 146, 148
done by variable forces, 143-151
due to frictional forces, 140, 142
due to normal forces, 141
due to variable force
differential displacement in polar notation, 148
work (con't.)
due to variable force
differential displacement in unit vector notation, 144, 150
sign significance, 143
work/energy theorem, 151-157